Complete Monopole Dominance of the Static Quark Potential

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Introduction

- We wish to identify the topological objects that lead to quark confinement.
- Our tool is the gauge-independent Abelian decomposition first proposed by Cho, Duan and Ge.
- This decomposes the non-Abelian gauge field $A_\mu$ into an Abelian (restricted) part, $\hat{A}$ and a coloured field $\hat{X}$,

$$A_\mu(x) = \hat{A}_\mu(x) + X_\mu(x) \quad U_{\mu,x} = \hat{U}_{\mu,x} \hat{X}_{\mu,x+\hat{\mu}},$$

where $U_{\mu}$ is the lattice gauge link $P[e^{ig\int A_\mu dx_\mu}]$ $\hat{U}$ and $\hat{X}$ the links corresponding to the $\hat{A}$ and $X$ fields.
The decomposition is performed independent of the gauge by choosing:
1. an SU($N$) field $\theta(x)$,
2. a subsequent colour direction $n_j = \theta \lambda_j \theta^\dagger$, $j = 3, 8, \ldots$

$$D_\mu[\hat{A}] n_j = 0 \quad \text{tr} \ n_j X_\mu = 0$$

$$\hat{A}_\mu = \frac{1}{2} n_j \text{tr} \left( n_j A_\mu + \frac{i}{g} \lambda_j \theta^\dagger \partial_\mu \theta \right).$$

The two parts of $\hat{A}$ are known as the Maxwell part and $\theta$-part.
The field strength $F_{\mu\nu}[\hat{A}]$ is a function of $n_j$.
There are $N - 1$ redundant d.o.f. in $\theta$: $(\theta \rightarrow \theta e^{idj\lambda_j})$

$$\theta = \begin{pmatrix} \cos a & i \sin ae^{ic} \\ i \sin ae^{-ic} & \cos a \end{pmatrix} e^{id\lambda_3},$$

$0 \leq a \leq \frac{\pi}{2}$ and $c, d \in \mathbb{R}$
• We can choose a $\theta$ field and perform this decomposition regardless of the gauge.

• If under a gauge transformation $\Lambda \in SU(2)$

\[
U_{\mu,x} \rightarrow \Lambda_x U_{\mu,x} \Lambda_{x+\hat{\mu}}^\dagger \quad \theta_x \rightarrow \Lambda_x \theta_x,
\]

then $\hat{U}$ and $\hat{X}$ will transform gauge covariantly,

\[
\hat{U}_{\mu,x} \rightarrow \Lambda_x \hat{U}_{\mu,x} \Lambda_{x+\hat{\mu}}^\dagger \quad \hat{X}_{\mu,x} \rightarrow \Lambda_x \hat{X}_{\mu,x} \Lambda_{x}^\dagger.
\]

• $\hat{A}$ transforms as a U(1) field. In SU(2),

\[
\frac{1}{2} \text{tr} \left( n_3 A_{\mu} + \frac{i}{g} \lambda_3 \theta^\dagger \partial_{\mu} \theta \right) \rightarrow \frac{1}{2} \text{tr} \left( n_3 A_{\mu} + \frac{i}{g} \lambda_3 \theta^\dagger \partial_{\mu} \theta \right) + \frac{1}{g} \partial_{\mu}(d' - d).
\]
• We can construct gauge invariant observables from the restricted field alone.
• \( \hat{U} \) is Abelian, so is much easier for analytical studies.
• We choose \( \theta \) so that it contains the eigenvalues of the Wilson Loop operator
• Along the Wilson loop (in xt plane), this implies \( \hat{U} = U \).

\[
\text{tr } W_L = \text{tr } \prod U_{\mu,x} = \text{tr } \prod \hat{U}_{\mu,x} = \text{tr } e^{ig\lambda_3 \oint dx_\mu \frac{1}{2} \text{tr} \left( n_3 A_\mu + \frac{i}{g} \lambda_3 \theta^+ \partial_\mu \theta \right)}.
\]

In SU(2),

\[
\theta^+ \partial_\mu \theta = i\phi \partial_\mu a + i\bar{\phi} \sin a \cos a \partial_\mu c - i\lambda_3 \sin^2 a \partial_\mu c,
\]

with

\[
\phi = \begin{pmatrix} 0 & e^{ic} \\ e^{-ic} & 0 \end{pmatrix} \quad \bar{\phi} = \begin{pmatrix} 0 & ie^{ic} \\ -ie^{-ic} & 0 \end{pmatrix}.
\]
• The expression for the Wilson Loop contains the factor

\[ \text{tr}W_L = \text{tr} \prod U_{\mu,x} = \text{tr} e^{i \lambda_3 \oint dx_\mu \sin^2 a \partial_\mu c} + \ldots \]

• The parameter \( c \) is ill-defined at \( a = 0 \) or \( \pi/2 \)
• We can have non-zero winding number around these points.

