O($N$) model with Nienhuis action

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Lattice 2016, Southampton, July 26

• action ‘optimized’ for simple loop formulation (↔ worm)
• investigate transfer matrix and Monte Carlo
• result: ‘effectively’ in the right universality class
**O(N) model**

Partition function for standard lattice action:

\[
Z = \int \left[ \prod_z d\mu(s(z)) \right] e^{\beta \sum_{l=\langle xy \rangle} s(x) \cdot s(y)}, \quad s \in S_{N-1}
\]

Nienhuis truncation (same universality class?!):

\[
Z = \int \left[ \prod_z d\mu(s(z)) \right] \prod_l \left[ 1 + \tilde{\beta} s(x) \cdot s(y) \right]
\]

- **exactly solved** in \( D = 2 \) for \(-2 \leq N \leq 2\) on honeycomb lattice
  - in these cases critical region covered for \( \tilde{\beta} \leq 1 \)
- **includes** XY, Kosterlitz Thouless
- **Ising, \( N = 1 \): equivalent** for \( \tilde{\beta} = \tanh(\beta) \)
- **\( N = 3 \) will require** \( \tilde{\beta} > 1 \) ⇒ serious sign problem for spins, **not for worm**
- **truncation desirable**: simpler worm simulation
Transfer matrix

\[ T[s', s] = T_0[s', s] T_1[s] \]

\[ T_0[s', s] = \prod_{\bar{z}} \left( 1 + \tilde{\beta}s' (\bar{z}) \cdot s(\bar{z}) \right), \quad T_1[s] = \prod_{\langle \bar{x}, \bar{y} \rangle} \left( 1 + \tilde{\beta}s(\bar{x}) \cdot s(\bar{y}) \right) \]

- \( T \) acts on \( \psi[s] \) defined on a row of spins, \( L \), periodic
- \( T_0 \): finite dimensional projection, \( \text{dim} = (1 + N)^L \)
- \( \Rightarrow \) build finite matrix, find spectrum [possible: \( L \leq 14 \)]
- \( \exists K, \quad KTK^{-1} = \text{real symmetric} \)
- eigenvalues real, \( \lambda < 0 \) possible
- two-step matrix \( T^2 \): real, positive
spectrum for $\bar{g}^2 = m(L)L = 1.0595$

coupling vs. $\tilde{\beta}$ for $L = \text{odd, even}$

- $L = \text{odd}, m < 0$: triplet becomes groundstate!
- $L = \text{even}$: minimal $\bar{g}^2$ reached
- standard action: $\bar{g}^2 = 1/\beta + c(L)/\beta^2 + \ldots \searrow 0$ as $\beta \to \infty$  $\text{PT}$
Worm Simulation

start from:

\[ Z = \sum_{u,v,c} \rho^{-1}(u-v) \int Ds \prod_{l=\langle xy \rangle} \left[ 1 + \tilde{\beta} s(x) \cdot s(y) \right] s_c(u) s_c(v) \]

- \( k_l = 0 \) or \( 1 \ldots N \) on each link for term \( 1(k=0) \) or \( \tilde{\beta}s_k(x)s_k(y) \)

\[ Z = \sum_{u,v,c,\{k\}} \rho^{-1}(u-v) \tilde{\beta} \sum_{l} (1-\delta_{k(l),0}) \prod_{z} C[q(z)] \]

with

\[ q_a(z) = \sum_{l,\partial l \ni z} k_a(l) + \delta_{a,c}(\delta_{z,u} + \delta_{z,v}), \quad C[q] = \ldots. \]

\[ N = 3(\text{red, blue, black}), \quad L = 12, \quad T = 96, \quad \tilde{\beta} = 2.6804, \quad \bar{g}^2 = 0.9 \]

\[ \text{Diagram} \]
Monte Carlo Results

- $\tilde{\beta} > 1$ clearly required $\rightarrow$ sign problem in $s(x)$, not $k_a(l)$
- falling branches: $g^2(2L) > g^2(L)$, sign for as. freedom
- minima rising with $L$
Measuring SSFs \[ \Sigma(u, L^{-2}) = \bar{g}^2(2L) \bar{g}^2(L) = u \]

\[ \sum(1.0595, L^{-2}) \]

\[ \sum(0.9, L^{-2}) \]

‘traditional’ \( u = 1.0595 \) \( u = 0.9 \) close to min for \( L = 48 \)

- run up to \( L = 48 \rightarrow 2L = 96, * = \) exact
- data seem to ‘know’ *, deviations small
- cut-off effects look non-monotonic at our precision
Balog, Niedermayer, Weisz:

\[
\Sigma - \sigma = \frac{1}{L^2}[A\ln^3 L + B\ln^2 L + C\ln L + D]
\]

for a wide class of actions **not including Nienhuis**

\[
A = 0.97, B = -6.8, C = 13, D = -7.3
\]

\[
A = 0, B = 0.36, C = -1.6, D = 1.7
\]

- reasonable fits, competing logs
Conclusions

- for a given $\bar{g}^2$ only $(L/a) \leq c(\bar{g}^2)$ can be realized
  - $\rightarrow$ no complete continuum limit $a/L \rightarrow 0$
  - larger $L/a$ possible for larger $\bar{g}^2$ ($c' > 0$) (all $\bar{g}^2$ ??)
- perturbatively small $\bar{g}^2$ not reached; $mL \rightarrow \infty$ okay?
- BUT: for $\bar{g}^2=1.0595, 0.9$ we got very close to exact answer up to tiny cutoff effects
- small $\bar{g}^2$ via ODD $L$: totally off...
- effective theory?
- analogue: triviality in 4d $\lambda \phi^4$: for given $\lambda_R > 0$
  - no complete limit $am_R \rightarrow 0$
- opposite to here: smaller $\lambda_R$ allows smaller $am_R$