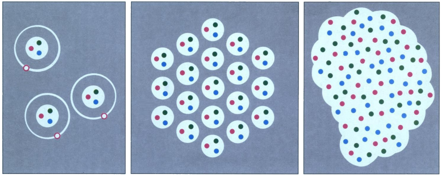
The Nuclear and Chiral Transition in the Strong Coupling Regime of Lattice QCD

Wolfgang Unger, Bielefeld University

Lattice 2016

University of Southampthon, 29.07.2016



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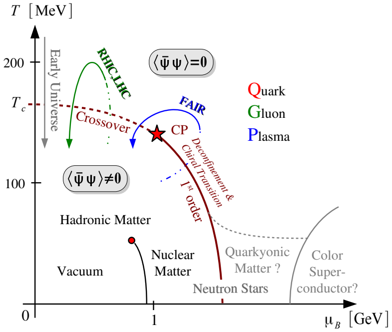
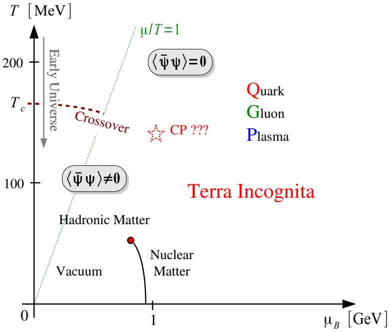
Overview

1. Lattice QCD at strong coupling: **dual approach** to QCD
2. The phase diagram at leading order in β
3. Towards higher order gauge corrections
4. Preliminary results on the nuclear/chiral transition

QCD Phase Diagram and Sign Problem



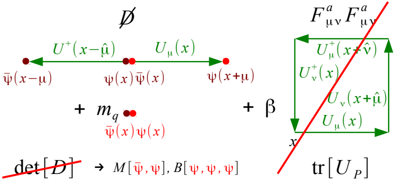


- **Sign problem:** no direct RHMC simulations at finite μ
- **Complex Langevin**: not (yet) ready to address confined phase,
  also needs to be crosschecked (convergence issue)
- **Lefshetz Thimbles**: challenging to adapt to SU(3) and 4 dimensions

Sign problem is **representation dependent:** Dual Representation of QCD

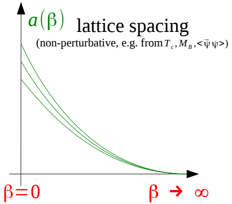
Lattice QCD at strong coupling

Alternative approach:



$\boxed{\text{Limit of strong coupling: } \beta = \frac{6}{g^2} \rightarrow 0}$

- gauge fields $U_\mu(x)$ can be integrated out
- **"dual" representation:** via color singlets on the links!
- at strong coupling: **mesons** and **baryons**



Advantage:

- very mild sign problem
- fast simulations (no supercomputers necessary)

  $\Rightarrow$ **complete phase diagram** can be calculated

Caveat:

- was limited to **infinitely strong coupling** $\rightarrow$ **coarse lattices**
- continuum limit?!?

$\boxed{\text{Necessary to extend to } \beta > 0}$

Strong Coupling Partition Function

Dualization for staggered fermions: Mapping onto **discrete system:**

[Rossi & Wolff, 1984], [Karsch & Mütter, 1989]

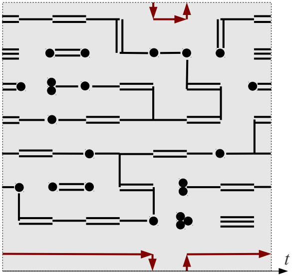
$$Z_F(m_q,\mu) = \sum_{\{k,n,\ell\}} \underbrace{\prod_{b=(x,\mu)} \frac{(N_c-k_b)!}{N_c!k_b!}}_{\text{meson hoppings } M_x M_y} \underbrace{\prod_x \frac{N_c!}{n_x!}(2am_q)^{n_x}}_{\text{chiral condensate } \bar\psi\psi} \underbrace{\prod_\ell w(\ell,\mu)}_{\text{baryon hoppings } \bar B_x B_y}$$

$k_b \in \{0,\dots N_c\}$, $n_x \in \{0,\dots N_c\}$, $\ell_b \in \{0,\pm 1\}$, QCD: $N_c = 3$

- Grassmann constraint:

$$n_x + \sum_{\hat\mu=\pm\hat 0,\dots\pm\hat d} \left(k_{\hat\mu}(x) + \frac{N_c}{2}|\ell_{\hat\mu}(x)| \right) = N_c$$

- weight $w(\ell,\mu)$ and sign $\sigma(\ell)\in\{-1,+1\}$ for oriented baryonic loop ℓ depends on loop geometry
- next talk by **Jangho Kim**: finite m_q



finite quark mass

Strong Coupling Partition Function

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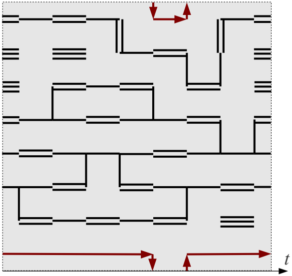
- this talk: consider chiral limit



chiral limit: monomers absent

Strong Coupling Partition Function

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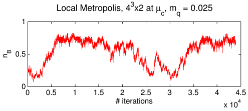
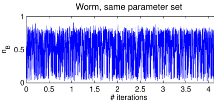
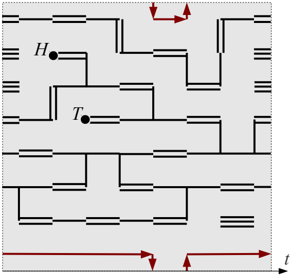
$k_b \in \{0,\dots N_c\}$, $n_x \in \{0,\dots N_c\}$, $\ell_b \in \{0,\pm 1\}$, QCD: $N_c = 3$

