The QCD equation of state at finite density from analytical continuation

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The \((T, \mu_B)\)-phase diagram of QCD

Our observables:
Last Year: \(T_c\)


This year: The Equation of State along trajectories of constant \(\frac{S}{N_B}\) and its Taylor coefficients determined by the method of analytical continuation

\[ J. \text{Günther et al., arXiv:1607.02493} \]
Analytic continuation

- Roberge-Weiss
- Continuation
- \( d(p/T^4)/d\mu \)
- \( Tc(\mu) \)
- \( \mu^2/T^2 \)
- Lattice simulations
- Real chemical potentials

\( \mu^2/T^2 \): 2.4, 2.0, 1.6, 1.2
Overview over the Analysis

1. Do the simulations at $\langle n_s \rangle \approx 0$
2. Extrapolate to $\langle n_s \rangle = 0$ and $\langle n_Q \rangle = 0.4 \langle n_B \rangle$
3. Make a fit in the $T$ direction
4. Determine everything you need for the observables
5. Make a fit in the $\mu_B$ direction
6. Make a fit in the $\frac{1}{N_f^2}$ direction
7. Determine the observables
Action: tree-level Symanzik improved gauge action, with four times stout smeared staggered fermions

2+1+1 flavour, on LCP with pion and kaon mass

Simulation at $\langle n_S \rangle = 0$ (as for heavy ion collisions, in contrast to simulations with $\mu_s = 0$ or $\mu_S = 0$ where $\mu_S = \frac{1}{3} \mu_B - \mu_s$)

Lattice sizes: $40^3 \times 10$, $48^3 \times 12$ and $64^3 \times 16$

$\frac{\mu_B}{T} = i\frac{j\pi}{8}$ with $j = 0, 3, 4, 5, 6, 6.5$ and $7$

Two methods of scale setting: $f_\pi$ and $w_0$, $Lm_\pi > 4$
Tuning to $\langle n_S \rangle = 0$

Aim: For a given $\mu_B$ determine $\mu_S$ so that $\langle n_S \rangle = 0$. This means solving the differential equation

$$\langle n_S \rangle = 0 \iff \frac{\partial \log Z}{\partial \mu_S} = 0$$

Notation:

$$\chi_{udsc} = -\frac{1}{T^4} \frac{\partial^4}{\partial (\mu_u/T) \partial (\mu_d/T) \partial (\mu_s/T) \partial (\mu_c/T)} \frac{T}{V} \log Z$$

Assuming we know the value for $\mu_S(\mu_B)$ so that $\langle n_s \rangle = 0$ for $\mu_S(\mu_B^0)$ and $\mu_S(\mu_B^0 - \Delta \mu_B)$ with all the derivatives. Then (Runge-Kutta):

$$\mu_S(\mu_B^0 + \Delta \mu_B) = \mu_S(\mu_B^0 - \Delta \mu_B) + 2\Delta \mu_B \frac{d\mu_S}{d\mu_B}(\mu_B^0)$$

In the simulations with $\mu_B^0$ and $\mu_B^0 - \Delta \mu_B$, $\mu_S$ might not precisely tuned. There we want to extrapolate to a better value. We assume that correct value of $\mu_S$ is $\tilde{\mu}_S = \mu_S' + \Delta \mu_S'$. Then:

$$\langle n_S \rangle = \frac{\partial \log Z}{\partial \tilde{\mu}_S} = \frac{\partial \log Z}{\partial \mu_S'} + \frac{\partial^2 \log Z}{\partial \mu_S'^2} \Delta \mu_S' = 0$$
Tuning to $\langle n_S \rangle = 0$

This yields

$$\Delta \mu'_S = -\frac{\chi_S}{\chi_{SS}}$$

Similar for the derivative we get:

$$\frac{d\tilde{\mu}_S}{d\mu_B} = -\frac{\tilde{\chi}_{SB}}{\tilde{Z}_{SS}} |\langle n_S \rangle = 0 = -\frac{\chi_{SB}}{\chi_{SS}} - \frac{\chi_{SSB}\chi_{SS} - \chi_{SSS}\chi_{SB}}{(\chi_{SS})^2} \Delta \mu'_S + O(\Delta \mu'_S^2)$$

