The Nf=3 gradient flow coupling running from 4GeV to 200MeV

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ALPHA Collaboration

"Slow running of the Gradient Flow coupling from 200 MeV to 4 GeV in $N_f = 3$ QCD"
arXiv:1607.06423

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Overview

CONNECTING WITH THE HADRONIC REGIME OF $N_f = 3$ QCD

- Result of the high energy SF running gives at $g^2_{SF}(L_0) = 2.012$ the result [A. Sint Talk]

$$L_0\Lambda = 0.0308(8)$$

- Hadronic scale ($g^2_{GF} \equiv$ Gradient Flow, Finite Volume, SF boundary conditions)

$$g^2_{GF}(L_{had}) = 11.31$$

  $L_{had} \times F_{K,\pi}$ can be computed using CLS ensambles [R. Sommer talk].

- Main result:

$$\frac{L_{had}}{L_0} = 21.86(42)$$

  NOT very interesting... How we compute it, **is very interesting**.

CONNECTING WITH THE HADRONIC REGIME OF $N_\mathrm{f} = 3$ QCD

Strategy

1. Determine the value $g^2_{\mathrm{GF}}(L_0)$ at $g^2_{\mathrm{SF}}(L_0) = 2.012$
2. Determine the β-function. Massless FV scheme: $-L\frac{\partial g_{\mathrm{GF}}(L)}{\partial L} = \beta(g_{\mathrm{GF}})$

$$\frac{L_{\mathrm{had}}}{L_0} = \exp\left\{-\int_{g_{\mathrm{GF}}(L_0)}^{g_{\mathrm{GF}}(L_{\mathrm{had}})} \frac{dx}{\beta(x)}\right\}$$

3. Determine the step scaling function $\sigma(u) = g^2_{\mathrm{GF}}(2L)|_{g^2_{\mathrm{GF}}(L)=u}$

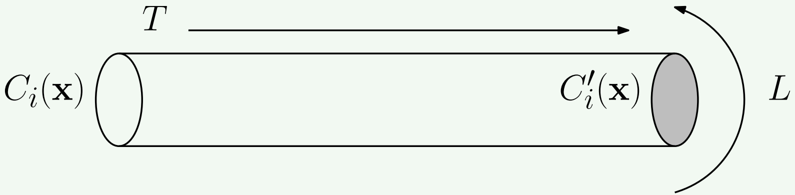
$$\log 2 = -\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} \frac{dx}{\beta(x)}$$

4. Determine the lattice step scaling function $\Sigma(u, a/L) = g^2_{\mathrm{GF}}(2L)|_{g^2_{\mathrm{GF}}(L)=u}$

$$\sigma(u) = \lim_{a/L\to 0} \Sigma(u, a/L)$$

Overview

Lattice details



- Schrödinger Functional boundary conditions [Lüscher et al. '92; S. Sint '94]

$$U_k(x)|_{x_0=0,T} = 1; \qquad P_+\psi(0,\mathbf{x}) = 0 = \bar{\psi}(0,\mathbf{x})P_-\,,$$
$$\psi(x+L\hat{k}) = e^{\imath\theta}\psi(x); \qquad P_-\psi(T,\mathbf{x}) = 0 = \bar{\psi}(T,\mathbf{x})P_+\,.$$

  - Symanzik tree-level $\mathcal{O}(a^2)$ improved gauge action [Lüscher, Weisz '85].
  - `option B` near time boundaries [Aoki et al. '99].
  - One-loop improvement coefficients $c_t, \tilde{c}_t$ [S. Takeda et al. '03, P. Vilaseca '15]
- Three massless NP-$\mathcal{O}(a)$ improved Wilson fermions [Bulava, Schaefer '13].

