Complex Langevin Dynamics In 1+1d QCD At Non-Zero Densities

Sebastian Schmalzbauer[†‡] & Jacques Bloch[†]

[†] Institut für Theoretische Physik, Universität Regensburg, Germany

[‡] Institut für Theoretische Physik, Goethe Universität, Frankfurt am Main, Germany

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We present our results obtained from gauge cooled complex Langevin simulations in 1+1d QCD at non-zero densities in the strong coupling regime with unrooted staggered fermions. For small quark masses there are regions of the chemical potential where this method fails to reproduce correct results. In these parameter ranges we studied the effect of different gauge cooling schemes on the distributions of the fermion determinant as well as of observables.

Theoretical Background

1+1d QCD

- Strong coupling partition function for 1 flavor

$$Z = \int \mathcal{D}[U]\, \det D[U]$$

- 1+1 dimensional staggered Dirac operator for quark mass m at chemical potential μ

$$D_{x,y} = m\delta_{x,y} + \frac{1}{2}\left[e^{\mu}U_{0,x}\delta_{x+\hat{0},y} - e^{-\mu}U^{-1}_{0,y}\delta_{x-\hat{0},y}\right] + \frac{1}{2}(-1)^{x_0}\left[U_{1,x}\delta_{x+\hat{1},y} - U^{-1}_{1,y}\delta_{x-\hat{1},y}\right]$$

- Sign problem already severe enough for a meaningful test of the complex Langevin method
- Validation of results with subset method [1] or reweighted phase-quenched simulations still possible
- Less computational effort than full QCD $\Rightarrow$ ideal toy model
- Presented results are obtained from small lattices 4×4

Complex Langevin (CL) Dynamics

- Evolve gauge fields according to the complex Langevin equation [2] (stochastic quantization [3])
- Requires complexification of gauge fields $\mathrm{SU}(3) \to \mathrm{SL}(3,\mathbb{C})$
- Sample correct configurations after thermalization? See [4, 5] for problems with
  (1) Large excursions in the imaginary direction
  (2) Existence of singular drifts and local non-holomorphic observables/action
- Discretized CL update step = rotation (Gell-Mann representation)

$$U'_{n\nu} = R_{n\nu}U_{n\nu} \qquad R_{n\nu} = \exp\left[i\sum_{a=1}^{8}\left(\epsilon K_{n\nu a} + \sqrt{\epsilon}\eta_{n\nu a}\right)\lambda_a\right]$$

with deterministic drift term $K_{n\nu a}$ depending on the discretization scheme (Euler/Runge-Kutta), real gaussian distributed noise $\eta_{n\nu a}$ with variance 2 and time discretization ϵ

Gauge Cooling

- Use gauge invariance of theory to apply gauge transformations (GT) after each CL update

$$U_{n\nu} \to U^{(G)}_{n\nu} = G_n U_{n\nu} G^{-1}_{n+\hat{\nu}}$$

- GT influences the CL trajectory: drift $K_{n\nu} = \sum_a K_{n\nu a}\lambda_a$ transforms as

$$K_{n\nu}[U^{(G)}] = G_n K_{n\nu}[U] G_n^{-1}$$

- Desired effect: counter (1) and (2) during the Langevin evolution
- Implementation: minimize some $\mathrm{SL}(3,\mathbb{C})$ gauge variant norm $\mathcal{N}$ by gradient descent [6]
- Parameters: #cooling iterations and cooling stepsize α (chosen adaptively)

Results with Unitarity Cooling

Unitarity Cooling

- Idea: reduce distance to corresponding SU(3) configuration to counter (1) [6]

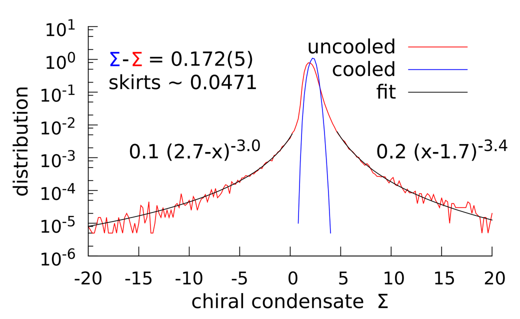
$$\mathcal{N}_{\mathrm{u}} = \sum_{n,\nu} \mathrm{tr}\left[U^{\dagger}_{n\nu}U_{n\nu} + \left(U^{\dagger}_{n\nu}U_{n\nu}\right)^{-1} - 2\right]$$

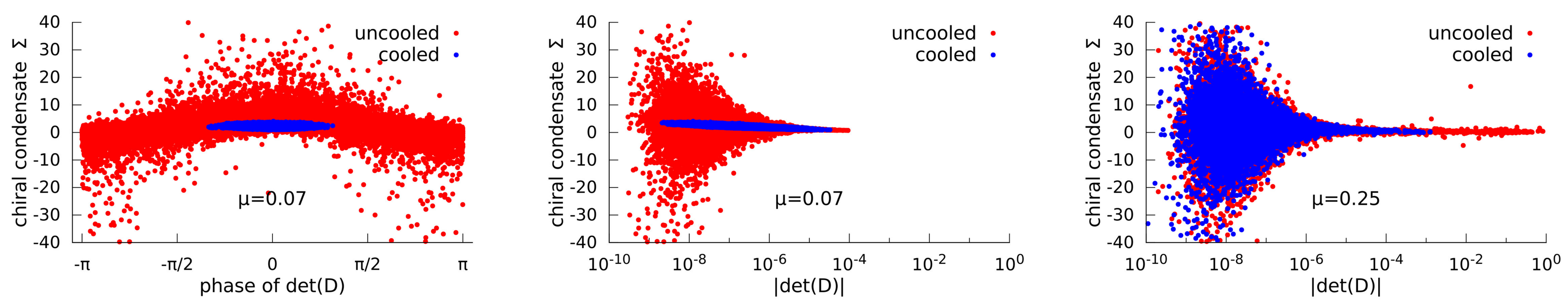
- Performs minimally better than non-symmetric definition