- Worm algorithm [Prokof'ev & Svistunov 2001]: sampling 2-monomer sector (for U(3): [Adams & Chandrasekharan, 2003])
- SU(3): Worm both in mesonic and baryonic sector



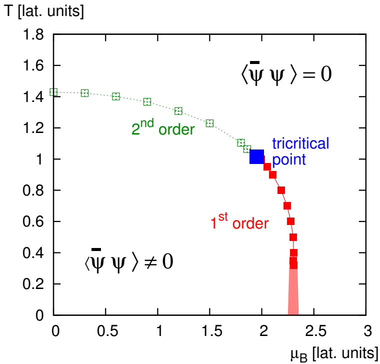




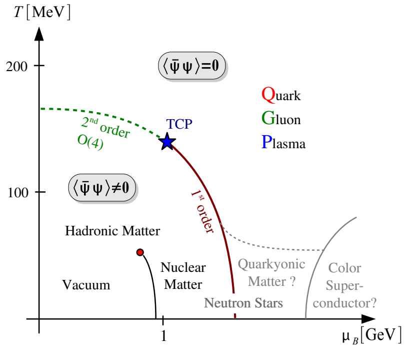
during Worm evolution

The Phase Diagram in the Strong Coupling Limit

Comparison of phase boundaries (T_c, μ_c) for **massless quarks** [de Forcrand & U. (2011)]:



measured at strong coupling



speculated in continuum

- Similar to **standard scenario of continuum QCD** [Stephanov *et al.* PRL 81 (1998)]
- However, nuclear and chiral transition coincide at $\beta = 0$

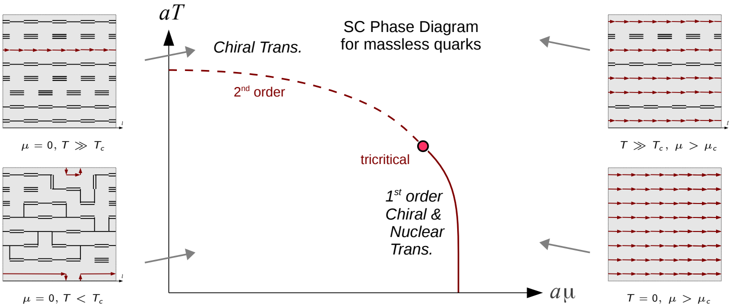
Relation Between Chiral and Nuclear Transition at $\beta = 0$

Chiral Transition at Strong Coupling:

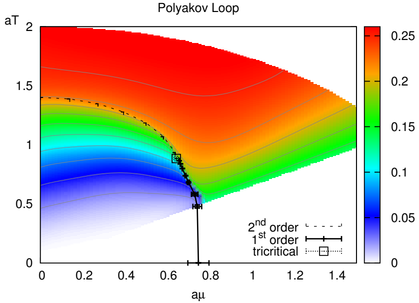
- **chiral symmetry**: $U(1)_{55}$: $\quad \psi(x) \mapsto e^{i\epsilon(x)\theta_{55}}\psi(x), \quad \epsilon(x) = (-1)^{x_1+x_2+x_3+x_4}$
  is spontaneously broken at low temperatures/densities
- chiral transition: **spatial dimers vanish**, at small μ: 2nd order with O(2) exponents

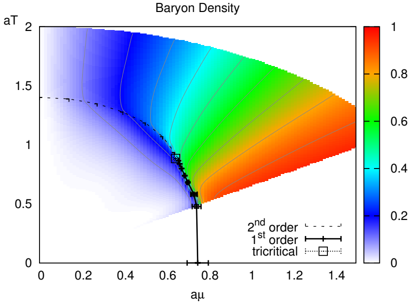
Nuclear Transition (below TCP):

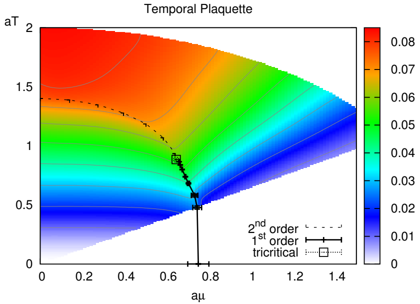
- **baryon crystal** forms (Pauli saturation)
- coincides with chiral transition: $\langle \bar{\psi}\psi \rangle$ vanishes as baryonic crystal forms