Same ensembles used to compute running mass [D. Preti Tue B67 14:40]

Gradient flow coupling definition [Lüscher '10; P. Fritzsh, A. Ramos '13]

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t); \quad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \,.$$

$$g^2_{\mathrm{GF}}(L) = \hat{\mathcal{N}}^{-1}(c, a/L)\,\frac{t^2}{4}\,\frac{\langle G^a_{ij}(t,x) G^a_{ij}(t,x)\,\delta_{Q,0}\rangle}{\langle \delta_{Q,0}\rangle}\Big|_{\sqrt{8t}=cL,\; x_0=T/2},$$

- Coupling defined at $x_0 = T/2$ and with only $G_{ij}(t,x) \Longrightarrow$ less boundary effects.
- $c = 0.3$ ratio between smearing radius ($\sqrt{8t}$) and lattice size (L).
- $\hat{\mathcal{N}}^{-1}(c, a/L)$ Computed with our choices of discretization: $g^2_{\mathrm{GF}} = g^2_0 + \mathcal{O}(g^4_0)$.
- $\delta_{Q,0}$ Projects to the $Q = 0$ topological sector $\Longrightarrow$ overcome "topology freezing" [P. Frizsch et al. '13]

How do the links $V_\mu(x,t) = \exp[B_\mu(t,x)]$ change with the t?

$$a^2 \frac{d}{dt} V_\mu(x,t) = -g_0^2 \frac{\delta S^{\mathrm{latt}}[V]}{\delta V_\mu(x,t)} V_\mu(x,t)$$

- Is this the best option?
- Which lattice action S^{latt}?

Gradient flow coupling definition [Lüscher '10; P. Fritzsh, A. Ramos '13]

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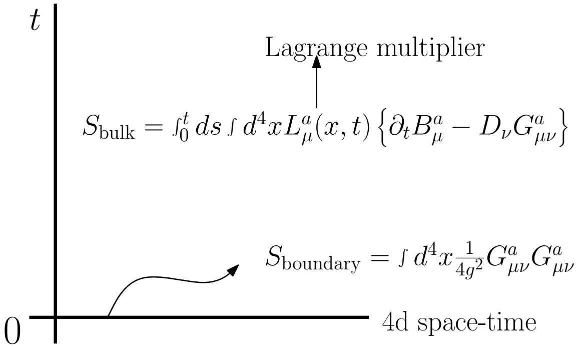
The Zeuthen flow

$$a^2 \frac{d}{dt} V_\mu(x,t) = -g_0^2 \left(1 + \frac{a^2}{12} D_\mu D^*_\mu\right) \frac{\delta S^{\rm LW}[V]}{\delta V_\mu(x,t)} V_\mu(x,t)$$

Improvement of flow quantities [A. Ramos, S. Sint '15]

Four sources of cutoff effects in flow quantities

- Quantum effects at $t = 0$. Very complicated dependence on g_0^2
  - Choice of action.
  - Choice of initial condition in the flow equation (i.e. $V_\mu(0, x) = U_\mu(x)$)
- Integrating the flow equation
- Evaluating an observable



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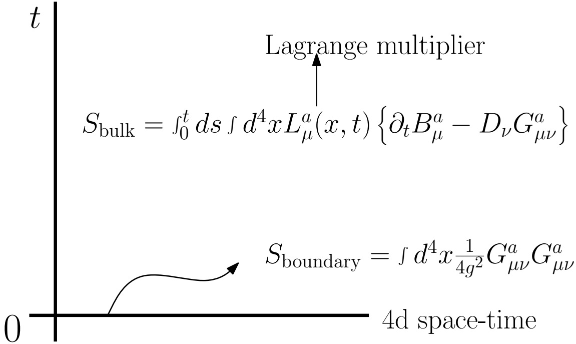
- ~~Integrating the flow equation~~ Zeuthen flow
- ~~Evaluating an observable~~ Classically improved discretization.



