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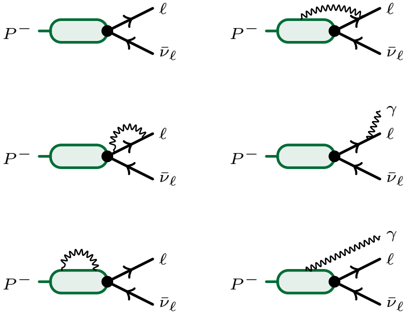
QED corrections to $\mathrm{P}^- \to \ell\bar{\nu}(\gamma)$: finite volume effects

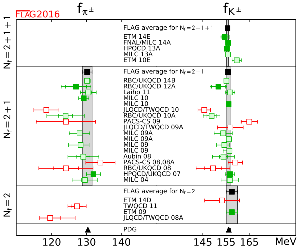
southampton, 27-07-2016

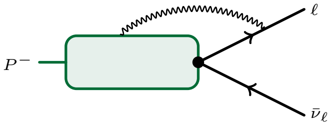
- phenomenological motivation
- infrared-safe measurable observables
- the RM123-SOTON strategy
- universality of IR logs and $1/L$ terms
- sums approaching integrals
- analytical result for $\Delta\Gamma_0^{pt}(L)$
- conclusions & outlooks



$$\Delta\Gamma_0(L) - \Delta\Gamma_0(\infty) = c_{IR}\,\log\left(L^2 m_P^2\right) + \frac{c_1}{L m_P} + O\left(\frac{1}{L^2}\right)$$

FLAG, arXiv:1607.00299
PDG review, j.rosner, s.stone, r.van de water, 2016
v.cirigliano et al., Rev.Mod.Phys. 84 (2012)





- from the last FLAG review we have

$$f_{\pi^\pm} = 130.2(1.4)\ \text{MeV}\ , \qquad \delta = 1.1\%\ ,$$

$$f_{K^\pm} = 155.6(0.4)\ \text{MeV}\ , \qquad \delta = 0.3\%\ ,$$

$$f_+(0) = 0.9704(24)(22)\ , \qquad \delta = 0.3\%$$

- QED corrections are currently estimated in χ-pt

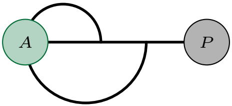
$$\delta_{QED}\Gamma[\pi^- \to \ell\bar{\nu}] = 1.8\%\ ,$$

$$\delta_{QED}\Gamma[K^- \to \ell\bar{\nu}] = 1.1\%\ ,$$

$$\delta_{QED}\Gamma[K \to \pi\ell\bar{\nu}] = [0.5, 3]\%$$

- let's consider the extraction of a matrix-element from an euclidean correlator

$$C(t,\boldsymbol{p}) = \langle 0|A(t)\,P(0,\boldsymbol{p})|0\rangle\;, \qquad t>0$$

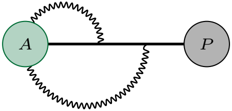


- we know very well what we have to do when there is a mass-gap

$$C(t,\boldsymbol{0}) = \langle 0|A|P(\boldsymbol{0})\rangle\,\frac{\langle P(\boldsymbol{0})|P|0\rangle}{2m_P}\,e^{-tm_P} + R(t)\;,$$

$$R(t) = R_1\,e^{-tE_1} + \cdots$$

- in presence of electromagnetic interactions, the states



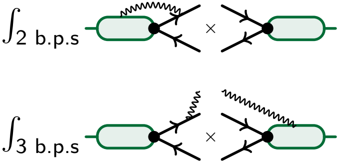
$$\mathbb{H}\,|P\gamma_1\cdots\gamma_n\rangle =$$

$$\left\{\sqrt{m_P^2 + (\boldsymbol{k}_1 + \cdots \boldsymbol{k}_n)^2} + |\boldsymbol{k}_1| + \cdots + |\boldsymbol{k}_n|\right\}|P\gamma_1\cdots\gamma_n\rangle$$

are *degenerate* with $|P(\boldsymbol{0})\rangle$ in the limit $\boldsymbol{k}_i \mapsto \boldsymbol{0}$

f.bloch, a.nordsieck, Phys.Rev. 52 (1937)
t.d.lee, m.nauenberg, Phys.Rev. 133 (1964)
p.p.kulish, l.d.faddeev, Theor.Math.Phys. 4 (1970)