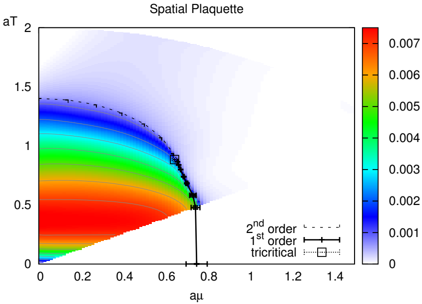


Polyakov Loop, Baryon Density and Plaquettes

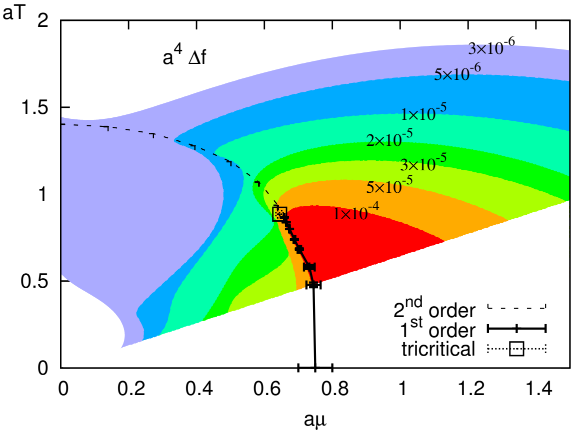








Sign Problem in Dual Representation (Strong Coupling)



- average sign: $\langle \mathrm{sign} \rangle \simeq e^{-\frac{V}{T}\Delta_f}$
- volumes $32^3 \times N_\tau$ can be easily simulated at tricritical point
- sign problem more **severe at low temperatures**

Including the gauge corrections

- QCD Partition function via **strong coupling expansion in β:**

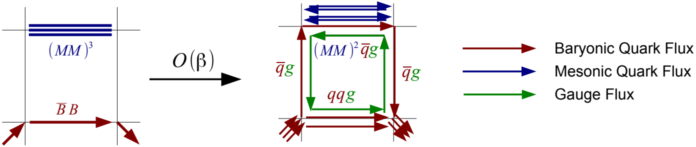
$$Z_{QCD} = \int d\psi d\bar{\psi} dU e^{S_G + S_F} = \int d\psi d\bar{\psi} Z_F \left\langle e^{S_G} \right\rangle_{Z_F}$$

$$\langle O \rangle_{Z_F} = \frac{1}{Z_F} \int dU O e^{-S_F}, \qquad Z_F = \int dU e^{-S_F} = \prod_{l=(x,\mu)} z(x,\mu)$$

- expand gauge action to some order in β:

$$\left\langle e^{S_G} \right\rangle_{Z_F} \simeq 1 + \langle S_G \rangle_{Z_F} + \mathcal{O}(\beta^2) = 1 + \frac{\beta}{2N_c} \sum_P \left\langle \mathrm{tr}[U_P + U_P^\dagger] \right\rangle_{Z_F} + \mathcal{O}(\beta^2)$$

$\rightarrow$ **additional color singlet link states**



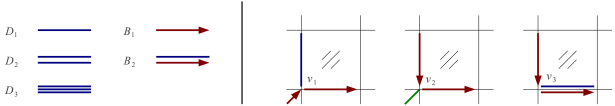
Link Integrations for $\mathcal{O}(\beta)$ diagrams

One-Link integrals for links on the edge of an excited plaquette:

[Azakov & Aliev, Physica Scripta 38 (1988)]

$$J_{ij} = \sum_{k=1}^{N_c} \underbrace{\frac{(N_c - k)!}{N_c!(k-1)!}(M_\chi M_\varphi)^{k-1}\bar{\chi}_j\varphi_i}_{D_k = mesons + \bar{q}g} - \underbrace{\frac{1}{N_c!(N_c-1)!}\epsilon_{ii_1i_2}\epsilon_{jj_1j_2}\bar{\varphi}_{i_1}\bar{\varphi}_{i_2}\chi_{j_1}\chi_{j_2}}_{B_1 = qqg} - \underbrace{\frac{1}{N_c}\bar{B}_\varphi B_\chi \bar{\chi}_j \varphi_i}_{\substack{B_2 = meson + qqg \\ baryon + \bar{q}g}}$$

- determine plaquette link product $P = \mathrm{tr}[J_{ik}J_{kl}J_{lm}J_{mi}]$
- result can be consistently re-expressed via
  link weights: $w(D_k) = \frac{(N_c - k)!}{N_c!(k-1)!}$, $w(B_1) = \frac{1}{N_c!(N_c-1)!}$, $w(B_2) = \frac{(N_c-1)!}{N_c!}$
  and **site weights:** $v_1 = N_c!$, $v_2 = (N_c - 1)!$, $v_3 = 1$

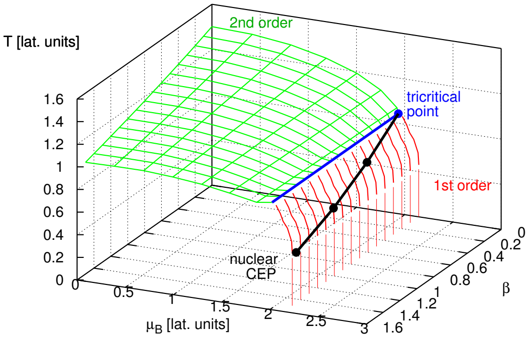


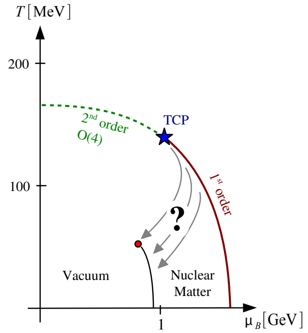
- Grassman constraint on sites touching a plaquette altered $N_c \to N_c + 1$

