A variational method for spectral functions

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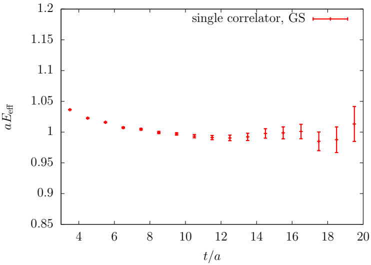
Outline

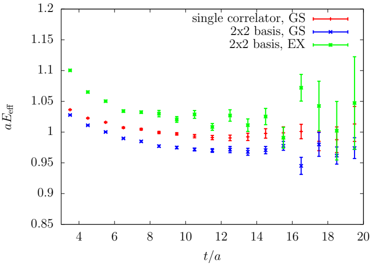
- Usual GeVP: short review
- Backus-Gilbert method
- One variational example in frequency space: Υ-channel from NRQCD
- Conclusions

GeVP in configuration space

- What can one do in order to resolve better a dense spectrum of states??

- Well established technique to reliably extract energy-levels from Euclidean Correlators in Lattice QCD.
- of special importance for states living near a multi-particle production threshold.





Data from Mohler *et. al*, *Phys. Rev. Lett.* **111** 222001 (2013)
"D^*_{s0} (2317) Meson and D-Meson-Kaon scattering from Lattice QCD"

A short reminder ...

The central ideal consists in replacing the Euclidean correlation function by a matrix of correlators

$$G(\tau) \to G_{ij}(\tau) = \left\langle O_i(\tau) O_j^\dagger(0) \right\rangle$$

where $\{O_i(\tau)\}_{i=1,\ldots,N}$ is a set of operators with common quantum numbers:

- one can choose *different gamma structures, covariant derivatives, different smearings, multi particle operators, ...* in order to construct the basis. The only restriction is that they have to couple to the same energy states.
- *"the more linearly independent they are chosen at the beginning, the better the subspace will be spanned."*

Where is the gain?

Using the spectral decomposition we can write

$$G_{ij}(\tau) = \sum_{n=1}^{\mathcal{N}} Z_{ij}^{(n)} e^{-E_n \tau}, \qquad Z_{ij} = Z_i^{(n)} Z_j^{(n)*}, \qquad Z_i^{(n)} = \langle 0 | O_i(0) | n \rangle$$

- Notice that $G = G^\dagger$ by construction and has therefore N^2 real d.o.f.
- On the other hand, Z has $2N - 1$ real d.o.f. because it is the direct product of one complex vector and its h.c. (rank$(Z) = 1$, this will be important later on.)
- Therefore, a counting of accessible d.o.f. tells us that

$$\#d.o.f.\,[G_{ij}(\tau)] = N^2 N_T, \qquad \#d.o.f.\left[\sum_{n=1}^{\mathcal{N}} Z_{ij}^{(n)} e^{-E_n \tau}\right] = (2N-1+1)\mathcal{N}$$

- as we increase the value of N we are constraining more the problem than if we considered a single operator. It is common to set $N = \mathcal{N}$.

GeVP in coordinate space

Once the matrix $G_{ij}(\tau)$ is constructed, one solves the Generalized eigenvalue Problem

$$G_{ij}(\tau)v_j^{(n)}(\tau,\tau_0) = \lambda^{(n)}(\tau,\tau_0)G(\tau_0)_{ij}v_j^{(n)}(\tau,\tau_0), \qquad n = 1, ..., N$$

with $G(\tau_0)$ acting like a metric in this subspace. Assuming non-degenerate eigenvalues, their form is shown to be

$$\lambda^{(n)}(\tau,\tau_0) = e^{-E_n\tau} + O(e^{-(E_{N+1}-E_n)\tau}), \qquad \text{if } \tau_0 \geq \frac{\tau}{2}$$

Blossier *et. al* [0902.1265]

and the energy-levels are extracted in the usual way

$$E_{eff,n}(\tau + a/2) = \log\left(\frac{\lambda^{(n)}(\tau)}{\lambda^{(n)}(\tau + a)}\right)$$