\[(t, x, y, z) = r(\cos \varphi_1, \sin \varphi_1 \cos \varphi_2, \sin \varphi_1 \sin \varphi_2 \cos \varphi_3, \sin \varphi_1 \sin \varphi_2 \sin \varphi_3)\]

• Then write \( c = \nu \varphi_1 \) for winding number \( \nu \).
• \( \oint dx_\mu \sin^2 a \partial_\mu c \) will contain a contribution proportional to the sum of the winding number within the loop
• It should scale with the number of monopoles within the loop
• This will be proportional to the area of the loop.
• My goal is to show that the topological objects discussed above are
  – Present in the configuration
  – Contribute to quark confinement.
• \( \theta \) is gauge dependent – we need to fix the gauge.
• But whatever causes confinement should not be gauge dependent.
• Under a gauge transformation:

\[
\text{tr}(n_3(gA_\mu)) \rightarrow \text{tr}(\lambda_3(g\theta^\dagger A_\mu \theta + i\theta^\dagger \partial_\mu (\Lambda^\dagger)\Lambda \theta))
\]
\[
\text{tr}(i\lambda_3(\theta^\dagger \partial_\mu \theta)) \rightarrow \text{tr}(i\lambda_3(\theta^\dagger \partial_\mu \theta - \theta^\dagger \partial_\mu (\Lambda^\dagger)\Lambda \theta)) + i\partial_\mu (d' - d).
\]

• \( \Lambda \) has the same structure as \( \theta \)
• A gauge transformation moves features from one part of \( \hat{A} \) to the other part
• The objects themselves are not gauge dependent
• We can search for the objects within the $\theta^\dagger \partial_\mu \theta$ part of $\hat{A}$.
• This is harder in the Maxwell part ($\text{tr} \, n A_\mu$).
• Our proposal is to fix the gauge so that the $\theta$ term alone accounts for the whole Wilson Loop.
• We can then isolate the topological objects and study them.

So far, we just have numerical results in SU(2), $\beta = 3.4$, stout smeared Lüscher-Weisz gauge action, $16^3 \times 32$ lattice.
• We used a set of nested Wilson Loops in the $xt$ plane to construct $\theta$ and to measure the string tension.
• These loops are then stacked on top of each other in the $y$ and $z$ directions.
• $\theta$ is loop-dependent.
• The topological objects contributing to confinement depend on the quarks we are attempting to confine.
• The continuum version of the gauge fixing needs to satisfy three requirements:
  1. $\theta$ should be smooth and differentiable;
  2. The $\theta$ term should dominate the Wilson Loop;
  3. We should avoid a Fadeev-Poppov ghost determinant
• The lattice equivalent of this gauge fixing procedure is to
  1. Choose $d$ so that $\theta^\dagger_x \hat{U}_{x,\mu} \theta_{x+\mu} \sim 1 + iO(\epsilon)$.
  2. Fix the gauge so that

\[ e^{i\epsilon \lambda_3 \text{tr} \lambda_3 \theta^\dagger \partial_\mu \theta} = \theta^\dagger \hat{U}_\mu \theta \quad d = 0 \]

where the direction $\mu = x, t$, $\epsilon$ is the lattice spacing.
• We apply the gauge fixing condition exactly along the Wilson Loop, and as close as we can for the other links.
Finding the Monopoles

\[
\text{tr } W_L \sim \text{tr } e^{i \oint \lambda_3 \sin^2 a \partial_\mu c dx_\mu}
\]
$$\text{tr} \ W_L \sim \text{tr} \ e^{i \int \lambda_3 \sin^2 a \partial_\mu c \, dx_\mu}$$
\[ \text{tr } W_L \sim \text{tr } e^{i \oint \lambda_3 \sin^2 a \partial_\mu c \, dx_\mu} \]
• Are these objects (minimum in $a$, winding in $c$) distributed according to an area law?
• Is the winding number responsible for the Wilson Loop?

\[ L = \int dx_\mu \frac{1}{2} \text{tr} \left( \lambda_j \theta^\dagger A_\mu \theta + \frac{i}{g} \lambda_j \theta^\dagger \partial_\mu \theta \right) \]

\[ = \int dx_\mu \sin^2 a \partial_\mu c \sim \left( \frac{1}{2} \right) (2\pi \nu) \]
Conclusion

- Through a combination of an Abelian decomposition and gauge fixing, we can isolate the topological objects responsible for confinement in SU(2) pure Yang Mills gauge theory.
- The $\theta$ term is wholly responsible for the static quark potential.
- We are able to see the emergence of a localised topological winding number at minimum values of $\alpha$.
- These objects are distributed according to an area law.
- The value of the Wilson Loop is proportional to the winding number, and changes where there is winding.
- These topological objects are present in the Yang Mills gauge fields, and a good candidate for studies into the cause of confinement in SU(2) Yang-Mills theory.