Gauge corrections to the phase diagram at strong coupling

State of the art: $\mathcal{O}(\beta)$ **corrections for SU(3)**

[Langelage, de Forcrand, Philipsen & U., *PRL* **113** (2014)]





Questions we want to address:

- Do the **nuclear and chiral transition split?**
- Does the **tricritical point** move to smaller or larger μ as β is increased?

First step: obtain higher order gauge integrals: $\mathcal{O}(\beta^{r+s})$

- need to determine one-link integrals, with quark matrices $\mathcal{M}_{ij} = \sum_{f=1}^{N_\mathrm{f}} \psi_i^f(x)\bar{\psi}_j^f(y)$:

$$\mathcal{J}^{r;s}_{(i,j)_{1:r}(k,l)_{1:s}} = \int_{SU(3)} dU \underbrace{e^{\mathrm{tr}[U\mathcal{M}^\dagger + \mathcal{M}U^\dagger]}}_{\text{from quark action}} \underbrace{U_{i_1j_1}\ldots U_{i_rj_r}(U^\dagger)_{k_1l_1}\ldots(U^\dagger)_{k_sl_s}}_{\text{from gluon action}}$$

- in the strong coupling limit, **link integration for $\mathcal{J}^{0,0}$ factorizes!**
- $\beta > 0$: **tensorial structure**, but still true that gauge integrals can be decomposed into linear combinations of **invariants**
- strategy: expand exponential of fermion action $\mathcal{J}^{r;s}_{(i,j)_{1:r}(k,l)_{1:s}} = \sum_{\kappa_1,\kappa_2} \mathcal{K}^{\kappa_1,r;\kappa_2,s}_{(i,j)_{1:r}(k,l)_{1:s}}$

$$\mathcal{K}^{\kappa_1,r;\kappa_2,s}_{(i,j)_{1:r}(k,l)_{1:s}} = \frac{1}{\kappa_1!\kappa_2!} \sum_{\{i_a,j_a,k_b,l_b\}} \left(\prod_{a=1}^{\kappa_1}(\mathcal{M}^\dagger)_{i_aj_a}\right)\left(\prod_{b=1}^{\kappa_2}\mathcal{M}_{k_bl_b}\right) \mathcal{I}^{\kappa_1+r;\kappa_2+s}_{(i,j)_{1:\kappa_1+r}(k,l)_{1:\kappa_2+s}}$$

- κ_1 quark hoppings, κ_2 anti-quark hoppings: $|\kappa_1 - \kappa_2 + r - s| \in \{0, N_\mathrm{c}, 2N_\mathrm{c}, \ldots\}$
- color and flavor structure intimately linked!
- Integrals $\mathcal{I}^{a;b}_{(i,j)_{1:a}(k,l)_{1:b}} = \int_{SU(3)} dU\, U_{i_1j_1}\ldots U_{i_aj_a}(U^\dagger)_{k_1l_1}\ldots(U^\dagger)_{k_bl_b}$ are known, recursive [M. Creutz, 1980], or expressed by **Young projectors** [J. Myers, 2014]:

Gauge Integrals and Characters of S_n

At strong coupling, general result for **any number of flavors N_f**:

$$\mathcal{K}_0^{\kappa;\kappa} = \sum_{\lambda,\tau} C_\lambda^\tau \prod_{i=1}^{\kappa} \mathrm{Tr}[(M_{xy})^i]^{t_i}$$

$$C_\lambda^\tau = \frac{c_\tau \chi_\tau^\lambda}{D_\lambda}$$

- χ_τ^λ is character of symmetric group S_n, D_λ is dimension of SU(N) representation λ, c_τ is degeneracy of cycle structure
- λ encodes N_c-dependence, τ for N_f-dependence, e.g.

$$\mathcal{K}_0^{3;3} = \frac{1}{6}\frac{1}{N_\mathrm{c}({N_\mathrm{c}}^2-1)({N_\mathrm{c}}^2-4)}\left[({N_\mathrm{c}}^2-2)\mathrm{Tr}[M_{xy}]^3 + 3N_\mathrm{c}\mathrm{Tr}[M_{xy}]\mathrm{Tr}[(M_{xy})^2] + 4\mathrm{Tr}[(M_{xy})^3]\right]$$

- Generalization for higher order gauge corrections possible, e.g.