Continuum limit of flow quantities

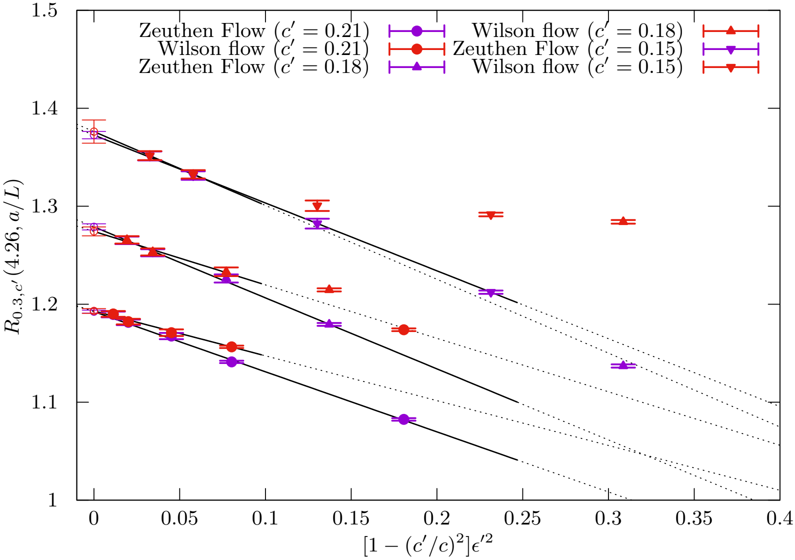
Getting some insight in the scalig of the $\Sigma(u, a/L)$

$$R_{c,c'}(u, a/L, s) = \left.\frac{g_c^2(L)}{g_{c'}^2(sL)}\right|_{g_c^2(L)=u} = R_{c,c'}(u, 0, s)\left\{1 + A_{c,c'}(u)[\epsilon^2 - \epsilon'^2] + \ldots\right\},$$

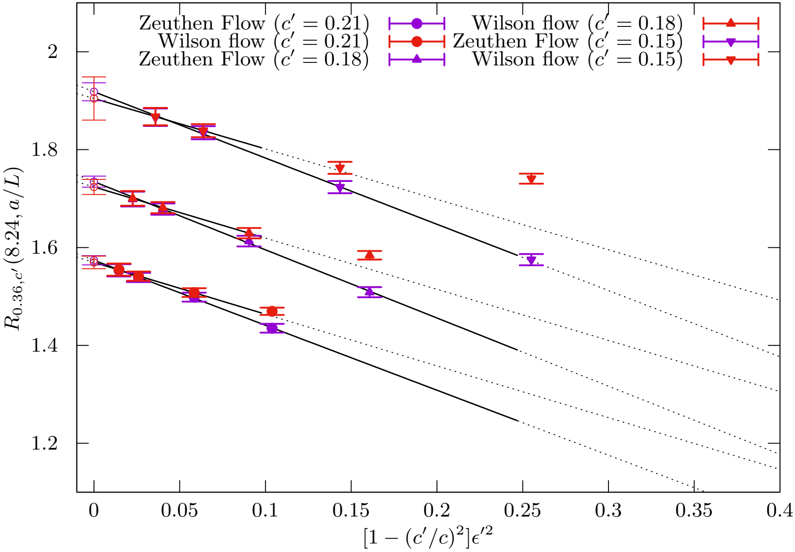
with $\epsilon = a/(cL)$ and $\epsilon' = a/(c'sL)$.

- $R_{c,c'}(u, a/L, s)$ is mainly a function of sc'.
- Step scaling function $\Longrightarrow R_{c,c}(u, a/L, 2) = u/\Sigma(u, a/L)$.
- Instead study $R_{c,c'}(u, a/L, 1) \Longrightarrow$ we can use $L/a = 8, 12, 16, 24, 32$

Scaling of the ratios $R_{c,c'}(u, a/L, 1)$



Scaling of the ratios $R_{c,c'}(u, a/L, 1)$



Scaling of the ratios $R_{c,c'}(u, a/L, 1)$

Four sources of cutoff effects in flow quantities [A. Ramos, S. Sint '15]

- Quantum effects at $t = 0$. Very complicated dependence on g_0^2
  - Choice of action.
  - Choice of initial condition in the flow equation (i.e. $V_\mu(0,x) = U_\mu(x)$)
- ~~Integrating the flow equation~~ Zeuthen flow
- ~~Evaluating an observable~~ Classically improved discretization.

Conclusions: Still lot to understand!