- the infrared problem has been analyzed by many authors over the years
- electrically-charged asymptotic states are *not* eigenstates of the photon-number operator
- the perturbative expansion of decay-rates and cross-sections with respect to α is cumbersome because of the degeneracies
- the block & nordsieck approach consists in lifting the degeneracies by introducing an infrared regulator, say m_γ, and in computing infrared-safe observables
- at any fixed order in α, infrared-safe observables are obtained by adding the appropriate number of photons in the final states and by integrating over their energy in a finite range, say $[0, \Delta E]$
- in this framework, infrared divergences appear at intermediate stages of the calculations and cancel in the sum of the so-called *virtual* and *real* contributions



$$(p+k)^2 + m_P^2 = 2p\cdot k + k^2 \sim 2p\cdot k \ ,$$

$$\int \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^2\,(2p\cdot k)\,(2p_\ell\cdot k)} \sim c_{IR}\log\left(\frac{m_P}{m_\gamma}\right) \ ,$$

$$c_{IR}\left\{\log\left(\frac{m_P}{m_\gamma}\right) + \log\left(\frac{m_\gamma}{\Delta E}\right)\right\} = c_{IR}\,\log\left(\frac{m_P}{\Delta E}\right)$$

RM123, Phys.Rev. D87 (2013)
RM123+SOTON, Phys.Rev. D91 (2015)

- we have proposed to compute the leptonic decay-rate of a pseudoscalar meson at $O(\alpha)$; in this case the infrared-safe observable is obtained by considering the real contributions with a single photon in the final state

$$\Gamma(\Delta E) = \Gamma_0^{\text{tree}} + e^2 \lim_{L\to\infty} \{\Delta\Gamma_0(L) + \Delta\Gamma_1(L, \Delta E)\}$$

- given a formulation of QED on the finite volume, L acts as an infrared regulator in the previous formula

- the finite-volume calculation of the real contribution is challenging: momenta are quantized and one would need very large volumes in order to perform the three-body phase space integral in the soft-photon region with an acceptable resolution; for this reason we have rewritten the previous formula as

$$\Gamma(\Delta E) = \Gamma_0^{\text{tree}} + e^2 \lim_{L\to\infty} \left\{\Delta\Gamma_0(L) - \Delta\Gamma_0^{pt}(L) + \Delta\Gamma_0^{pt}(L) + \Delta\Gamma_1(L, \Delta E)\right\}$$

- $\Delta\Gamma_0^{pt}(L)$ is the virtual decay rate calculated in the effective theory in which the meson is treated as a point-like particle; the so-called structure dependent contributions are given by

$$\Delta\Gamma_0^{SD}(L) = \Delta\Gamma_0(L) - \Delta\Gamma_0^{pt}(L)$$

RM123+SOTON, Phys.Rev. D91 (2015)
h.georgi, Ann.Rev.Nucl.Part.Sci. 43 (1993)

- the lagrangian of the point-like effective theory is

$$\mathcal{L}_{pt} = \phi_P^\dagger(x)\left\{-D_\mu^2 + m_P^2\right\}\phi_P(x) + \left\{2iG_F V_{CKM}\, f_P D_\mu \phi_P^\dagger(x)\,\bar{\ell}(x)\gamma^\mu \nu(x) + \text{h.c.}\right\} \;,$$

$$D_\mu = \partial_\mu - ieA_\mu(x)$$

- the matching with the full theory is obtained by using Γ_0^{tree}

$$\Gamma_0^{\text{tree},pt} = \Gamma_0^{\text{tree}} = \frac{G_F^2\,|V_{CKM}|^2 f_P^2}{8\pi}\, m_P^3 r_\ell^2 \left(1-r_\ell^2\right)^2 \;, \qquad r_\ell = \frac{m_\ell}{m_P}$$

- properly matched effective theories have the *same* infrared behaviour of the full theory: $\Delta\Gamma_0^{pt}(L)$ has *exactly* the same infrared divergence of $\Delta\Gamma_0(L)$ and we can write

$$\Gamma(\Delta E) = \Gamma_0^{\text{tree}} + e^2 \lim_{L\to\infty} \Delta\Gamma_0^{SD}(L) + e^2 \lim_{m_\gamma\to\infty}\left\{\Delta\Gamma_0^{pt}(m_\gamma) + \Delta\Gamma_1^{pt}(m_\gamma,\Delta E)\right\} + O\left(\frac{\Delta E}{\Lambda_{QCD}}\right)$$

- we have shown that the neglected terms are phenomenologically irrelevant for $P=\{\pi,K\}$ and $\Delta E \sim 20$ MeV

RM123+SOTON, Phys.Rev. D91 (2015)

- in our original proposal we have *not* performed an analysis of the finite volume corrections affecting $\Delta\Gamma_0(L)$: we are now going to fill the gap!