$$\mathcal{K}_{i_1 j_1, i_2 j_2}^{1,2,3,0} = \frac{1}{N_\mathrm{c}({N_\mathrm{c}}^2-1)({N_\mathrm{c}}^2-4)}\Big[({N_\mathrm{c}}^2+N_\mathrm{c}-2)\mathrm{Tr}[Q^{i_1 j_1}]\mathrm{Tr}[Q^{i_2 j_2}]\mathrm{Tr}[M_{xy}]$$
$$+ N_\mathrm{c}\mathrm{Tr}[Q^{i_1 j_1}Q^{i_2 j_2}]\mathrm{Tr}[M_{xy}] + N_\mathrm{c}\mathrm{Tr}[M_{xy}Q^{i_1 j_1}]\mathrm{Tr}[Q_{i_2 j_2}] + 4\mathrm{Tr}[M_{xy}Q^{i_1 j_1}Q^{i_2 j_2}]\Big]$$

Plaquette and Flux Variables

New interpretation of **dual representation:**

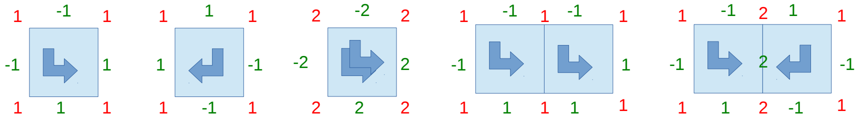
- at strong coupling limit: dimers=meson hoppings, 3-fluxes=baryons
- away from strong coupling limit: **dimers = color singlets** (=U(3) sector),
  3-fluxes = color triplets
- in principe: also 6-flux, 9-flux,. . . sectors, but neglected here

Plaquette occupation numbers at plaquette coorinate P:

- **equivalence classes** of difference of fundamental plaquettes $\mathrm{Tr}[U_P]$ and anti-fundamental plaquettes $\mathrm{Tr}[U_P^\dagger]$ from gauge action:

$$n_P = n_f(P) - n_a(P) \qquad \Rightarrow \qquad \beta \mapsto \text{U(3) sectors within } u(\beta)$$

- plaquette fluxes induce **links fluxes** f_b and defines **flux sites** f_x:



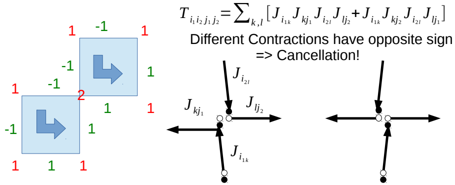
Second step: Grassmann integration

Recall: gauge integration before Grassmann integration (no fermion determinant)

- free color indices need to be **contracted** at each site (for given ensemble of plaquettes)
- in general, gives rise to tensor networks/vertex model $\rightarrow$ (too) difficult!

Simplification in the U(3) sector:

- plaqutte occupations $n_p, n_{p'} \in \mathbb{Z}$ can **only differ by** ± 1 if p, p' adjacent
- plaqutte occupations $n_p, n_{p'} \in \mathbb{Z}$ cannot share a site if they don't share a link



$\Rightarrow$ $\boxed{\text{Gauge fluxes } f_b \text{ form self-avoiding loops!}}$

- Does not apply to the additional SU(3) contributions: so far restricted to first non-trivial contribution (3-flux sector)

MDP+P partition function:

$$Z_F(m_q,\mu) = \sum_{\{k,n,\ell,n_p\}} \underbrace{\prod_{b=(x,\mu)} \frac{(N_c-k_b)!}{N_c!(k_b-|f_b|)!}}_{\text{singlet hoppings } M_x M_y} \underbrace{\prod_x \frac{N_c!}{n_x!}(2am_q)^{n_x}}_{\text{chiral condensate } \bar\psi\psi} \underbrace{\prod_\ell w(\ell, f_b, \mu)}_{\text{triplet hoppings } \bar B_x B_y} \underbrace{\prod_P \frac{1}{|n_P|!}\left(\frac{\beta}{2N_c}\right)^{|n_P|}}_{\text{gluon propagation}}$$

$k_b \in \{0,\dots N_c\}$, $n_x \in \{0,\dots N_c\}$, $\ell_b \in \{0,\pm 1\}$, $f_b = \partial n_p$, $f_x = \frac{1}{2}\sum_b f_b$

- color constraint:

$$n_x + \sum_{\hat\mu=\pm\hat 0,\dots\pm\hat d} \left(k_{\hat\mu}(x) + \frac{N_c}{2}|\ell_{\hat\mu}(x)|\right) = N_c + f_x$$

- 3-flux weight involves additional **site weights** v_i and **link weights**:

$$w(B_3) = \frac{1}{N_c!(N_c-1)!(N_c-2)!}, \quad w(B_4) = \frac{(N_c-1)!(N_c-2)!}{N_c!}$$

- sign: combine gauge flux f_b with triplet flux ℓ_b to identify **fermionic loops** $\tilde\ell$:

$$\sigma(C) = (-1)^{L(C)+W(C)+N_-(C)} \prod_{\tilde\ell} \eta_\mu(x)$$

QCD lattice partition function correct up to $\mathcal{O}(\beta^3)$

MDP+P Ensembles (2-dim for visualization)

$\beta = 1.0$, $\mu = 0.5$ (liquid phase)

MDP ensemble:

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~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
 → .    .    . ⇐ .    . → .    . ← .    .    .    .    . ← .    .    .
| + ⇓   |||            ↑ + ⇓   ↓        ↓    ↓    ↓    ↓ - ↓    ↓    ↓
. ← .    .    . ⇒ .    . ← .    . → .    .    .    .    . → .    .    .
↓             ↑ + ↓   |        | + ⇓    ↓    ↓    ↓    |    ↓    ↓    ↓
. → .— .— . ← .    . → .    . ← .    .    .    .    .    .    .    .
| + ↓    |        |   ↑ + ⇓   ↓        ↓    ↓    ↓    ||   ↓    ↓    ↓
. ← .— .— . → .    . ← .    . → .    .    .    .    .    .    .    .
↓             ⇑+ ⇓    |        | + ⇓    ↓    ↓    ↓    |    ↓    ↓    ↓
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Flux ensemble:

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1 → 1   .   2 ⇐ 2   1 → 1   1 ← 1   .   .   .   1 → 1   .   .
↑ 1 ↓               ↑ 1 ↓                       ↑−1 ↓
1 ← 1   .   1 → 1   1 ← 1   1 → 1   .   .   .   1 ← 1   .   .
            ↑ 1 ↓           ↑ 1 ↓
1 → 1   .   1 ← 1   1 → 1   1 ← 1   .   .   .   .   .   .   .
↑ 1 ↓               ↑ 1 ↓
1 ← 1   .   2 ⇒ 2   1 ← 1   1 → 1   .   .   .   .   .   .   .
            ⇑ 2 ⇓           ↑ 1 ↓
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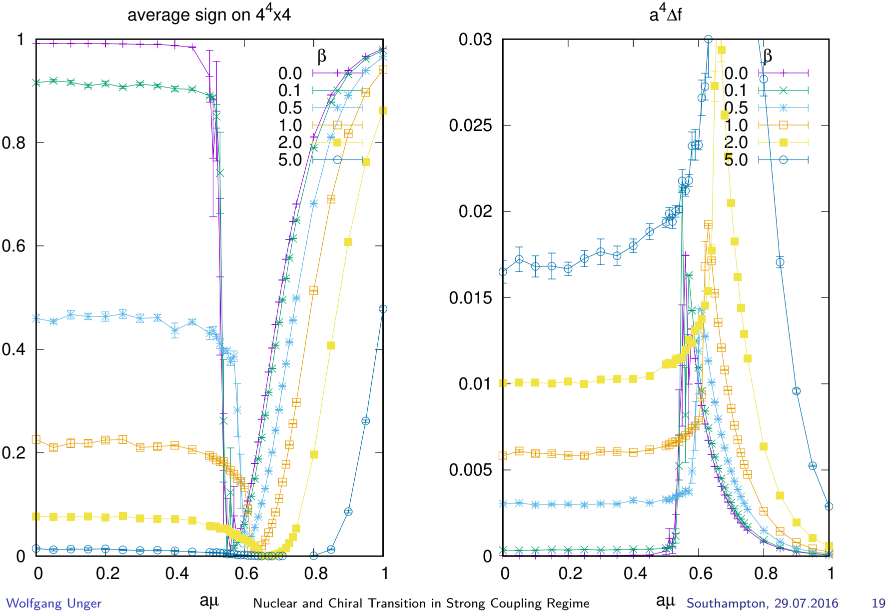
Nuclear and Chiral Transition at $T = 0$ - ongoing analysis



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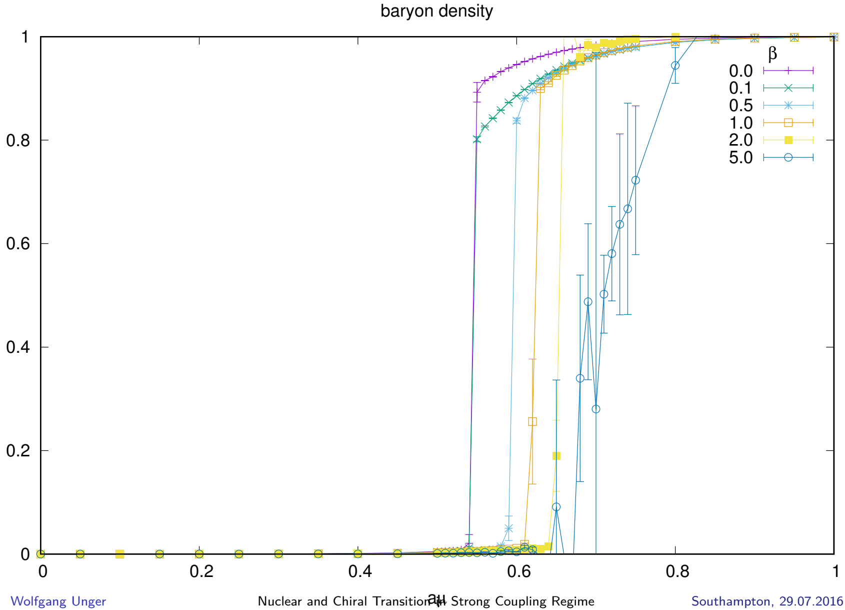
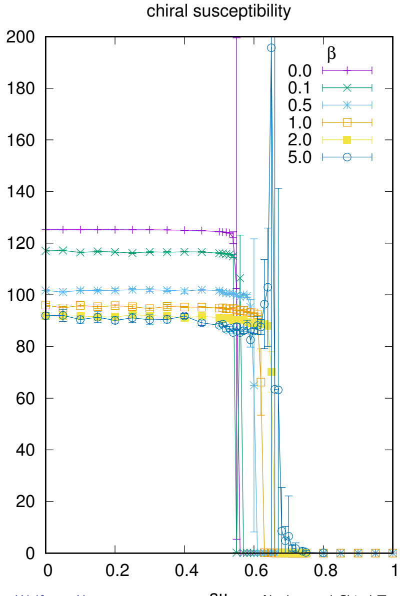
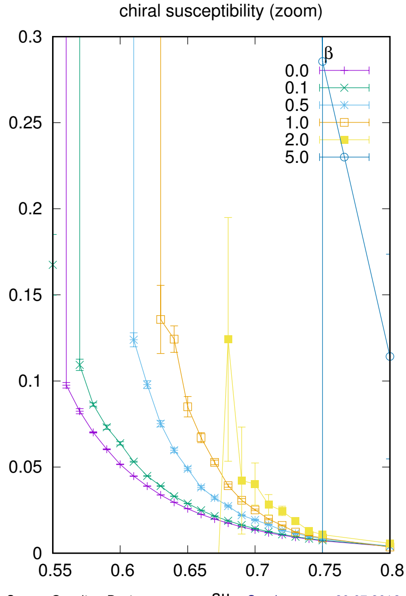