- In our data:
  - Wilson Flow: Breaking of scaling at $(a/cL)^2 = 0.15$
  - Zeuthen Flow: Breaking of scaling at $(a/cL)^2 = 0.3$
  - We use $L/a = 8, c = 0.3 \implies (a/cL)^2 = 0.17$
- Zeuthen flow **not** cooked for this!
- $\mathcal{O}(a^2)$ effects still significant!
- Main suspect: The initial condition of the flow equation.

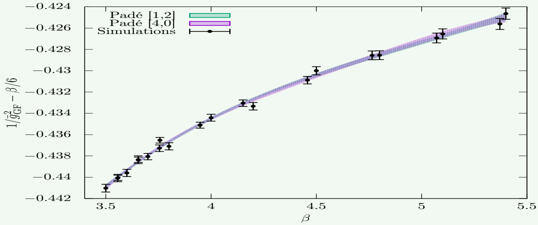
Overview

Dataset

- Tuning to critical mass vert precise $Lm_{\rm cr}(\beta, a/L) < 0.005$. Negligible effect.
- Simulate at 9 values of β and $L/a = 16 \Longrightarrow$ 9 target couplings

$$v_i = 2.1257, 2.3898, 2.7263, 3.22040, 3.8636, 4.4848, 5.3009, 5.8670, 6.5485$$

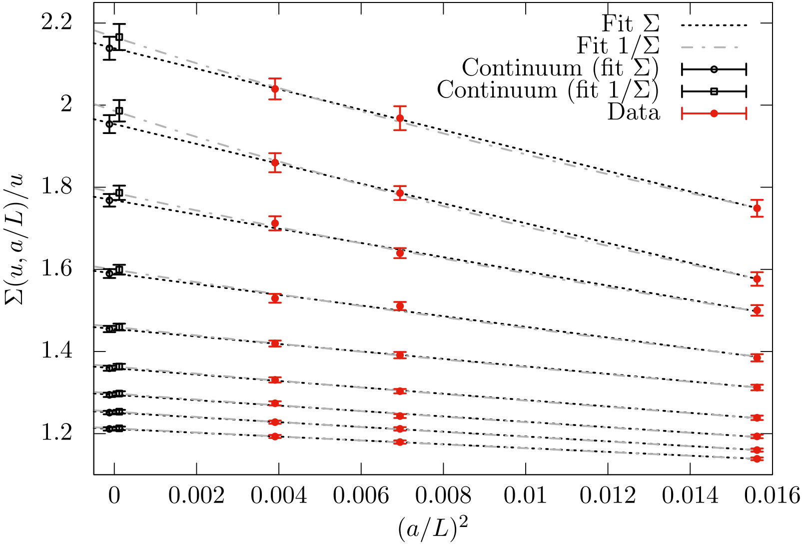
- At $L/a = 8, 12$ perform several simulation and find interpolating β.



Value for $g^2_{\rm GF}$ at some value of β taken from fit.

- Simulate at same bare parameters $2L/a = 16, 24, 32$ lattices to obtain $\Sigma(v_i, a/L)$.
- **FACT:** non perturbative data follows very closely $\frac{1}{\Sigma(v_i,a/L)} - \frac{1}{v_i} = \text{constant}$
- Remaining $\mathcal{O}(a)$ boundary effects: Propagate full 1-loop effect as error in $\Sigma(u, a/L)$. Subdominant effect. Checked ok on $L/a = 8, g^2_{\rm GF} \sim 4.5$

Step scaling function $L/a = 8, 12, 16 \to 16, 24, 32$



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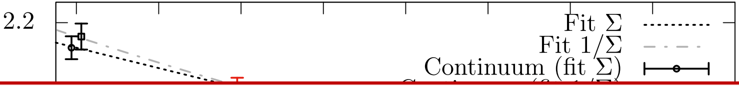