- the $L \mapsto \infty$ asymptotic expansion of the decay rate can be written as

$$\Delta\Gamma_0(L) - \Delta\Gamma_0(\infty) = c_{IR}\,\log\left(L^2 m_P^2\right) + \frac{c_1}{L m_P} + O\left(\frac{1}{L^2}\right)$$

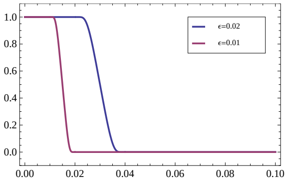
$$\Delta\Gamma_0^{pt}(L) - \Delta\Gamma_0^{pt}(\infty) = c_{IR}\,\log\left(L^2 m_P^2\right) + \frac{c_1}{L m_P} + O\left(\frac{1}{L^2}\right)$$

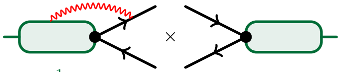
- in the following, we shall show that the coefficients c_{IR} and c_1 are *universal*, i.e. they are the same in the full theory and in the point-like approximation

- therefore, the finite volume effects on the non-perturbative structure-dependent contributions are

$$\Delta\Gamma_0^{SD}(L) - \Delta\Gamma_0^{SD}(\infty) = O\left(\frac{1}{L^2}\right)$$

- than we shall give an explicit analytical expression for $\Delta\Gamma_0^{pt}(L)$





- to see how this works, let's consider the contribution to the decay rate coming from the diagrams shown in the figure

$$\Delta\Gamma_{P\ell}(L) - \Delta\Gamma_{P\ell}(\infty) = \left\{ \frac{1}{L^3}\sum_{\boldsymbol{k}\neq 0} - \int \frac{d^3k}{(2\pi)^3} \right\} \int \frac{dk^0}{2\pi}\, H^{\alpha\mu}(k,p)\, \frac{1}{k^2}\, \frac{\mathcal{L}_{\alpha\mu}(k)}{2p_\ell\cdot k + k^2}$$

- infrared divergences and power-law finite volume effects come from the singularity at $k^2 = 0$ of the integrand and from the QED$_L$ prescription $\boldsymbol{k} \neq \boldsymbol{0}$

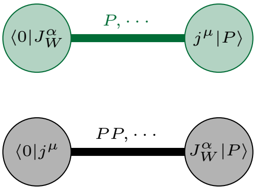
- the tensor $\mathcal{L}_{\alpha\mu}$ is a regular function, it contains the numerator of the lepton propagator and the appropriate normalization factors

$$\mathcal{L}_{\alpha\mu}(k) \equiv \mathcal{L}_{\alpha\mu}(k, p_\nu, p_\ell) = O(1)$$

- the hadronic tensor is a QCD quantity that, by neglecting exponentially suppressed finite volume effects, is given by

$$H^{\alpha\mu}(k,p) = i \int d^4x\, e^{ik\cdot x}\, T\langle 0|\, J_W^\alpha(0)\, j^\mu(x)\, |P(\mathbf{0})\rangle \ ,$$

$$H_{pt}^{\alpha\mu}(k,p) = f_P \left\{ \delta^{\alpha\mu} - \frac{(p+k)^\alpha\, (2p+k)^\mu}{2p\cdot k + k^2} \right\}$$



- the point like effective theory is built in such a way to satisfy the same WIs of the full theory

$$k_\mu\, H^{\alpha\mu}(k,p) = -f_P\, p^\alpha \ , \qquad H_{SD}^{\alpha\mu}(k,p) = H^{\alpha\mu}(k,p) - H_{pt}^{\alpha\mu}(k,p) \ , \qquad k_\mu\, H_{SD}^{\alpha\mu}(k,p) = 0$$

- the structure dependent contributions are regular and, since there is no constant two-index tensor orthogonal to k,

$$H_{SD}^{\alpha\mu}(k,p) = \left(p\cdot k\, \delta^{\alpha\mu} - k^\alpha p^\mu\right) F_A + \epsilon^{\alpha\mu\rho\sigma} p_\rho k_\sigma F_V + \cdots = O(k)$$

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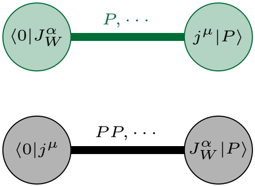

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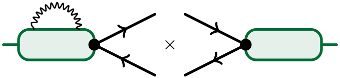
- structure-dependent terms can be also understood in the effective field theory language by adding all the operators compatible with the symmetries of the full-theory, e.g.