Results

- Inconsistency with benchmark for light quarks around phase transition [7], where the distribution of fermion determinant includes the origin, i.e. (2) is not suppressed enough and distributions of the observables develop "skirts", both observed for e.g. $m = 0.1$, $\mu \in [0.1, 0.5]$
- Skirts themselves are not responsible for the wrong values, but can be used as an indicator:
- Skirts are not related to branch-cut crossings [8], as they happen independently of the determinant phase, but correlate with small determinant values, eg. explicitly for the chiral condensate via

$$\det D = \prod_i d_i \quad \Leftrightarrow \quad \sigma = \frac{1}{V}\sum_i d_i^{-1}$$





- Does not perform well enough in some parameter regions $\Rightarrow$ study other cooling schemes

New Cooling Schemes

Polyakov Cooling

- Idea: Increase unitarity of Polyakov loops P_x

$$\mathcal{N}_P = \sum_x \mathrm{tr}\left[P^{\dagger}_x P_x + \left(P^{\dagger}_x P_x\right)^{-1} - 2\right]$$

- Motivation: it worked extremely well in Polyakov loop formulation of 0+1d QCD [7]
- Problem: only first timeslice can be modified due to almost gauge invariant definition
- Results: not changed perceptively compared to uncooled case (in 1+1d)

Antihermiticity Cooling

- Idea: restore/increase antihermiticity of Dirac operator [9] (given for $m = 0$ and $\mu = 0$ or $\mu \in i\mathbb{R}$)

$$\mathcal{N}_{\bar{\dagger}} = \mathrm{tr}\left[\left(D + D^{\dagger}\right)^2\right] = \frac{1}{2}\sum_{n,\nu}\mathrm{tr}\left[e^{2\mu\delta_{\nu 0}}U^{\dagger}_{n\nu}U_{n\nu} + e^{-2\mu\delta_{\nu 0}}\left(U^{\dagger}_{n\nu}U_{n\nu}\right)^{-1} - 2\right] + 12Vm^2$$

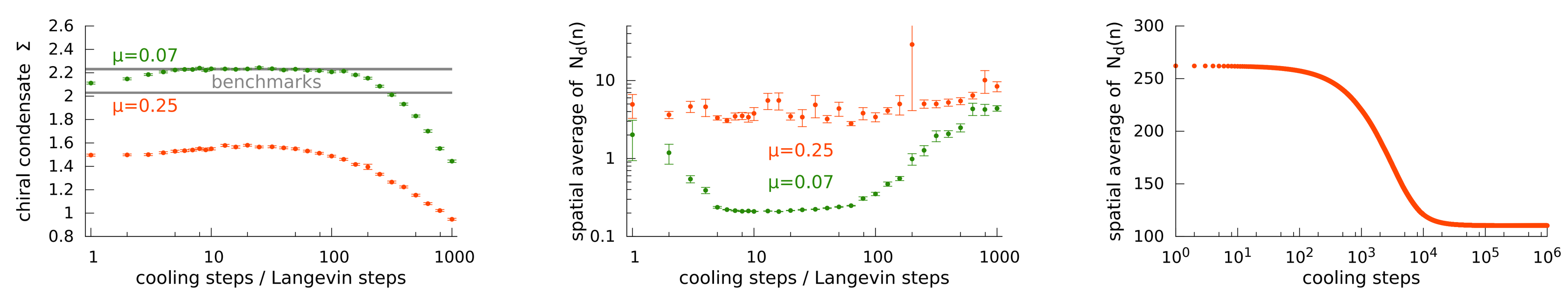
- Similar to unitarity norm (neglecting the irrelevant offset), including a μ-dependence
- Results: not changed perceptively compared to unitarity cooled case

Maximum Drift Cooling

- Idea: reduce the absolute value of the local maximum drift to suppress (2) [5]

$$\mathcal{N}_d(n) = \max_{\nu}\ \mathrm{tr}\left[K^{\dagger}_{n\nu}K_{n\nu}\right]$$

- In contrast to the cooling schemes above, the results somehow depend on #iterations, although we observe no unusual behavior by cooling a single configuration excessively



- Does not perform better than unitarity cooling, but is more expensive

Concluding Remarks

Summary

- Gauge cooled CL simulations produce correct results for heavy quarks
- However, for light quarks, the considered gauge cooling schemes are not enough to return to the correct values in a significant range of the chemical potential around the phase transition
- Similar behaviors are shown by cooling with different norms, even if they are not formally related

Perspectives

- Missing understanding why gauge cooling cannot improve the results enough in the problematic cases
- Perform same study including the gauge action
- Increase applicability of CL by combining it with other methods, eg. reweighting [10] or Taylor expansions from $\mu > 0$

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