The Nuclear and Chiral Transition in the Strong Coupling Regime of Lattice QCD

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Overview

1. Lattice QCD at strong coupling: dual approach to QCD
2. The phase diagram at leading order in $\beta$
3. Towards higher order gauge corrections
4. Preliminary results on the nuclear/chiral transition
QCD Phase Diagram and Sign Problem

- **Sign problem**: no direct RHMC simulations at finite $\mu$
- **Complex Langevin**: not (yet) ready to address confined phase, also needs to be crosschecked (convergence issue)
- **Lefshetz Thimbles**: challenging to adapt to SU(3) and 4 dimensions

Sign problem is representation dependent: Dual Representation of QCD
Lattice QCD at strong coupling

Alternative approach:

Limit of strong coupling: \( \beta = \frac{6}{g^2} \rightarrow 0 \)

- gauge fields \( U_\mu(x) \) can be integrated out
- “dual” representation: via color singlets on the links!
- at strong coupling: mesons and baryons

Advantage:

- very mild sign problem
- fast simulations (no supercomputers necessary)
  \( \Rightarrow \) complete phase diagram can be calculated

Caveat:

- was limited to infinitely strong coupling \( \rightarrow \) coarse lattices
- continuum limit?!

\( \beta = 0 \rightarrow \beta \rightarrow \infty \)

Necessary to extend to \( \beta > 0 \)
**Strong Coupling Partition Function**

**Dualization** for staggered fermions: Mapping onto discrete system:

\[ Z_F(m_q, \mu) = \sum_{\{k,n,\ell\}} \prod_{b=(x,\mu)} \frac{(N_c - k_b)!}{N_c! k_b!} \prod_{x} \frac{N_c!}{n_x!} (2am_q)^{n_x} \prod_{\ell} w(\ell, \mu) \]

- meson hoppings \( M_x M_y \)
- chiral condensate \( \bar{\psi} \psi \)
- baryon hoppings \( \bar{B}_x B_y \)

\( k_b \in \{0, \ldots N_c\} \), \( n_x \in \{0, \ldots N_c\} \), \( \ell_b \in \{0, \pm 1\} \), \( QCD: N_c = 3 \)

- Grassmann constraint:

\[ n_x + \sum_{\hat{\mu}=\pm \delta, \ldots \pm \bar{d}} \left( k_{\hat{\mu}}(x) + \frac{N_c}{2} |\ell_{\hat{\mu}}(x)| \right) = N_c \]

- weight \( w(\ell, \mu) \) and sign \( \sigma(\ell) \in \{-1, +1\} \)
  for oriented baryonic loop \( \ell \) depends on loop geometry

- next talk by **Jangho Kim**: finite \( m_q \)

**finite quark mass**
Strong Coupling Partition Function

Dualization for staggered fermions: Mapping onto discrete system:

\[ Z_F(m_q, \mu) = \sum_{\{k,n,\ell\}} \prod_{b=(x,\mu)} \frac{(N_c - k_b)!}{N_c! k_b!} \prod_{x} \frac{N_c!}{n_x!} (2am_q)^{n_x} \prod_{\ell} w(\ell, \mu) \]

\( k_b \in \{0, \ldots N_c\}, \ n_x \in \{0, \ldots N_c\}, \ \ell_b \in \{0, \pm 1\}, \ \text{QCD: } N_c = 3 \)

- Grassmann constraint:
  \[ \sum_{\hat{\mu}=\pm 0, \ldots \pm d} \left( k_{\hat{\mu}}(x) + \frac{N_c}{2} |\ell_{\hat{\mu}}(x)| \right) = N_c \]

- weight \( w(\ell, \mu) \) and sign \( \sigma(\ell) \in \{-1, +1\} \)
  for oriented baryonic loop \( \ell \) depends on loop geometry

- this talk: consider chiral limit

chiral limit: monomers absent
Strong Coupling Partition Function

Dualization for staggered fermions: Mapping onto discrete system:

\[ Z_F(m_q, \mu) = \sum_{\{k,n,\ell\}, b=(x,\mu)} \prod_{b=(x,\mu)} \frac{(N_c - k_b)!}{N_c! k_b!} \prod_x \frac{N_c!}{n_x!} (2am_q)^{n_x} \prod_\ell w(\ell, \mu) \]