$$\mathcal{O}_V(x) = F_V\, \epsilon^{\mu\nu\rho\sigma}\, D_\mu \phi_P(x)\, F_{\nu\rho}(x)\, \bar{\ell}(x)\gamma_\sigma \nu(x)$$

$$\mathcal{L}_{\alpha\mu}(k) = O(1) \ , \qquad H^{\alpha\mu}_{SD}(k,p) = O(k)$$



- from the regularity of $\mathcal{L}_{\alpha\mu}$ and from the previous relation we get

$$\Delta\Gamma^{SD}_{P\ell}(L) - \Delta\Gamma^{SD}_{P\ell}(\infty) = \left\{ \frac{1}{L^3} \sum_{\boldsymbol{k}\neq 0} - \int \frac{d^3k}{(2\pi)^3} \right\} \int \frac{dk^0}{2\pi} \frac{\mathcal{L}_{\alpha\mu}(k)\, H^{\alpha\mu}_{SD}(k,p)}{k^2\,(2p_\ell\cdot k + k^2)}$$

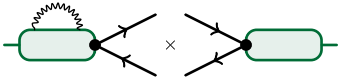
$$= \left\{ \frac{1}{L^3} \sum_{\boldsymbol{k}\neq 0} - \int \frac{d^3k}{(2\pi)^3} \right\} \int \frac{dk^0}{2\pi} \frac{O(k)}{k^2\,(2p_\ell\cdot k)}$$

$$= O\left(\frac{1}{L^2}\right)$$

- the other contributions, represented in the figure, can be analyzed by using similar arguments and we get our result

$$\Delta\Gamma^{SD}_{0}(L) = \Delta\Gamma^{SD}_{0}(\infty) + O\left(\frac{1}{L^2}\right)$$

$$\left\{ \frac{1}{L^3} \sum_{\boldsymbol{k}\neq 0} - \int \frac{d^3k}{(2\pi)^3} \right\} \int \frac{dk^0}{2\pi} \frac{1}{k^n} = O\left(\frac{1}{L^{4-n}}\right)$$



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$$\Delta\Gamma^{SD}_0(L) = \Delta\Gamma^{SD}_0(\infty) + O\left(\frac{1}{L^2}\right)$$

m.hayakawa, s.uno, Prog.Theor.Phys. 120 (2008)
BMW, Science 347 (2015)
b.lucini et al., JHEP 1602 (2016)

- in order to get an analytical expression for $\Delta\Gamma_0^{pt}(L)$ we have evaluated infrared divergent sums as the following

$$C_{P\ell}(L) = -\frac{8p\cdot p_\ell}{L^3}\sum_{\boldsymbol{k}\neq 0}\int \frac{dk^0}{2\pi}\,\frac{1}{k^2\,\left[2p\cdot k + k^2\right]\,\left[2p_\ell\cdot k + k^2\right]}$$

- that we managed to rewrite as

$$C_{P\ell}(L) = -\frac{(1+r_\ell^2)\log(r_\ell^2)}{16\pi^2(1-r_\ell^2)}\left\{2\log\left(L^2 m_P^2\right) + \log(r_\ell^2)\right\} + \zeta_C(\boldsymbol{\beta_\ell})$$

$$+\frac{1}{(m_P L)^3}\,\frac{(1+3r_\ell^2)(3+6r_\ell^2-r_\ell^4)}{4(1+r_\ell^2)^3}$$

- the $1/L^3$ term is peculiar of QED$_L$ and would be absent in a local formulation of the theory such as QED$_C$

- in the previous expression we have used the kinematics of the process, i.e. $p = p_\ell + p_\nu$, from which it follows

$$E_\ell = \frac{m_P}{2}(1+r_\ell^2)\,, \qquad \boldsymbol{p}_\ell = \hat{\boldsymbol{p}}_{\boldsymbol{\ell}}\,\frac{m_P}{2}(1-r_\ell^2)\,, \qquad \boldsymbol{\beta}_\ell = \frac{\boldsymbol{p}_\ell}{E_\ell}\,, \qquad r_\ell = \frac{m_\ell}{m_P}$$

- we have introduced generalized ζ-functions that depend upon an external spatial momentum ($\Omega' = 2\pi\mathbb{Z}^3 - \{\mathbf{0}\}$)

$$\zeta_C(\boldsymbol{\beta_\ell}) = \frac{1}{2\beta_\ell}\log\left(\frac{1+\beta_\ell}{1-\beta_\ell}\right)\frac{\log(u_\star)+\gamma_E}{4\pi^2} - \frac{4u_\star^{3/2}}{3\sqrt{\pi}}$$