Nuclear and Chiral Transition at $T = 0$ - ongoing analysis





Conclusions

Results:

- all gauge integrals needed for $\mathcal{O}(\beta^3)$ and related to group characters
- Grassmann integration simplifies in U(3) sector
- sign problem mild enough to go beyond $\beta > 1$, but hard at $T = 0$ (even at $\beta = 0$)
- **simulations with $\mathcal{O}(\beta)^3$ corrections** included but not conclusive yet concerning nuclear vs. chiral transition (weak dependence on β)
- split between chiral and nuclear transition might be very small in nature

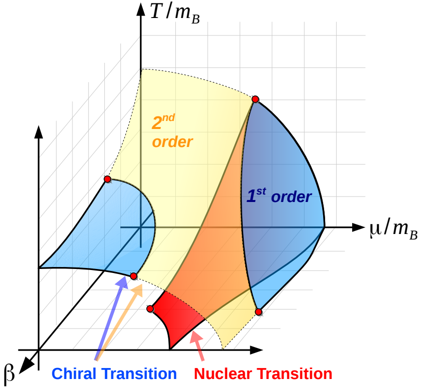
Goals:

- improve plaquette algorithm $\rightarrow$ incorporate **character expansion**
- generalize plaquette algorithm to non-trivial anisotropy $a/a_t = f(\gamma, \beta_s/\beta_t)$ to study various T ($\rightarrow$ talk by **Hélvio Vairinhos**: $a/a_t = f(\gamma)$ at $\beta = 0$)
- surpass the **roughening transition** at $\beta \simeq 5.9$: sampling of all orders needed
- other dualizations? ($\rightarrow$ talk by **Carla Marchis**)

Backup: Connection Between Strong Coupling and Continuum Limit?

One of several **possible scenarios** for the extension to the continuum:

- back plane: strong coupling phase diagram ($\beta = 0$), $N_{\rm f} = 1$
- front plane: continuum phase diagram ($\beta = \infty$, $a = 0$)
- due to fermion doubling, corresponds to $N_{\rm f} = 4$ in continuum (no rooting)



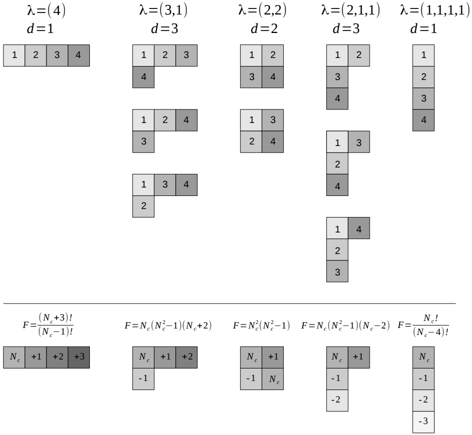
Backup: Young Tableaux - Color Structure

So, what are the C_λ^τ (and for $\mathcal{O}(\beta)$, $\mathcal{O}(\beta^2)$: C_λ^{τ,ρ_1}, $C_\lambda^{\tau,\rho_1,\rho_2}$, etc) ?