- Worm algorithm [Prokof’ev & Svistunov 2001]:
  sampling 2-monomer sector
  (for U(3): [Adams & Chandrasekharan, 2003])
- SU(3): Worm both in mesonic and baryonic sector

Worm algorithm [Prokof’ev & Svistunov 2001]:
sampling 2-monomer sector
(for U(3): [Adams & Chandrasekharan, 2003])

SU(3): Worm both in mesonic and baryonic sector
The Phase Diagram in the Strong Coupling Limit

Comparison of phase boundaries \((T_c, \mu_c)\) for \textbf{massless quarks} \cite{de Forcrand & U. (2011)}:

- \(\langle \bar{\psi} \psi \rangle = 0\)
- \(\langle \bar{\psi} \psi \rangle \neq 0\)

\(1^\text{st}\) order
\(2^\text{nd}\) order

\(<\bar{\psi} \psi> = 0\)
\(<\bar{\psi} \psi> \neq 0\)

\(\mu_B\) [lat. units]

\(T\) [MeV]

\(T\) [lat. units]

\(\mu_B\) [GeV]

- measured at strong coupling
- speculated in continuum

- Similar to \textbf{standard scenario of continuum QCD} \cite{Stephanov et al. PRL 81 (1998)}
- However, nuclear and chiral transition coincide at \(\beta = 0\)
Relation Between Chiral and Nuclear Transition at $\beta = 0$

Chiral Transition at Strong Coupling:
- **chiral symmetry**: $U(1)_{55} : \psi(x) \mapsto e^{i\epsilon(x)\theta_{55}} \psi(x)$, $\epsilon(x) = (-1)^{x_1+x_2+x_3+x_4}$ is spontaneously broken at low temperatures/densities
- chiral transition: **spatial dimers vanish**, at small $\mu$: 2nd order with $O(2)$ exponents

Nuclear Transition (below TCP):
- **baryon crystal** forms (Pauli saturation)
- coincides with chiral transition: $\langle \bar{\psi}\psi \rangle$ vanishes as baryonic crystal forms

SC Phase Diagram for massless quarks

1st order Chiral & Nuclear Trans.

2nd order

tricritical

$\mu = 0, T \gg T_c$

$\mu = 0, T < T_c$

$T \gg T_c, \mu > \mu_c$

$T = 0, \mu > \mu_c$
Polyakov Loop, Baryon Density and Plaquettes

- **Polyakov Loop**: Graph showing the Polyakov loop as a function of $aT$ and $a\mu$.
- **Baryon Density**: Graph showing the baryon density as a function of $aT$ and $a\mu$.
- **Temporal Plaquette**: Graph showing the temporal plaquette as a function of $aT$ and $a\mu$.
- **Spatial Plaquette**: Graph showing the spatial plaquette as a function of $aT$ and $a\mu$.

Parameters:
- 2\textsuperscript{nd} order
- 1\textsuperscript{st} order
- Tricritical
Sign Problem in Dual Representation (Strong Coupling)

- average sign: $\langle \text{sign} \rangle \sim e^{-\frac{\nu}{\tau} \Delta f}$
- volumes $32^3 \times N_{\tau}$ can be easily simulated at tricritical point
- sign problem more severe at low temperatures

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Including the gauge corrections

- QCD Partition function via strong coupling expansion in $\beta$:

$$Z_{QCD} = \int d\psi d\bar{\psi} dU e^{S_G + S_F} = \int d\psi d\bar{\psi} Z_F \langle e^{S_G} \rangle_{Z_F}$$

$$\langle O \rangle_{Z_F} = \frac{1}{Z_F} \int dU O e^{-S_F}, \quad Z_F = \int dU e^{-S_F} = \prod_{l=(x,\mu)} z(x,\mu)$$

- expand gauge action to some order in $\beta$:

$$\langle e^{S_G} \rangle_{Z_F} \simeq 1 + \langle S_G \rangle_{Z_F} + O(\beta^2) = 1 + \frac{\beta}{2N_c} \sum_P \langle \text{tr}[U_P + U_P^\dagger] \rangle_{Z_F} + O(\beta^2)$$