$$+\frac{2}{\sqrt{\pi}}\sum_{\boldsymbol{k}\in\Omega'}\frac{\Gamma\left(\frac{3}{2},u_\star\boldsymbol{k}^2\right)}{|\boldsymbol{k}|^3\left[1-(\hat{\boldsymbol{k}}\cdot\boldsymbol{\beta}_\ell)^2\right]}\left\{1+\frac{e^{u_\star(\boldsymbol{k}\cdot\boldsymbol{\beta}_\ell)^2}\,\bar{\Gamma}\left[\frac{3}{2},u_\star(\boldsymbol{k}\cdot\boldsymbol{\beta}_\ell)^2\right]}{|\hat{\boldsymbol{k}}\cdot\boldsymbol{\beta}_\ell|\,e^{u_\star\boldsymbol{k}^2}\,\Gamma\left(\frac{3}{2},u_\star\boldsymbol{k}^2\right)}\right\}$$

$$+\frac{1}{4\pi^2}\sum_{\boldsymbol{n}\neq\mathbf{0}}\int_0^{\frac{4u_\star}{\boldsymbol{n}^2}}\frac{du}{u}e^{-\frac{1}{u}}\int_0^{\frac{1}{1+\beta_\ell}}dy\,\frac{1-\frac{2y(\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta_\ell})}{\sqrt{u(1-2\beta_\ell y)}}\,\mathrm{D}\left(\frac{y(\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta_\ell})}{\sqrt{u(1-2\beta_\ell y)}}\right)}{(1-2\beta_\ell y)}$$

where $u_\star > 0$ is an arbitrary parameter, $\zeta_C(\boldsymbol{\beta_\ell})$ does not depend upon $u_\star$, and

$$\Gamma(\alpha,x)=\int_x^\infty du\,u^{\alpha-1}\,e^{-u}\;,\quad \bar{\Gamma}(\alpha,x)=\int_0^x du\,u^{\alpha-1}\,e^{-u}\;,\quad \mathrm{D}(x)=e^{-x^2}\int_0^x du\,e^{u^2}$$

- this is an horrible expression (we have other equivalent horrible expressions) but can be evaluated with remarkable numerical accuracy . . .

m_P (MeV)	m_ℓ	β_ℓ	$\hat{\boldsymbol{\beta}}_{\boldsymbol{\ell}}$	$\zeta_B(\boldsymbol{\beta}_\ell)$	$\zeta_C(\boldsymbol{\beta}_\ell)$
m_{π^+}	m_μ	0.27138338825	$(1,1,1)/\sqrt{3}$	-0.05791071589	-0.06331584128
m_{K^+}	m_μ	0.91240064548	$(1,1,1)/\sqrt{3}$	-0.10350847338	-0.09037019089
319.94	m_μ	0.80332680614	$(1,1,1)/\sqrt{3}$	-0.08090777589	-0.07877650869
382.36	m_μ	0.85811529992	$(1,1,1)/\sqrt{3}$	-0.08960375038	-0.08359870731
439.50	m_μ	0.89072556952	$(1,1,1)/\sqrt{3}$	-0.09706060796	-0.08737355417
273.50	m_μ	0.74027641641	$(1,1,1)/\sqrt{3}$	-0.07428926453	-0.07477600535
256.19	m_μ	0.70926754699	$(1,1,1)/\sqrt{3}$	-0.07184408338	-0.07321735266
299.65	m_μ	0.77883567253	$(1,1,1)/\sqrt{3}$	-0.07801627478	-0.07706625341
433.26	m_μ	0.88773322628	$(1,1,1)/\sqrt{3}$	-0.09627652081	-0.08699199510
221.79	m_μ	0.63006264555	$(1,1,1)/\sqrt{3}$	-0.06711881612	-0.07006731685
252.97	m_μ	0.70292547354	$(1,1,1)/\sqrt{3}$	-0.07139283129	-0.07292458544
573.28	m_μ	0.93429632487	$(1,1,1)/\sqrt{3}$	-0.11167875480	-0.09376593376
607.84	m_μ	0.94134202978	$(1,1,1)/\sqrt{3}$	-0.11470049030	-0.09488055773

$$\zeta_B(\mathbf{0}) = -0.05644623986\ , \qquad \zeta_C(\mathbf{0}) = -0.06215473226$$

- notice that the ζ-functions are functions of a single variable

$$\boldsymbol{\beta}_\ell = \frac{\boldsymbol{p}_\ell}{E_\ell} = \hat{\boldsymbol{p}}_{\boldsymbol{\ell}}\,\frac{1-r_\ell^2}{1+r_\ell^2}\ , \qquad r_\ell = \frac{m_\ell}{m_P}$$