Systematic difference between ansatz for continuum extrapolation

$$\Sigma(u_i, a/L) = \sigma_i + r(a/L)^2 \qquad \text{and} \qquad \frac{1}{\Sigma(u_i, a/L)} = \frac{1}{\sigma_i} + \tilde{r}(a/L)^2$$

u_i	σ_i		$(1/\sigma_i - 1/u_i) \times 10^2$	
6.5489	14.005(175)	14.184(197)	−8.13(10)	−8.22(12)
5.8673	11.464(123)	11.654(146)	−8.32(10)	−8.46(13)
5.3013	9.371(79)	9.468(89)	−8.19(11)	−8.30(12)
4.4901	7.139(47)	7.181(51)	−8.26(11)	−8.34(12)
3.8643	5.622(28)	5.641(30)	−8.09(10)	−8.15(14)
3.2029	4.354(19)	4.367(21)	−8.25(12)	−8.32(13)
2.7359	3.541(14)	3.550(15)	−8.31(12)	−8.38(13)
2.3900	2.991(10)	2.996(10)	−8.40(12)	−8.46(13)
2.1257	2.575(9)	2.578(9)	−8.21(14)	−8.26(14)
Constant fit:			−8.233(37)	−8.316(42)

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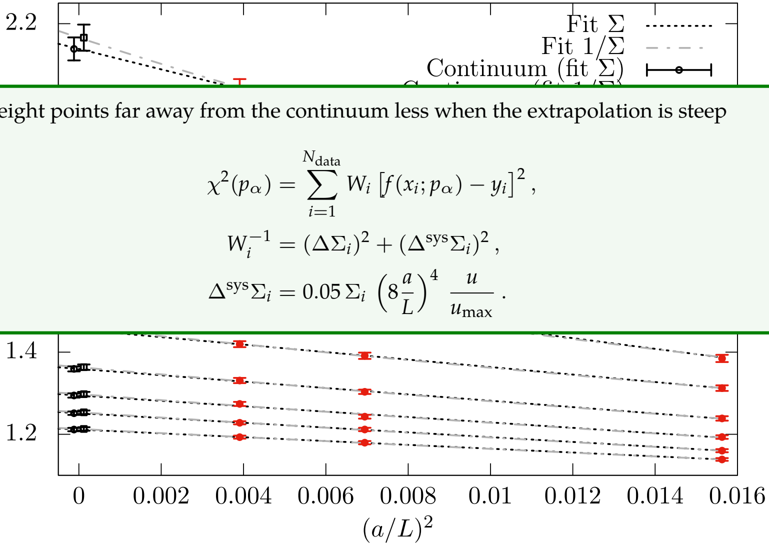


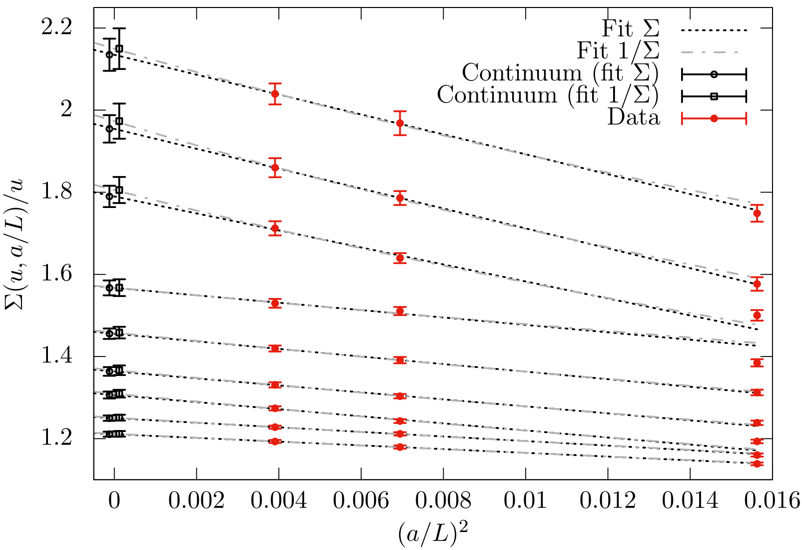
Weight points far away from the continuum less when the extrapolation is steep

$$\chi^2(p_\alpha) = \sum_{i=1}^{N_{\text{data}}} W_i \left[f(x_i; p_\alpha) - y_i \right]^2 ,$$

$$W_i^{-1} = (\Delta\Sigma_i)^2 + (\Delta^{\text{sys}}\Sigma_i)^2 ,$$

$$\Delta^{\text{sys}}\Sigma_i = 0.05\,\Sigma_i \left(8\frac{a}{L}\right)^4 \frac{u}{u_{\max}} .$$