→ additional color singlet link states

Baryonic Quark Flux
Mesonic Quark Flux
Gauge Flux
Link Integrations for $\mathcal{O}(\beta)$ diagrams

One-Link integrals for links on the edge of an excited plaquette:

$J_{ij} = \sum_{k=1}^{N_c} \frac{(N_c - k)!}{N_c!(k - 1)!} \left( M_\chi M_\varphi \right)^{k-1} \bar{\chi}_j \varphi_i - \frac{1}{N_c!(N_c - 1)!} \epsilon_{i_1 i_2 j_1 j_2} \bar{\varphi}_{i_1} \bar{\varphi}_{i_2} \chi_{j_1} \chi_{j_2} - \frac{1}{N_c} \bar{B}_\varphi B_\chi \bar{\chi}_j \varphi_i$

- determine plaquette link product $P = \text{tr} [ J_{ik} J_{kl} J_{lm} J_{mi} ]$
- result can be consistently re-expressed via

**link weights:** $w(D_k) = \frac{(N_c - k)!}{N_c!(k - 1)!}$,  $w(B_1) = \frac{1}{N_c!(N_c - 1)!}$,  $w(B_2) = \frac{(N_c - 1)!}{N_c!}$

and **site weights:** $v_1 = N_c!$,  $v_2 = (N_c - 1)!$,  $v_3 = 1$

- Grassman constraint on sites touching a plaquette altered $N_c \rightarrow N_c + 1$
Gauge corrections to the phase diagram at strong coupling

State of the art: $O(\beta)$ corrections for SU(3)

[State of the art: $O(\beta)$ corrections for SU(3)]

Questions we want to address:

- Do the nuclear and chiral transition split?
- Does the tricritical point move to smaller or larger $\mu$ as $\beta$ is increased?
First step: obtain higher order gauge integrals: \( O(\beta^{r+s}) \)

- need to determine one-link integrals, with quark matrices \( M_{ij} = \sum_{f=1}^{N_f} \psi_i^f(x) \bar{\psi}^f_j(y) \):

\[
\mathcal{J}_{(i,j)_{1:r}(k,l)_{1:s}}^{r,s} = \int_{SU(3)} dU \ e^{\text{tr}[U M^\dagger + M U^\dagger]} \ U_{i_1j_1} \cdots U_{i_rj_r} (U^\dagger)^{k_1l_1} \cdots (U^\dagger)^{k_sl_s}
\]

- in the strong coupling limit, link integration for \( J^{0,0} \) factorizes!

- \( \beta > 0 \): tensorial structure, but still true that gauge integrals can be decomposed into linear combinations of invariants

- strategy: expand exponential of fermion action \( J_{(i,j)_{1:r}(k,l)_{1:s}}^{r,s} = \sum_{\kappa_1,\kappa_2} \mathcal{K}_{(i,j)_{1:r}(k,l)_{1:s}}^{\kappa_1,r;\kappa_2,s} \mathcal{I}_{(i,j)_{1:r}(k,l)_{1:s}}^{\kappa_1+r;\kappa_2+s} \)

\[
\mathcal{K}_{(i,j)_{1:r}(k,l)_{1:s}}^{\kappa_1,r;\kappa_2,s} = \frac{1}{\kappa_1! \kappa_2!} \ \sum_{\{i_a,j_a,k_b,l_b\}} \left( \prod_{a=1}^{\kappa_1} (M^\dagger)^{i_a j_a} \right) \left( \prod_{b=1}^{\kappa_2} M^{k_b l_b} \right) \mathcal{I}_{(i,j)_{1:r}(k,l)_{1:s}}^{\kappa_1+r;\kappa_2+s}
\]

- \( \kappa_1 \) quark hoppings, \( \kappa_2 \) anti-quark hoppings: \( |\kappa_1 - \kappa_2 + r - s| \in \{0, N_c, 2N_c, \ldots\} \)

- color and flavor structure intimately linked!