- our final result for $\Delta\Gamma_0^{pt}(L)$, to be used in order to apply our strategy in numerical simulations, is

$$\frac{\Delta\Gamma_0^{pt}(L) - \Delta\Gamma_0^{\ell\ell}(L)}{\Gamma_0^{\text{tree}}} = c_{IR}\log(L^2 m_P^2) + c_0 + \frac{c_1}{(m_P L)} + O\left(\frac{1}{L^2}\right)$$

where

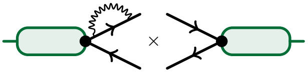
$$c_{IR} = \frac{1}{8\pi^2}\left\{\frac{(1+r_\ell^2)\log(r_\ell^2)}{(1-r_\ell^2)} + 1\right\},$$

$$c_0 = \frac{1}{16\pi^2}\left\{2\log\left(\frac{m_P^2}{m_W^2}\right) + \frac{(2-6r_\ell^2)\log(r_\ell^2) + (1+r_\ell^2)\log^2(r_\ell^2)}{1-r_\ell^2} - \frac{5}{2}\right\} + \frac{\zeta_C(\mathbf{0}) - 2\zeta_C(\boldsymbol{\beta}_\ell)}{2},$$

$$c_1 = -\frac{2(1+r_\ell^2)}{1-r_\ell^2}\zeta_B(\mathbf{0}) + \frac{8r_\ell^2}{1-r_\ell^4}\zeta_B(\boldsymbol{\beta}_\ell)$$

and we have shown only the universal terms

- notice that the lepton wave-function contribution to the decay rate, $\Delta\Gamma_0^{\ell\ell}(L)$, does not contribute to $\Delta\Gamma_0^{SD}(L)$



- our method to calculate $O(\alpha)$ QED radiative corrections to hadronic decay rates is based on the block & nordsieck approach and on the universality of infrared divergences

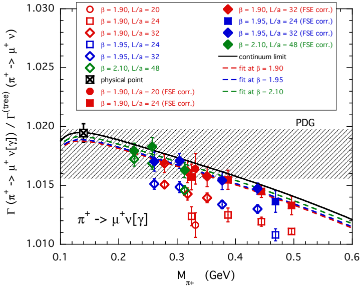
- the infrared divergent term in the non-perturbative virtual decay rate is cancelled by subtracting the same quantity calculated in the point-like effective theory

- we have now computed analytically $\Delta\Gamma_0^{pt}(L)$

- and shown that, together with the infrared divergence, also the leading $1/L$ finite volume effects are universal and cancel in the difference $\Delta\Gamma_0(L) - \Delta\Gamma_0^{pt}(L)$

- therefore, finite volume effects on the non-perturbative structure-dependent contributions start to contribute at $O(1/L^2)$

- with the results presented in this talk, all the ingredients are now in place for a non-perturbative calculation of the $O(\alpha)$ leptonic decay rate of pseudoscalar mesons



see next talk by s.simula

backup material

RM123+SOTON, Phys.Rev. D91 (2015)

- notice that $\Delta\Gamma_0(L)$ and $\Delta\Gamma_0^{pt}(L)$ are ultraviolet divergent

- the divergence can be reabsorbed into a renormalization of G_F, both in the full theory and in the point-like effective theory

- we have analized the renormalization of the four-fermion weak operator on the lattice in details and calculated the matching coefficients to the so-called W-regularization

$$\frac{1}{k^2} \mapsto \frac{1}{k^2} - \frac{1}{k^2 + m_W^2}$$

- indeed, this is the regularization conventionally used to extract G_F from the muon decay

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[1 - \frac{8m_e^2}{m_\mu^2}\right]\left[1 + \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right)\right]$$

- this is the reason why one has an ultraviolet divergent log depending upon m_W in the analytical result for $\Delta\Gamma_0^{pt}(L)$ shown above

- in order to calculate $C_{P\ell}(L)$ it is convenient to introduce a second infrared regulator and to separate the infrared-divergent infinite volume integral from the corresponding finite volume corrections

$$C_{P\ell}(L) = \lim_{\varepsilon\to 0} \{C_{P\ell}(\varepsilon) + \Delta C_{P\ell}(L,\varepsilon)\} \ ,$$

$$C_{P\ell}(\varepsilon) = -8p\cdot p_\ell \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2+\varepsilon^2]\ [2p\cdot k + k^2+\varepsilon^2]\ [2p_\ell\cdot k+k^2+\varepsilon^2]}$$

$$= \frac{(1+r_\ell^2)\log(r_\ell^2)}{16\pi^2(1-r_\ell^2)} \left\{2\log\left(\frac{\varepsilon^2}{m_P}\right) - \log(r_\ell^2)\right\} \ ,$$

$$\Delta C_{P\ell}(L,\varepsilon)$$

$$= \left\{\sum_{\boldsymbol{k}\in\Omega'} - \int\frac{d^3k}{(2\pi)^3}\right\} \int \frac{dk^0}{2\pi} \frac{8E_\ell m_P L^2}{[k^2+(L\varepsilon)^2]\ [2Lp\cdot k+k^2+(L\varepsilon)^2]\ [2Lp_\ell\cdot k+k^2+(L\varepsilon)^2]}$$

