Properties of non-local wave function equivalent potentials with generalized derivative expansion

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Collaborators

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July 28, 2016, Lattice 2016
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Nuclear Force from Lattice QCD

[N. Ishii et al., PRL**99** '07]

1) Choose Interpolating Field (IF) for N

e.g.
$$p_\alpha(\boldsymbol{r}) = \epsilon_{abc}\left(u_a^T(\boldsymbol{r})C\gamma_5 d_b(\boldsymbol{r})\right)u_{c\alpha}$$
$$n_\beta(\boldsymbol{r}) = \epsilon_{abc}\left(u_a^T(\boldsymbol{r})C\gamma_5 d_b(\boldsymbol{r})\right)d_{c\beta}$$

2) Compute Nambu-Bethe-Salpeter (NBS) wave function

$$\Psi_{\alpha\beta}(\boldsymbol{r},t) = \langle 0\,|\,n_\beta(\boldsymbol{x}+\boldsymbol{r},t)\,p_\alpha(\boldsymbol{x},t)\,|\,B=2\rangle$$

faithful to phase shift

3) Solve the Schrödinger equation for a non-local potential

$$\left[\frac{1}{2\mu}\frac{d^2}{dx^2}+E\right]\Psi(x;E) = \int dx'\,\underline{V(x,x')}\Psi(x';E)$$

derivative expansion

Derivative Expansion

$$V(x, x') = \left(V_0(x) + V_1(x)\frac{d}{dx} + V_2(x)\frac{d^2}{dx^2} + \cdots \right) \delta(x - x')$$

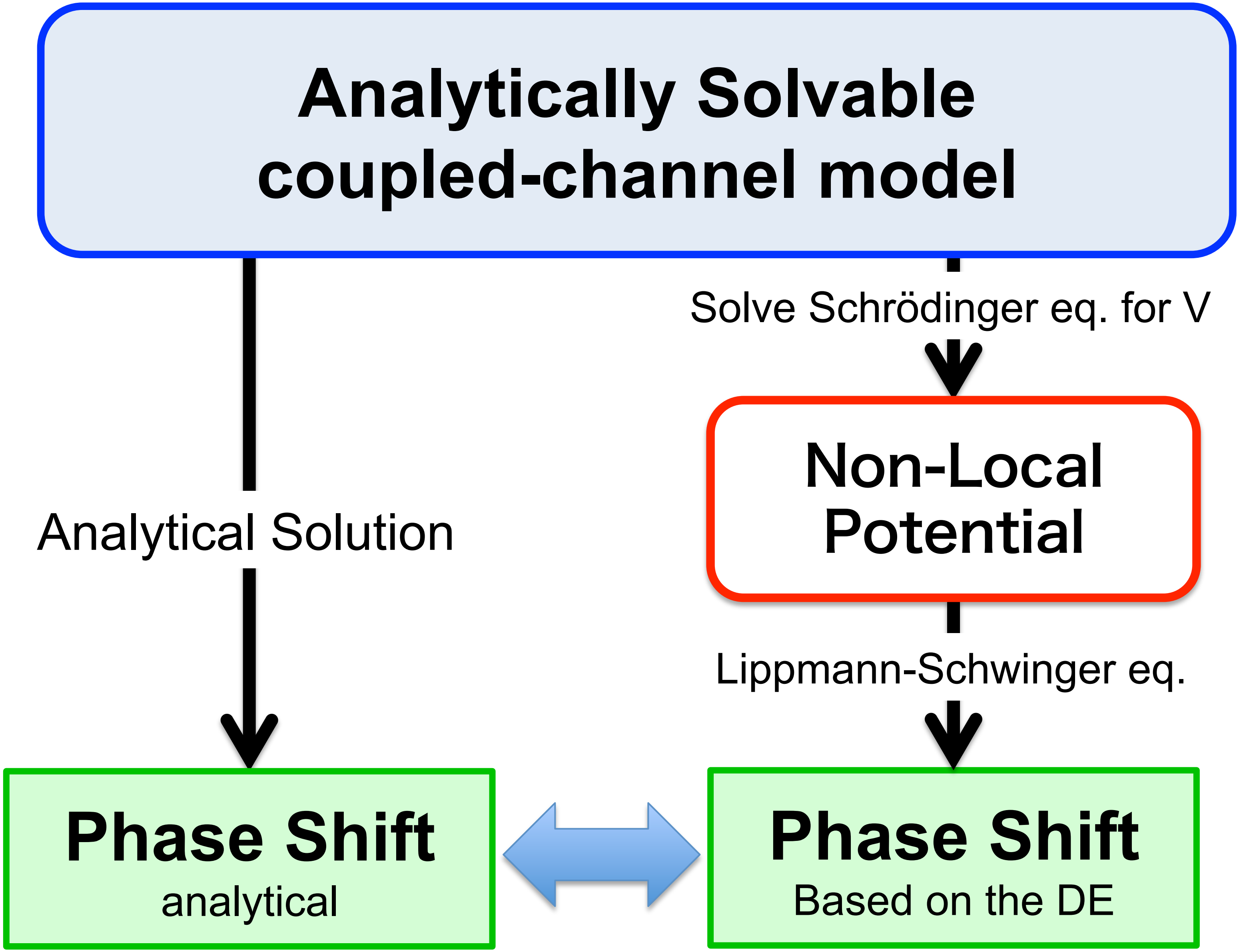
- Since the higher-order terms are of $\mathcal{O}(\hat{p}^n)$, the expansion will converge: low-energy scattering is well described by