Step scaling function $L/a = 8, 12, 16 \to 16, 24, 32$



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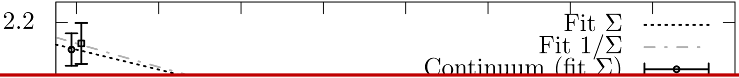

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2.1257	2.575(9)	2.578(9)	−8.21(14)	−8.26(14)
Constant fit:			−8.233(37)	−8.316(42)
Constant fit with new weights:			−8.24(5)	−8.30(6)

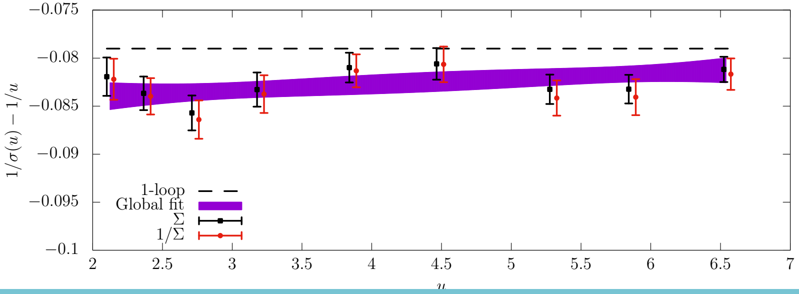
Determination of $\sigma(u)$: Combined analysis

Combine continuum extrapolation with parametrization of $\sigma(u)$

$$\frac{1}{\Sigma(u,a/L)} - \frac{1}{u} = \tilde{P}(u) + \rho(u)\left(\frac{a}{L}\right)^2$$

$$\tilde{P}(x) = \sum_{k=0}^{n_p} c_k x^k; \qquad \rho(x) = \sum_{k=0}^{n_p} r_k x^k$$

Flexible: No tuning, no shift: just simulations at L/a and $2L/a$ at matching bare parameters. We can use two additional simulations: $12, 16 \to 24, 32$ at $g^2_{\rm GF} \sim 11.31$



Determination of the β-function: "slow" running

Use the exact relation

$$\log 2 = -\int_{g(L)}^{g(2L)} \frac{dx}{\beta(x)}$$

with the ansatz

$$\beta(x;p) = -\frac{x^3}{P(x)}; \qquad P(x) = \sum p_k x^{2k}$$

Fit your data using

$$\chi^2(p) = \sum_{\text{data}} \frac{1}{\delta I^2} \left[\log 2 + \int_{\sqrt{u_i}}^{\sqrt{\sigma(u_i)}} \frac{dx}{\beta(x;p)} \right]^2$$

NOTE: Very flexible: No tunning, no fitting: just simulations at different L/a and matching g_0

$$\chi^2(p) = \sum_{\text{data}} \frac{1}{\delta I^2} \left[\log s + \int_{\sqrt{u_i}}^{\sqrt{\Sigma_s(u_i,a/L)+\rho(u,s)\left(\frac{a}{L}\right)^2}} \frac{dx}{\beta(x;p)} \right]^2$$

$$\chi^2(p) = \sum_{\text{data}} \frac{1}{\delta I^2} \left[\log s + \int_{\sqrt{u_i}}^{\sqrt{\Sigma_s(u_i,a/L)}} \frac{dx}{\beta(x;p)} + \tilde{\rho}(u,s) \left(\frac{a}{L}\right)^2 \right]^2$$