where we have made the change of variables $k \mapsto k/L$ and made explicit our choice of reference frame

$$p = (im_P, \mathbf{0}) \ , \qquad p_\ell = (iE_\ell, \boldsymbol{p}_\ell)$$

- we now combine the three denominators by introducing two Feynman's parameters

$$\frac{8E_\ell m_P L^2}{\left[k^2+(L\varepsilon)^2\right]\left[2Lp\cdot k+k^2+(L\varepsilon)^2\right]\left[2Lp_\ell\cdot k+k^2+(L\varepsilon)^2\right]}$$

$$=\int_0^1 dy \int_0^L dx\, x\, \frac{16E_\ell m_P}{\left\{(k+xp_y)^2+x^2m_y^2+(L\varepsilon)^2\right\}^3}$$

where we have defined

$$p_y = yp_\ell + (1-y)p \ ,$$

$$m_y^2 = -p_y^2 = y^2m_\ell^2 + (1-y)^2m_P^2 + 2y(1-y)E_\ell m_P \ > \ 0$$

- it is important to notice that $m_y^2 > 0$ and it is also useful to introduce the following quantities

$$e_y^2 = m_y^2 + y^2\boldsymbol{p}_\ell^2 > 0 \ , \qquad \boldsymbol{q}_y = \frac{y}{m_y}\,\boldsymbol{p}_\ell$$

- the k^0-integral appearing in the $\Delta C_{P\ell}(L,\varepsilon)$ formula can now be traded for a Schwinger's parameter integral

$$\int \frac{dk^0}{2\pi} \frac{1}{\left\{(k+xp_y)^2 + x^2 m_y^2 + \varepsilon^2\right\}^3} = \frac{1}{4\sqrt{\pi}} \int_0^\infty du\, u^{3/2}\, e^{-u\left\{(\boldsymbol{k}+x\boldsymbol{p}_y)^2 + x^2 m_y^2 + (L\varepsilon)^2\right\}}$$

- an extremely useful trick to evaluate this kind of sums consists in splitting the Schwinger's parameter integral at an arbitrary scale $u_\star > 0$

$$\int_0^\infty du = \int_0^{u_\star} du + \int_{u_\star}^\infty du$$

the contribution to the sum corresponding to $u \in [u_\star, \infty]$ is then calculated in momentum space

$$\left\{ \sum_{\boldsymbol{k}\in\Omega'} - \int \frac{d^3k}{(2\pi)^3} \right\} \int_{u_\star}^\infty du\, f(u,\boldsymbol{k}) \quad \to \quad C^+ + \hat{C}^0$$

while the other contribution is calculated in coordinate space by using Poisson's summation formula

$$\left\{ \sum_{\boldsymbol{k}\in\Omega'} - \int \frac{d^3k}{(2\pi)^3} \right\} \int_0^{u_\star} du\, f(u,\boldsymbol{k})$$

$$= -\int_0^{u_\star} du\, f(u,\boldsymbol{0}) + \sum_{\boldsymbol{n}\neq\boldsymbol{0}} \int_0^{u_\star} du \int \frac{d^3k}{(2\pi)^3} f(u,\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{n}} \quad \to \quad C^- + C^0$$

- by applying this trick we have

$$\Delta C_{P\ell}(L,\varepsilon) = C^0 + \hat{C}^0 + C^+ + C^- \ ,$$

$$C^0 = -\frac{4E_\ell m_P}{\sqrt{\pi}} \int_0^{u_\star} du\, u^{3/2} e^{-u(L\varepsilon)^2} \int_0^1 \frac{dy}{m_y^2} \int_0^{Lm_y} dx\, x\, e^{-ux^2\left[\boldsymbol{q}_y^2+1\right]} \ ,$$

$$\hat{C}^0 = -\frac{4E_\ell m_P}{\sqrt{\pi}} \int_{u_\star}^{\infty} du\, u^{3/2} e^{-u(L\varepsilon)^2} \int_0^1 \frac{dy}{m_y^2} \int_0^{Lm_y} dx\, x\, e^{-ux^2} \int \frac{d^3k}{(2\pi)^3} e^{-u(\boldsymbol{k}+x\boldsymbol{q}_y)^2}$$

$$C^+ = \frac{4E_\ell m_P}{\sqrt{\pi}} \sum_{\boldsymbol{k}\in\Omega'} \int_{u_\star}^{\infty} du\, u^{3/2} e^{-u(L\varepsilon)^2} \int_0^1 \frac{dy}{m_y^2} \int_0^{Lm_y} dx\, x\, e^{-u\left[(\boldsymbol{k}+x\boldsymbol{q}_y)^2+x^2\right]} \ ,$$

$$C^- = \frac{E_\ell m_P}{2\pi^2} \sum_{\boldsymbol{n}\neq 0} \int_0^{u_\star} du\, e^{-u(L\varepsilon)^2} \int_0^1 \frac{dy}{m_y^2} \int_0^{Lm_y} dx\, x\, e^{-u\left\{x^2+\frac{xi\boldsymbol{n}\cdot\boldsymbol{q}_y}{u}\right\}-\frac{\boldsymbol{n}^2}{4u}}$$