With similar arguments we get to extrapolate to $\langle n_Q \rangle = 0.4 \langle n_B \rangle$
Fit in the $T$ direction

\[ \frac{d\hat{P}}{d\hat{\mu}_B} = \mu = 0.785397i \]

\[ \chi^2 / N_{dof} = 0.977524 \]
\[ Q = 0.480529 \]

\[ T \text{ MeV} \]

\[ \frac{A_1(T)}{A_2(T)} = \frac{a + bT + c/T + d \arctan(e(T - f))}{a + bT + c/T + d/(1 + e(T - f)^g)^{1/g}}, \]

\[ A_3(T) = a + bT + cT^2 + d \arctan(e(T - f)) \]

\[ A_4(T) = a + bT + cT^2 + d/(1 + e(T - f)^g)^{1/g}. \]
Fit in the $\mu_B$ direction

\[ B_1(\hat{\mu}) = a + b\hat{\mu}^2 + c\hat{\mu}^4 \]

\[ B_2(\hat{\mu}) = \frac{(a + b\hat{\mu}^2)}{1 + c\hat{\mu}^2} \]

\[ B_3(\hat{\mu}) = a + b\hat{\mu}^2 + c\sin(\hat{\mu})/\hat{\mu} \]
Extrapolation from different fit functions

Analytical continuation on $N_t = 12$ raw data

$T \frac{d\langle p/T^4 \rangle}{d\mu_B}$

$T=170\text{MeV}$

$T=145\text{MeV}$

$(\mu_B/T)^2 = -\hat{\mu}^2$

$a + b\hat{\mu}^2 + c\hat{\mu}^4$

$(a + b\hat{\mu}^2)/(1 + c\hat{\mu}^2)$

$a + b\hat{\mu}^2 + c\sin(\hat{\mu})/\hat{\mu}$
Entropy

- \( S = \frac{\partial P}{\partial T} \bigg|_{\mu_i} \)
- We have \( \frac{d\hat{P}}{d\hat{\mu}_B} \)
- And we can only do a total derivative in \( T \)

\[
\frac{d\hat{P}}{dT} = \left. \frac{\partial P}{\partial T} \right|_{\mu_i} + \frac{d\mu_B}{dT} \frac{\hat{n}_B}{T} + \frac{d\mu_Q}{dT} \frac{\hat{n}_Q}{T}
\]

How we correct this:

\[
\frac{d\mu_B}{dT} = \frac{d(\hat{\mu}_B T)}{dT} = \hat{\mu}_B \\
\frac{d\mu_Q}{dT} = \frac{d(\hat{\mu}_Q T)}{dT} = \hat{\mu}_Q + T \frac{d\hat{\mu}_Q}{dT}
\]

\[
\hat{S} = 4\hat{P} + T \frac{d\hat{P}}{dT} - \hat{\mu}_B \hat{n}_B - \hat{\mu}_Q \hat{n}_Q - T \frac{d\hat{\mu}_Q}{dT}
\]

\[
= 4\hat{P} + T \frac{d\hat{P}}{dT} - \hat{n}_B (\hat{\mu}_B + 0.4\hat{\mu}_Q) - T \frac{d\hat{\mu}_Q}{dT}
\]

New terms we have to determine
Error estimation

- **Statistical error:**
  Bootstrap method

- **Systematic error:**
  Using different way of analysis, combining them in a histogram:
  - 4 fit functions for the $T$ direction
  - 3 fit functions in the $\mu_B$ direction
  - Doing continuum extrapolation and $\mu_B$-fit in one or two steps
  - 2 methods of scale setting: $f_\pi$ and $w_0$
  - 2 temperatures from where we use the extrapolated data

This adds up to 96 ways of analysis
The Taylor coefficients of \( \frac{P}{T^4} = c_0 + c_2 \left( \frac{\mu_B}{T} \right)^2 + c_4 \left( \frac{\mu_B}{T} \right)^4 + c_6 \left( \frac{\mu_B}{T} \right)^6 \)

Influence of different orders

\[ T = 145 \text{MeV} \]

\[ O(\mu_B) \]

\[ O(\mu_B^3) \]

\[ O(\mu_B^5) \]
Influence of different orders

$T = 170\text{MeV}$

$O(\mu_B)$

$O(\mu^3_B)$

$O(\mu^5_B)$

$n_B/T^3$ vs $\mu_B/T$
Trajectories

\begin{align*}
\frac{S}{N_B} &= 420 \ (200 \ \text{GeV}) \\
\frac{S}{N_B} &= 144 \ (62.4 \ \text{GeV}) \\
\frac{S}{N_B} &= 94 \ (39 \ \text{GeV}) \\
\frac{S}{N_B} &= 70 \ (27 \ \text{GeV}) \\
\frac{S}{N_B} &= 51 \ (19.6 \ \text{GeV}) \\
\frac{S}{N_B} &= 30 \ (14.5 \ \text{GeV})
\end{align*}
Equation of state

\[
\frac{P}{T^4} = 51 \quad \frac{S}{N} = 420
\]

\[
\frac{\epsilon - 3P}{T^4} = 51 \quad \frac{S}{N} = 420
\]