Spectral functions and their importance

- Spectral functions contain all information for a given a channel.
- Relevant at $T \neq 0$, they encode real-time properties of the medium (diffusion of conserved charges, differential production rates, ...)
- Formally, it is defined as

$$\rho(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \; e^{i\omega t} \; \mathrm{Tr}\left\{\hat{\rho}[O(t), O^{\dagger}(0)]\right\}, \qquad \hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}$$

- and its relation to the euclidean correlator $G(\tau)$ is defined via an integral equation. For the case of symmetric correlators one has,

$$G(\tau) = \int_{0}^{\infty} d\omega \rho(\omega) K(\omega, \tau), \qquad K(\omega, \tau) = \frac{\cosh(\omega(\beta/2 - \tau))}{\sinh(\omega\beta/2)} \overset{\beta \to \infty}{\to} e^{-\omega\tau}$$

The numerical inversion of the last equation is numerically an ill-posed problem. Regularization needed!

Talk of Mr. Haitao SHU on 28/7 at 15:00, *Stochastic approaches to extract spectral functions from Euclidean correlators*

The Backus-Gilbert method: an (old) linear method

Recently used in: B. Brandt, A. Francis, H. Meyer, DR [1506.05732]
A. Francis, O. Kaczmarek, M. Laine, T. Neuhaus, H. Ohno [1508.04543]
B. Brandt, A. Francis, B. Jäger, H. Meyer [1512.07249]

$$\boxed{G(\tau) = \int_0^\infty d\omega \rho(\omega) K(\omega, \tau)}$$

Define an estimator $\hat{\rho}(\bar{\omega})$ which is a "filtered" version of the true spectral function (it is just a linear combination of the input):

$$\hat{\rho}(\bar{\omega}) = \int_0^\infty d\omega \hat{\delta}(\bar{\omega}, \omega) \rho(\omega) = \sum_{\alpha=1}^{N_T} q_\alpha(\bar{\omega}) G(\tau_\alpha) \qquad \text{(linear method)}$$

The coefficients $q_\alpha(\bar{\omega})$ also define the resolution function:

$$\hat{\delta}(\bar{\omega}, \omega) = \sum_{\alpha=1}^{N_T} q_\alpha(\bar{\omega}) K(\omega, \tau_\alpha)$$

Variational method in frequency space

1. Construct from the matrix of euclidean correlators $G_{ij}(\tau)$, the matrix of spectral functions estimators $\hat{\rho}_{ij}(\bar{\omega})$ via the BG-method.
   - ... we use the same common resolution function $\forall i, j$

(i, j label the operator basis. α, β label the time slices).

2. Solve the Generalized Eigenvalue Problem

$$\hat{\rho}_{ij}(\bar{\omega}) v_j^{(n)}(\bar{\omega}, \tau_0) = \lambda^{(n)} \underbrace{G_{ij}(\tau_0)}_{\text{metric}} v_j^{(n)}(\bar{\omega}, \tau_0)$$

   - $G(\tau_0)$ serves to the purpose of having a reference value such that overall normalization factor of operators are irrelevant.

One example: the Υ- channel from NRQCD

- From perturbative results we know that these type of spectral functions go like $\sim \omega^{1/2}$ for $\omega \to \infty$. Burnier *et. al* [0711.1743].
- We therefore use a reweighting function to write

$$G(\tau) = \int_0^\infty \left(\frac{\rho(\omega)}{\sqrt{\omega}}\right) \underbrace{\left(e^{-\omega\tau}\sqrt{\omega}\right)}_{K(\tau,\omega)}$$

$$\hat{\rho}(\bar{\omega}) = \int_0^\infty d\omega \hat{\delta}(\bar{\omega},\omega) \left(\frac{\rho(\omega)}{\sqrt{\omega}}\right)$$

- Similar, to other methods that attempt to solve inverse-problems, the Backus-Gilbert method has a regulator ξ which trades off resolving power vs. error:

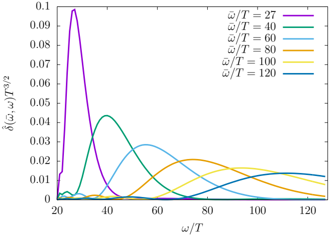
$$\Delta\hat{\rho}(\bar{\omega}) = \sqrt{q_\alpha(\bar{\omega}) S_{\alpha\beta} q_\beta(\bar{\omega})}, \qquad \text{(error on } \hat{\rho}(\bar{\omega})\text{)}$$