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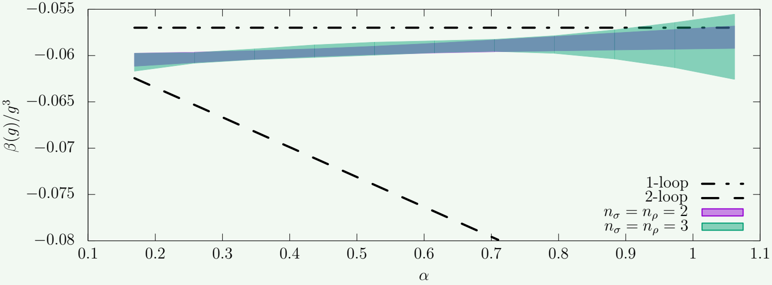


Agreement between all analysis techniques

- $u_0 = 11.31$, sequence of couplings

$$u_k = \sigma^{-1}(u_{k-1})\,, k = 1, \ldots, n_s$$

- Define $g_1^2 = 2.6723$ and $g_2^2 = 11.31$

$$s(g_1^2, g_2^2) = \exp\left\{-\int_{g_1}^{g_2} \frac{\mathrm{d}x}{\beta(x)}\right\}$$

Fit	$--$	n_ρ	W_i	u_1	u_2	u_3	u_4	$s(g_1^2, g_2^2)$
Σ,σ	3	–	$\Delta\Sigma_i^{-2}$	5.866(21)	3.955(17)	2.981(13)	2.392(11)	–
Σ, Q	3	–	$\Delta\Sigma_i^{-2}$	5.867(21)	3.956(16)	2.981(14)	2.391(12)	–
$1/\Sigma, Q$	3	–	$\Delta\Sigma_i^{-2}$	5.832(21)	3.927(17)	2.960(13)	2.374(11)	–
$1/\Sigma, P$	2	–	$\Delta\Sigma_i^{-2}$	5.832(21)	3.927(15)	2.959(13)	2.374(11)	10.82(14)
$1/\Sigma, P$	3	–	$\Delta\Sigma_i^{-2}$	5.831(21)	3.926(17)	2.959(13)	2.374(11)	10.82(15)
Σ, P	3	–	a^4	5.870(28)	3.954(22)	2.976(17)	2.385(15)	11.00(20)
$1/\Sigma, P$	1	3	a^4	5.843(20)	3.939(18)	2.971(16)	2.385(13)	10.96(18)
$1/\Sigma, P$	2	3	a^4	5.864(26)	3.944(19)	2.968(16)	2.378(14)	10.90(18)
$1/\Sigma, P$	3	3	a^4	5.864(27)	3.944(21)	2.968(17)	2.378(14)	10.90(19)
Global, P	2	2	a^4	5.872(27)	3.949(19)	2.971(16)	2.379(14)	10.93(19)
Global, P	3	3	a^4	5.874(28)	3.951(22)	2.972(17)	2.379(14)	10.93(20)

Matching with L_0 scale

L_0 defined via $g^2_{\mathrm{SF}}(L_0) = 2.012$

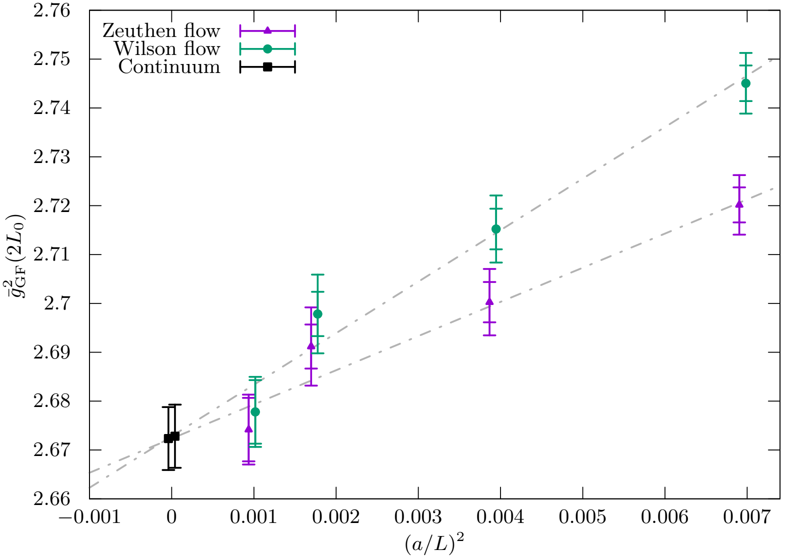
How to relate with g^2_{GF}?