- Integrals \( \mathcal{I}_{(i,j)_{1:a}(k,l)_{1:b}}^{a,b} = \int_{SU(3)} dU \ U_{i_1j_1} \cdots U_{i_a j_a} (U^\dagger)^{k_1l_1} \cdots (U^\dagger)^{k_bl_b} \) are known, recursive [M. Creutz, 1980], or expressed by Young projectors [J. Myers, 2014]:

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At strong coupling, general result for any number of flavors $N_f$:

$$K_0^{\kappa;\kappa} = \sum_{\lambda, \tau} C^\tau_\lambda \prod_{i=1}^\kappa \text{Tr}[(M_{xy})^i]^{t_i}$$

- $\chi^\lambda_\tau$ is character of symmetric group $S_n$, $D_\lambda$ is dimension of SU(N) representation $\lambda$, $c_\tau$ is degeneracy of cycle structure
- $\lambda$ encodes $N_c$-dependence, $\tau$ for $N_f$-dependence, e.g.

$$K_0^{3;3} = \frac{1}{6} \frac{1}{N_c(N_c^2 - 1)(N_c^2 - 4)} \left[ (N_c^2 - 2) \text{Tr}[M_{xy}]^3 + 3N_c \text{Tr}[M_{xy}] \text{Tr}[(M_{xy})^2] + 4 \text{Tr}[(M_{xy})^3] \right]$$

Generalization for higher order gauge corrections possible, e.g.

$$K_{i_1j_1,i_2j_2}^{1,2,3,0} = \frac{1}{N_c(N_c^2 - 1)(N_c^2 - 4)} \left[ (N_c^2 + N_c - 2) \text{Tr}[Q^{i_1j_1}] \text{Tr}[Q^{i_2j_2}] \text{Tr}[M_{xy}] \right.
+ N_c \text{Tr}[Q^{i_1j_1} Q^{i_2j_2}] \text{Tr}[M_{xy}] + N_c \text{Tr}[M_{xy} Q^{i_1j_1}] \text{Tr}[Q^{i_2j_2}] + 4 \text{Tr}[M_{xy} Q^{i_1j_1} Q^{i_2j_2}] \right]$$
Plaquette and Flux Variables

New interpretation of dual representation:

- at strong coupling limit: dimers = meson hoppings, 3-fluxes = baryons
- away from strong coupling limit: dimers = color singlets (=U(3) sector), 3-fluxes = color triplets
- in principle: also 6-flux, 9-flux, ... sectors, but neglected here

Plaquette occupation numbers at plaquette coordinate $P$:

- equivalence classes of difference of fundamental plaquettes $\text{Tr}[U_P]$ and anti-fundamental plaquettes $\text{Tr}[U_P^\dagger]$ from gauge action:

$$n_P = n_f(P) - n_a(P) \implies \beta \mapsto \text{U(3) sectors within } u(\beta)$$

- plaquette fluxes induce links fluxes $f_b$ and defines flux sites $f_x$:
Second step: Grassmann integration

Recall: gauge integration before Grassmann integration (no fermion determinant)
- free color indices need to be \textit{contracted} at each site (for given ensemble of plaquettes)
- in general, gives rise to tensor networks/vertex model → (too) difficult!

**Simplification** in the U(3) sector:
- plaquette occupations $n_p, n_{p'} \in \mathbb{Z}$ can **only differ by $\pm 1$** if $p, p'$ adjacent
- plaquette occupations $n_p, n_{p'} \in \mathbb{Z}$ cannot share a site if they don’t share a link

\[
T_{i_1 j_1 j_2} = \sum_{k,l} [J_{i_1 k} J_{k j_1} J_{i_2 l} J_{l j_2} + J_{i_1 k} J_{k j_2} J_{i_2 l} J_{l j_1}]
\]

Different Contractions have opposite sign \(\Rightarrow\) Cancellation!

⇒ Gauge fluxes $f_b$ form self-avoiding loops!

- Does not apply to the additional SU(3) contributions: so far restricted to first non-trivial contribution (3-flux sector)
MDP+P partition function:

$$Z_F(m_q, \mu) = \sum_{\{k, n, \ell, n_p\}} \prod_{b=(x, \mu)} \frac{(N_c - k_b)!}{N_c!(k_b - |f_b|)!} \prod_x \frac{N_c!}{n_x!} (2am_q)^{n_x} \prod_\ell w(\ell, f_b, \mu) \prod_P \frac{1}{|n_p|!} \left(\frac{\beta}{2N_c}\right)^{|n_p|}$$