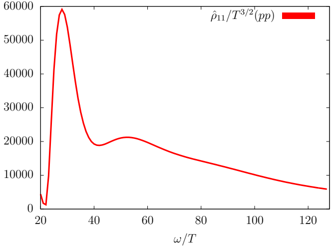
$$\Gamma(\bar{\omega}) = q_\alpha(\bar{\omega}) \underbrace{\int_0^\infty d\omega K(\tau_\alpha,\omega)(\omega-\bar{\omega})^2 K(\tau_\beta,\omega)}_{W_{\alpha\beta}(\bar{\omega})} q_\beta(\bar{\omega}) \quad \text{(width of } \hat{\delta}(\bar{\omega},\omega)\text{)}$$

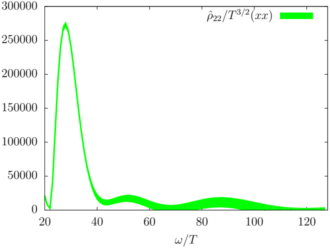
$$\boxed{W_{\alpha\beta}(\bar{\omega}) \overset{\text{reg.}}{\to} \xi W_{\alpha\beta}(\bar{\omega}) + (1-\xi) S_{\alpha\beta}, \qquad \text{tuning of } \xi \text{ necessary.}}$$

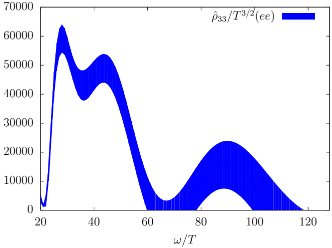
Preliminary results: Υ from NRQCD

- Three different operators were considered: a point source (p), a gaussian smeared (x) and an "excited" (e) one.

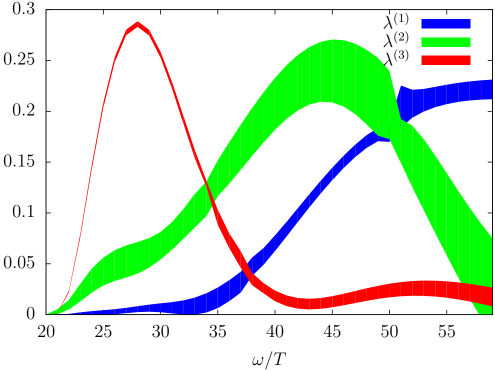








Eigenvalues from the GeVP



- Hierarchy between eigenvalues indicates a rank $= 1$ structure for the Z matrix ...
  - ...it is the case if the width of the resolution function is smaller than the separation between states (enough resolving power).
  - ...also when the basis of operators is not linearly independent enough (say all operators couple very strongly to one and the same state)

A perturbative inspired mock-data study

- Consider the Euclidean Lagrangian density with $\mathbb{Z}_2$-symmetry: $\phi \mapsto -\phi$

$$\mathcal{L} = \frac{1}{2}(\partial K)^2 + \frac{1}{2}m_K^2 K^2 + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m_\phi^2\phi^2 + \mathcal{L}_{\text{int}}$$

$$\mathcal{L}_{\text{int}} = \frac{g}{2}K\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\lambda_k}{4!}K^4 + \frac{g_K}{3!}K^3 + \frac{h}{4}K^2\phi^2$$

(a similar model was already used in L. Lellouch, M. Lüscher [hep-lat/0003023] in the context of finite volume $K \to \pi\pi$ transitions)

- and consider the operators:

$$O^{(1)}(x_0) = \int d^3x\, e^{i\mathbf{px}} K(x_0, \mathbf{x}), \quad O^{(2)}(y_0) = \int d^3y\, d^3z\, e^{i\mathbf{k}(\mathbf{y}-\mathbf{z})}\phi(y_0,\mathbf{y})\phi(y_0,\mathbf{z})$$

that create/annihilate a K-particle at time x_0 with momentum $\mathbf{p}$ and 2ϕ-particles at time y_0 with back-to-back momentum $\mathbf{k}$.

- The goal is to calculate the matrix of euclidean correlators

$$G_{ij}(x_0 - y_0) = \left\langle O^{(i)}(x_0) O^{(j)\dagger}(y_0) \right\rangle, \qquad i,j = 1,2$$

to some useful order in the couplings and feed it through the method to check our expectations.