- SF running done with Wilson gauge action!.
- Take a few of $\beta, L/a$ s.t. $g^2_{SF}(L) = 2.012$

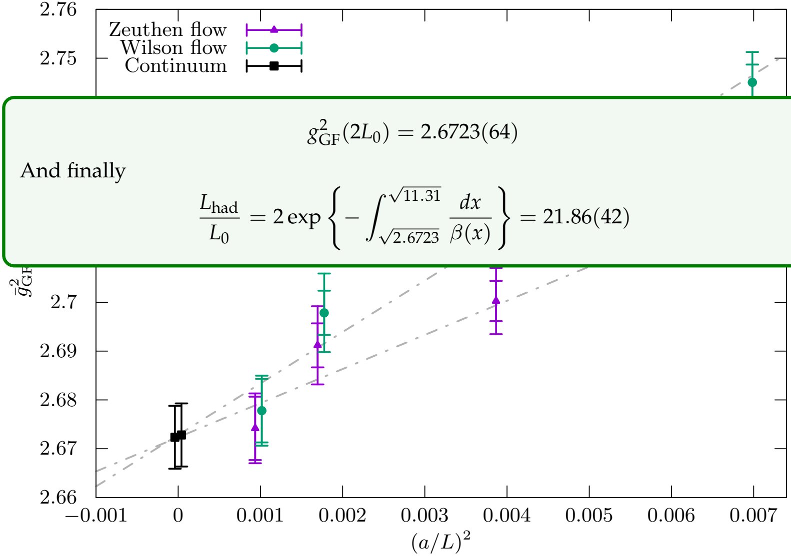
L/a	β	κ	$g^2_{\mathrm{SF}}(L)$	$g^2_{\mathrm{GF}}(2L)$	$\Phi(u, a/L)$
6	6.2735	0.1355713	2.0120(27)	2.7202(36)	2.7202(61)
8	6.4680	0.1352363	2.0120(30)	2.7003(41)	2.7003(68)
12	6.72995	0.1347582	2.0120(37)	2.6912(45)	2.6912(80)
16	6.9346	0.1344121	2.0120(17)	2.6742(65)	2.6742(72)
continuum limit					2.6723(64)

- Compute GF coupling in $\beta, 2L/a$ and obtain $g^2_{GF}(2L_0) = 2.6723(64)$
- Strategy avoids inconvenients of both schemes:
  - $g^2_{\mathrm{SF}}(L)$ noisy on large lattices.
  - $g^2_{\mathrm{GF}}(L)$ has large cutoff effects in small lattices.

Matching with L_0 scale



Matching with L_0 scale



$$g^2_{\mathrm{GF}}(2L_0) = 2.6723(64)$$

And finally

$$\frac{L_{\mathrm{had}}}{L_0} = 2\exp\left\{-\int_{\sqrt{2.6723}}^{\sqrt{11.31}} \frac{dx}{\beta(x)}\right\} = 21.86(42)$$

Overview

Conclusions

- Gradient Flow schemes ideal for matching non-perturbative regimes of strongly coupled QFT
- Main result $L_{\text{had}}/L_0 = 21.86(42)$
  - Careful continuum extrapolations: Better scaling with Zeuthen flow, still significant $\mathcal{O}(a^2)$ effects.
  - Via a deterination of β-function.
- Determination of β- function allows more flexibility than $\sigma(u)$.
- Non-perturbative β-function same functional form than 1-loop.
- Different coefficient.
- Perturbation seems broken in $\alpha \in 0.17 - 1$.
- Large 3-loop coefficient? [M. Dalla Brida Talk B2a Wed 9:40]

Conclusions