- singlet hoppings $M_x M_y$
- chiral condensate $\bar{\psi}\psi$
- triplet hoppings $\bar{B}_x B_y$
- gluon propagation

$k_b \in \{0, \ldots N_c\}, \ n_x \in \{0, \ldots N_c\}, \ \ell_b \in \{0, \pm 1\}, \ f_b = \partial n_p, \ f_x = \frac{1}{2} \sum_b f_b$

- color constraint:

$$n_x + \sum_{\hat{\mu}=\pm \hat{\delta}, \ldots \pm \hat{d}} \left( k_{\hat{\mu}}(x) + \frac{N_c}{2} |\ell_{\hat{\mu}}(x)| \right) = N_c + f_x$$

- 3-flux weight involves additional site weights $v_i$ and link weights:

$$w(B_3) = \frac{1}{N_c!(N_c-1)!(N_c-2)!}, \ w(B_4) = \frac{(N_c-1)!(N_c-2)!}{N_c!}$$

- sign: combine gauge flux $f_b$ with triplet flux $\ell_b$ to identify fermionic loops $\tilde{\ell}$:

$$\sigma(C) = (-1)^{L(C)+W(C)+N_-(C)} \prod_{\tilde{\ell}} \eta_{\ell}(x)$$

QCD lattice partition function correct up to $O(\beta^3)$
MDP+P Ensembles (2-dim for visualization)

\[ \beta = 1.0, \mu = 0.5 \text{ (liquid phase)} \]

MDP ensemble:

Flux ensemble:
Nuclear and Chiral Transition at $T = 0$ - ongoing analysis

![Graphs showing the average sign on 4$^4$×4 and $a^4\Delta f$ as functions of $\beta$.]
Nuclear and Chiral Transition at $T = 0$ - ongoing analysis
Nuclear and Chiral Transition at $T = 0$ - ongoing analysis

chiral susceptibility

chiral susceptibility (zoom)
Conclusions

Results:
- all gauge integrals needed for $\mathcal{O}(\beta^3)$ and related to group characters
- Grassmann integration simplifies in U(3) sector
- sign problem mild enough to go beyond $\beta > 1$, but hard at $T = 0$ (even at $\beta = 0$)
- simulations with $\mathcal{O}(\beta)^3$ corrections included but not conclusive yet concerning nuclear vs. chiral transition (weak dependence on $\beta$)
- split between chiral and nuclear transition might be very small in nature

Goals:
- improve plaquette algorithm $\rightarrow$ incorporate character expansion
- generalize plaquette algorithm to non-trivial anisotropy $a/a_t = f(\gamma, \beta_s/\beta_t)$ to study various $T$ (→ talk by Hélvio Vairinhos: $a/a_t = f(\gamma)$ at $\beta = 0$)
- surpass the roughening transition at $\beta \approx 5.9$: sampling of all orders needed
- other dualizations? (→ talk by Carla Marchis)
Backup: Connection Between Strong Coupling and Continuum Limit?

One of several possible scenarios for the extension to the continuum:

- back plane: strong coupling phase diagram ($\beta = 0$), $N_f = 1$
- front plane: continuum phase diagram ($\beta = \infty$, $a = 0$)
- due to fermion doubling, corresponds to $N_f = 4$ in continuum (no rooting)
So, what are the $C^\tau_\lambda$ (and for $O(\beta), O(\beta^2)$: $C^\tau_{\lambda^1,\rho_1}, C^\tau_{\lambda^1,\rho_1,\rho_2}$, etc) ?

- answer: related to irreducible representations of the symmetric group $S_n$ with $n = \kappa_1 + r = \kappa_2 + s$

Young Tableaux $\lambda = (\lambda_1, \ldots \lambda_k)$:

- Standard Young tableaux correspond to irreps of $S_n$ with dimension $d_\lambda = \frac{n!}{H_\lambda}$ with Hook lengths $H_\lambda$

- used to determine dimension $D_\lambda = \frac{F_\lambda}{H_\lambda}$ of irreps of SU($N_c$)
Backup: Young Tableaux - Flavor Structure

So, what are the $C^\tau_{\lambda}$ (and for $O(\beta)$, $O(\beta^2)$: $C^\tau_{\lambda, \rho_1}$, $C^\tau_{\lambda, \rho_1, \rho_2}$, etc) ?