The idea is ...

$$\left\langle \mathcal{O}^{(1)}(x_0,\mathbf{p})\mathcal{O}^{(1)}(y_0,\mathbf{k})\right\rangle = \frac{e^{-\omega_{\mathbf{p}}^{(K)}|x_0-y_0|}}{2\omega_{\mathbf{p}}} L^3\delta_{\mathbf{p},-\mathbf{k}} + \dots$$

$$\left\langle \mathcal{O}^{(1)}(x_0,\mathbf{p})\mathcal{O}^{(2)}(y_0,\mathbf{k})\right\rangle = g\,\frac{L^3\delta_{\mathbf{p},\mathbf{0}}}{4\omega_{\mathbf{p}}^K(\omega_{\mathbf{k}}^\pi)^2((\omega_{\mathbf{p}}^K)^2-4(\omega_{\mathbf{k}}^\pi)^2)}\left(\omega_{\mathbf{p}}^K e^{-2\omega_{\mathbf{k}}^\pi|x_0-y_0|} - 2\omega_{\mathbf{k}}^\pi e^{-\omega_{\mathbf{p}}^K|x_0-y_0|}\right) + \dots$$

$$\begin{aligned}\left\langle \mathcal{O}^{(2)}(x_0,\mathbf{p})\mathcal{O}^{(2)}(y_0,\mathbf{k})\right\rangle &= \frac{L^6}{4(\omega_{\mathbf{p}}^\pi)^2} e^{-2\omega_{\mathbf{p}}^\pi|x_0-y_0|}(\delta_{\mathbf{p},-\mathbf{k}}+\delta_{\mathbf{p},\mathbf{k}}) \\ &+ g^2\frac{L^3}{4\omega_{\mathbf{p}}^\pi\omega_{\mathbf{k}}^\pi}\left(\frac{2e^{-m_K|x_0-y_0|}}{m_K(4(\omega_{\mathbf{k}}^\pi)^2-m_K^2)(4(\omega_{\mathbf{p}}^\pi)^2-m_K^2)} + \frac{e^{-2\omega_{\mathbf{k}}^\pi|x_0-y_0|}}{4\omega_{\mathbf{k}}((\omega_{\mathbf{k}}^\pi)^2-(\omega_{\mathbf{p}}^\pi)^2)(4(\omega_{\mathbf{k}}^\pi)^2-m_K^2)}\right. \\ &\left. + \frac{e^{-2\omega_{\mathbf{p}}^\pi|x_0-y_0|}}{4\omega_{\mathbf{p}}((\omega_{\mathbf{p}}^\pi)^2-(\omega_{\mathbf{k}}^\pi)^2)(4(\omega_{\mathbf{p}}^\pi)^2-m_K^2)}\right) \\ &+ (\text{t, u-channels}) \\ &+ \lambda\frac{L^3}{4\omega_{\mathbf{p}}^\pi\omega_{\mathbf{k}}^\pi}\left(\frac{e^{-2\omega_{\mathbf{p}}^\pi|x_0-y0|}}{4\omega_{\mathbf{p}}^\pi((\omega_{\mathbf{k}}^\pi)^2-(\omega_{\mathbf{p}}^\pi)^2)} - \frac{e^{-2\omega_{\mathbf{k}}^\pi|x_0-y0|}}{4\omega_{\mathbf{k}}^\pi((\omega_{\mathbf{k}}^\pi)^2-(\omega_{\mathbf{p}}^\pi)^2)}\right)\end{aligned}$$

Conclusions

- Backus-Gilbert method proves to be useful compared to other reconstruction methods.
- GeVP analysis helps in the identification of states $\rightarrow$ hierarchy in the eigenvalues (rank $= 1$).
- Possibility of defining optimal operators that strongly overlap to local regions in frequency.

$$O^{(\mathrm{opt})}(\bar{\omega}, x) = v_i^{(n)}(\bar{\omega}) O_i(x)$$

  - Can we use this method to identify and resolve resonances?

- Perturbative calculations may help checking our expectations. (tree-level results completed).
- Currently constructing a QCD dataset both at finite temperature as well as zero temperature with ~ 6 operators.