- answer: related to **irreducible representations** of the **symmetric group S_n** with $n = \kappa_1 + r = \kappa_2 + s$

Young Tableaux $\lambda = (\lambda_1, \dots \lambda_k)$:

- Standard Young tableaux correspond to **irreps of S_n** with dimension $d_\lambda = \frac{n!}{H_\lambda}$ with Hook lengths H_λ
- used to determine dimension $D_\lambda = \frac{F_\lambda}{H_\lambda}$ of irreps of SU(N_c)



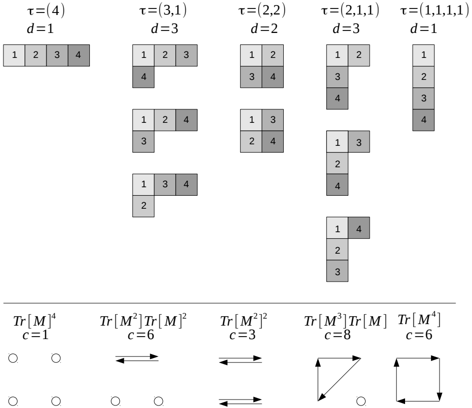
Backup: Young Tableaux - Flavor Structure

So, what are the C_λ^τ (and for $\mathcal{O}(\beta)$, $\mathcal{O}(\beta^2)$: C_λ^{τ,ρ_1}, $C_\lambda^{\tau,\rho_1,\rho_2}$, etc) ?

- answer: related to **irreducible representations** of the **symmetric group** S_n with $n = \kappa_1 + r = \kappa_2 + s$

Young Tableaux $\tau = (t_1, \ldots t_k)$:

- conjugacy classes
  $\leftrightarrow$ cycle structure of S_n
  $\leftrightarrow$ flavor permutations
- at a given order n, the **trace structure** $\tau = (t_1, \ldots t_k)$ with $T = \prod \mathrm{Tr}[M^i_{xy}]^{t_i}$ is equivalent to a partition of n: $\sum_i i t_i = n$
- example: $\pi = (136)(24)(58)(7)$
  $\rightarrow \quad t_1 = 1,\ t_2 = 2,\ t_3 = 1$



Backup: Table of Characters and Invariants

τ	$\lambda:$	$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)	Sum
(4,0,0,0)		1	3	2	3	1	10
(2,1,0,0)		1	1	0	-1	-1	0
(0,2,0,0)		1	-1	2	-1	1	2
(1,0,1,0)		1	0	-1	0	1	1
(0,0,0,1)		1	-1	0	1	-1	0
Sum		5	2	3	2	1	

Table: Characters χ_λ^τ for $n = 4$

τ	$\lambda:$	$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)	Sum
(4,0,0,0)		1	9	4	9	1	24
(2,1,0,0)		6	18	0	-18	-6	0
(0,2,0,0)		3	-9	12	-9	3	0
(1,0,1,0)		8	0	-16	0	8	0
(0,0,0,1)		6	-18	0	18	-6	0
Sum		24	0	0	0	0	

Table: Invariants C_λ^τ for $n = 4$

Backup: Relationship to the Character Expansion

- Define Bessel Determinants:

$$D_n^{(3,e)} = \begin{vmatrix} I_n(x) & I_{n+1}(x) & I_{n+2}(x) \\ I_{n-1}(x) & I_n(x) & I_{n+1}(x) \\ I_{n-2}(x) & I_{n-1}(x) & I_n(x) \end{vmatrix}, \; D_n^{(3,f)} = \begin{vmatrix} I_{n+1}(x) & I_{n+2}(x) & I_{n+3}(x) \\ I_{n-1}(x) & I_n(x) & I_{n+1}(x) \\ I_{n-2}(x) & I_{n-1}(x) & I_n(x) \end{vmatrix}$$

- fundamental character in $x = \frac{1}{g^2} = \frac{\beta}{2N_c}$:

for U(3): $u(\beta) = \frac{D_0^{(3,f)}(2x)}{D_0^{(3,e)}(2x)} = \frac{\frac{1}{0!1!}x + \frac{2}{1!2!}x^3 + \frac{6}{2!3!}x^5 + \frac{23}{3!4!}x^7 + \frac{103}{4!5!}x^9 + \ldots}{1 + \frac{1}{1!^2}x^2 + \frac{2}{2!^2}x^4 + \frac{6}{3!^2}x^6 + \frac{23}{4!^2}x^8 + \frac{103}{5!^2}x^{10} + \ldots}$

for SU(3): $u(\beta) = \frac{\sum_{n=-\infty}^{\infty} D_n^{(3,f)}(2x)}{\sum_{n=-\infty}^{\infty} D_n^{(3,e)}(2x)} = \frac{\frac{1}{0!1!}x + \frac{1}{0!2!}x^2 + \frac{2}{1!2!}x^3 + \frac{5}{8}x^4 + \frac{2\times 6+1}{24}x^5 + \frac{77}{240}x^6 + \frac{5\times 23+24}{720}x^7 + \ldots}{1 + \frac{1}{1!^2}x^2 + \frac{1+1}{3!}x^3 + \frac{2}{2!^2}x^4 + \frac{1}{4}x^5 + \frac{2\times 6+1}{72}x^6 + \frac{23}{4!^2}x^8 + \ldots}$

- character expansion recovered, but not limited to Wilson loops!

	1 (mesonic)	L^3	L^6	L^9	L^{12}
1 (baryonic)	1	1	5	42	462
(LL^*)	1	3	21	210	2574
$(LL^*)^2$	2	11	98	1122	15015
$(LL^*)^3$	6	74	498	6336	91091
$(LL^*)^4$	23	225	2709	37466	571428