- answer: related to irreducible representations of the symmetric group $S_n$ with $n = \kappa_1 + r = \kappa_2 + s$

Young Tableaux $\tau = (t_1, \ldots t_k)$:

- conjugacy classes
  $\leftrightarrow$ cycle structure of $S_n$
  $\leftrightarrow$ flavor permutations

- at a given order $n$, the trace structure $\tau = (t_1, \ldots t_k)$ with $T = \prod Tr[M^i_{xy}]^{t_i}$ is equivalent to a partition of $n$: $\sum_i it_i = n$

- example: $\pi = (136)(24)(58)(7)$
  $\rightarrow$ $t_1 = 1$, $t_2 = 2$, $t_3 = 1$
## Backup: Table of Characters and Invariants

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\lambda$:</th>
<th>(1, 1, 1, 1)</th>
<th>(2, 1, 1)</th>
<th>(2, 2)</th>
<th>(3, 1)</th>
<th>(4)</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4, 0, 0, 0)</td>
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<td>2</td>
<td>3</td>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>(2, 1, 0, 0)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(0, 2, 0, 0)</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>(1, 0, 1, 0)</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(0, 0, 0, 1)</td>
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<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
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<td>3</td>
<td>2</td>
<td>1</td>
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</table>

**Table: Characters $\chi_{\lambda}^{\tau}$ for $n = 4$**

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\lambda$:</th>
<th>(1, 1, 1, 1)</th>
<th>(2, 1, 1)</th>
<th>(2, 2)</th>
<th>(3, 1)</th>
<th>(4)</th>
<th>Sum</th>
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<td>4</td>
<td>9</td>
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<td>(2, 1, 0, 0)</td>
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<td>18</td>
<td>0</td>
<td>-18</td>
<td>-6</td>
<td>0</td>
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<td>18</td>
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</table>

**Table: Invariants $C_{\lambda}^{\tau}$ for $n = 4$**
Define Bessel Determinants:

\[ D_n^{(3,e)} = \begin{vmatrix} I_n(x) & I_{n+1}(x) & I_{n+2}(x) \\ I_{n-1}(x) & I_n(x) & I_{n+1}(x) \\ I_{n-2}(x) & I_{n-1}(x) & I_n(x) \end{vmatrix}, \quad D_n^{(3,f)} = \begin{vmatrix} I_{n+1}(x) & I_{n+2}(x) & I_{n+3}(x) \\ I_{n+1}(x) & I_{n+1}(x) & I_{n+1}(x) \\ I_{n-2}(x) & I_{n-1}(x) & I_n(x) \end{vmatrix} \]

fundamental character in \( x = \frac{1}{g^2} = \frac{\beta}{2N_c} \):

for U(3):

\[
u(\beta) = \frac{D_0^{(3,f)}(2x)}{D_0^{(3,e)}(2x)} = \frac{1}{0!} x + \frac{1}{1!} x^2 + \frac{4}{2!} x^3 + \frac{6}{3!} x^4 + \frac{23}{4!} x^5 + \frac{103}{5!} x^6 + \cdots
\]

for SU(3):

\[
u(\beta) = \sum_{n=-\infty}^{\infty} D_n^{(3,f)}(2x) = \frac{1}{0!} x + \frac{1}{1!} x^2 + \frac{2}{1!} x^3 + \frac{5}{2!} x^4 + \frac{2\times 6^1}{3!} x^5 + \frac{77}{4!} x^6 + \frac{5\times 23+24}{5!} x^7 + \cdots
\]

character expansion recovered, but not limited to Wilson loops!

<table>
<thead>
<tr>
<th>1 (mesonic)</th>
<th>(L^3)</th>
<th>(L^6)</th>
<th>(L^9)</th>
<th>(L^{12})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (baryonic)</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>42</td>
</tr>
<tr>
<td>((LL^*))</td>
<td>1</td>
<td>3</td>
<td>21</td>
<td>210</td>
</tr>
<tr>
<td>((LL^*)^2)</td>
<td>2</td>
<td>11</td>
<td>98</td>
<td>1122</td>
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<tr>
<td>((LL^*)^3)</td>
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<td>74</td>
<td>498</td>
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<td>((LL^*)^4)</td>
<td>23</td>
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