- notice that, except for $\hat{C}^0$, the x-integral can be extended up to ∞ at the price of neglecting exponentially suppressed finite volume effects (remember that $m_y > 0$)

- moreover, except for C^0, one can set $\varepsilon = 0$ in the remaining integrals by neglecting regular terms

- indeed, the infrared divergence is contained in C^0

$$C^0 = -\frac{(1+r_\ell^2)\log(r_\ell^2)}{2(1-r_\ell^2)}\,\frac{\log(L^2\varepsilon^2)+\log(u_\star)+\gamma_E}{4\pi^2}$$

- also the evaluation of the integrals entering in the expression of $\hat{C}^0$ is straightforward,

$$\hat{C}^0 = -\frac{4u_\star^{3/2}}{3\sqrt{\pi}} + \frac{1}{(m_P L)^3}\,\frac{(1+3r_\ell^2)(3+6r_\ell^2-r_\ell^4)}{4(1+r_\ell^2)^3}$$

notice the $1/L^3$ terms generated by the QED$_L$ prescription

- the remaining contributions can be evaluated by starting from the following formulae

$$C^+ = \frac{4E_\ell m_P}{\sqrt{\pi}} \sum_{\boldsymbol{k}\in\Omega'} \int_{u_\star}^{\infty} du\, \sqrt{u}\, e^{-u\boldsymbol{k}^2} \int_0^\infty dx\, x\, e^{-x^2} \int_0^1 \frac{dy}{e_y^2}\, e^{-2\frac{xy\sqrt{u}(\boldsymbol{k}\cdot\boldsymbol{p_\ell})}{e y}}$$

$$C^- = \frac{E_\ell m_P}{2\pi^2} \sum_{\boldsymbol{n}\neq 0} \int_0^{u_\star} du \int_0^1 \frac{dy}{m_y^2} \int_0^\infty dx\, x\, e^{-u\left\{x^2 + \frac{x i \boldsymbol{n}\cdot\boldsymbol{q}_y}{u}\right\} - \frac{\boldsymbol{n}^2}{4u}}$$

that can be eventually be reexpressed in terms of Jacobi's θ-functions

$$\theta_3\,(a,b) = 1 + 2\sum_{n=1}^{\infty} \cos(2na)\, b^{n^2}$$

- or, after some algebra, in terms of incomplete Γ-functions and the Dawson-function

$$C^{+} = \frac{2}{\sqrt{\pi}} \sum_{\boldsymbol{k}\in\Omega'} \frac{\Gamma\left(\frac{3}{2}, u_\star \boldsymbol{k}^2\right)}{|\boldsymbol{k}|^3 \left[1 - (\hat{\boldsymbol{k}}\cdot\boldsymbol{\beta}_\ell)^2\right]} \left\{ 1 + \frac{e^{u_\star (\boldsymbol{k}\cdot\boldsymbol{\beta}_\ell)^2}\, \bar{\Gamma}\left[\frac{3}{2}, u_\star (\boldsymbol{k}\cdot\boldsymbol{\beta}_\ell)^2\right]}{|\hat{\boldsymbol{k}}\cdot\boldsymbol{\beta}_\ell|\, e^{u_\star \boldsymbol{k}^2}\, \Gamma\left(\frac{3}{2}, u_\star \boldsymbol{k}^2\right)} \right\} ,$$

$$C^{-} = \frac{1}{4\pi^2} \sum_{\boldsymbol{n}\neq\boldsymbol{0}} \int_0^{\frac{4u_\star}{\boldsymbol{n}^2}} \frac{du}{u} e^{-\frac{1}{u}} \int_0^{\frac{1}{1+\beta_\ell}} dy\, \frac{1 - \frac{2y(\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}_{\boldsymbol{\ell}})}{\sqrt{u(1-2\beta_\ell y)}}\, \mathrm{D}\left(\frac{y(\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}_{\boldsymbol{\ell}})}{\sqrt{u(1-2\beta_\ell y)}}\right)}{(1 - 2\beta_\ell y)}$$

- by putting all the contributions together one gets the expression of $C_{P\ell}(L)$ given in the main part of the talk