$$V^{(N)}(x, x') = \left(V_0(x) + \cdots + V_N(x)\frac{d^N}{dx^N} \right) \delta(x - x').$$

- The validity of local approx. (N=0) has been confirmed in an indirect manner. [K. Murano et al., PTP**125** '11 1225-]

- Our purpose is to demonstrate the convergence directly when the higher-order terms are explicitly considered.

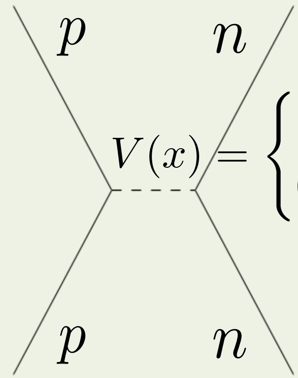
Outline



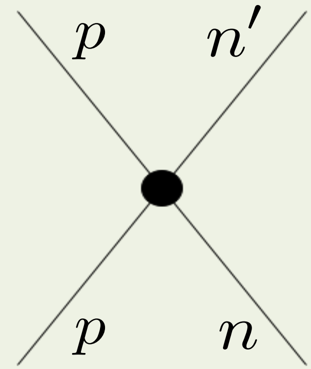
Analytically Solvable coupled-channel model
Solve Schrödinger eq. for V
Non-Local Potential
Analytical Solution
Lippmann-Schwinger eq.
Phase Shift
analytical
Phase Shift
Based on the DE

The Birse Model (1/2)

$$\hat{H} = \int dx\, p^\dagger(x) \left(-\frac{1}{2M} \nabla^2 \right) p(x)$$
$$+ \int dx\, n^\dagger(x) \left(-\frac{1}{2M} \nabla^2 \right) n(x)$$
$$+ \int dx\, n'^\dagger(x) \left(\Delta - \frac{1}{2M} \nabla^2 \right) n'(x)$$
$$+ \iint dx dy\, n^\dagger(x) p^\dagger(y)\, V(x-y)\, p(y) n(x)$$
$$+2g \int dx\, n^\dagger(x) p^\dagger(x) p(x) n'(x) + h.c.$$



$$V(x) = \begin{cases} -V_0 & \text{for } |x| < R \\ 0 & \text{for } |x| > R \end{cases}$$



NBS wave functions

$$\psi_0(x-y) \equiv \langle 0 \,|\, p(x) n(y) \,|\, \Psi(E) \rangle\,,$$
$$\psi_1(x-y) \equiv \langle 0 \,|\, p(x) n'(y) \,|\, \Psi(E) \rangle$$

2-particle eigenstate

$$\hat{H}\,|\Psi(E)\rangle = E\,|\Psi(E)\rangle$$

The Birse Model (2/2)

- $\psi_0(x)$ and $\psi_1(x)$ satisfy the coupled-channel eqs.

$$\left(\frac{1}{M}\frac{d^2}{dx^2}+E\right)\begin{bmatrix}\psi_0(x;E)\\ \psi_1(x;E)\end{bmatrix}=\begin{pmatrix}V(x) & 2g\delta(x)\\ 2g\delta(x) & \Delta\end{pmatrix}\begin{bmatrix}\psi_0(x;E)\\ \psi_1(x;E)\end{bmatrix}$$

[M.Birse, 1208.4807]

Parameters: $MV_0 = 1/R^2,\ M\Delta = 6/R^2,\ Mg = 6/R$ and $R = 1$

Twisted Boundary Condition: $\psi_i(x+2L) = i\psi_i(x)$, $\psi_i^*(x) = \psi_i(-x)$ with $L = 10$

- General "neutron" interpolating field $N_q(x)$ and the pN_q NBS wave function $\Psi_q(x;E)$ read

$$N_q(x) = n(x) + q n'(x)$$

$$\Psi_q(x) = \psi_0(x) + q\,\psi_1(x)$$

Generalized Derivative Expansion

$$\left[\frac{1}{M}\frac{d^2}{dx^2} + E_i\right]\Psi_q(x;E_i) = \int dx'\, V(x,x')\Psi_q(x';E_i)$$

- Naïve derivative expansion

$$V(x,x') = \sum_{n=0}^{N} w_n(x)\left(\frac{d}{dx}\right)^n \delta(x-x')$$

- *Generalized derivative expansion*

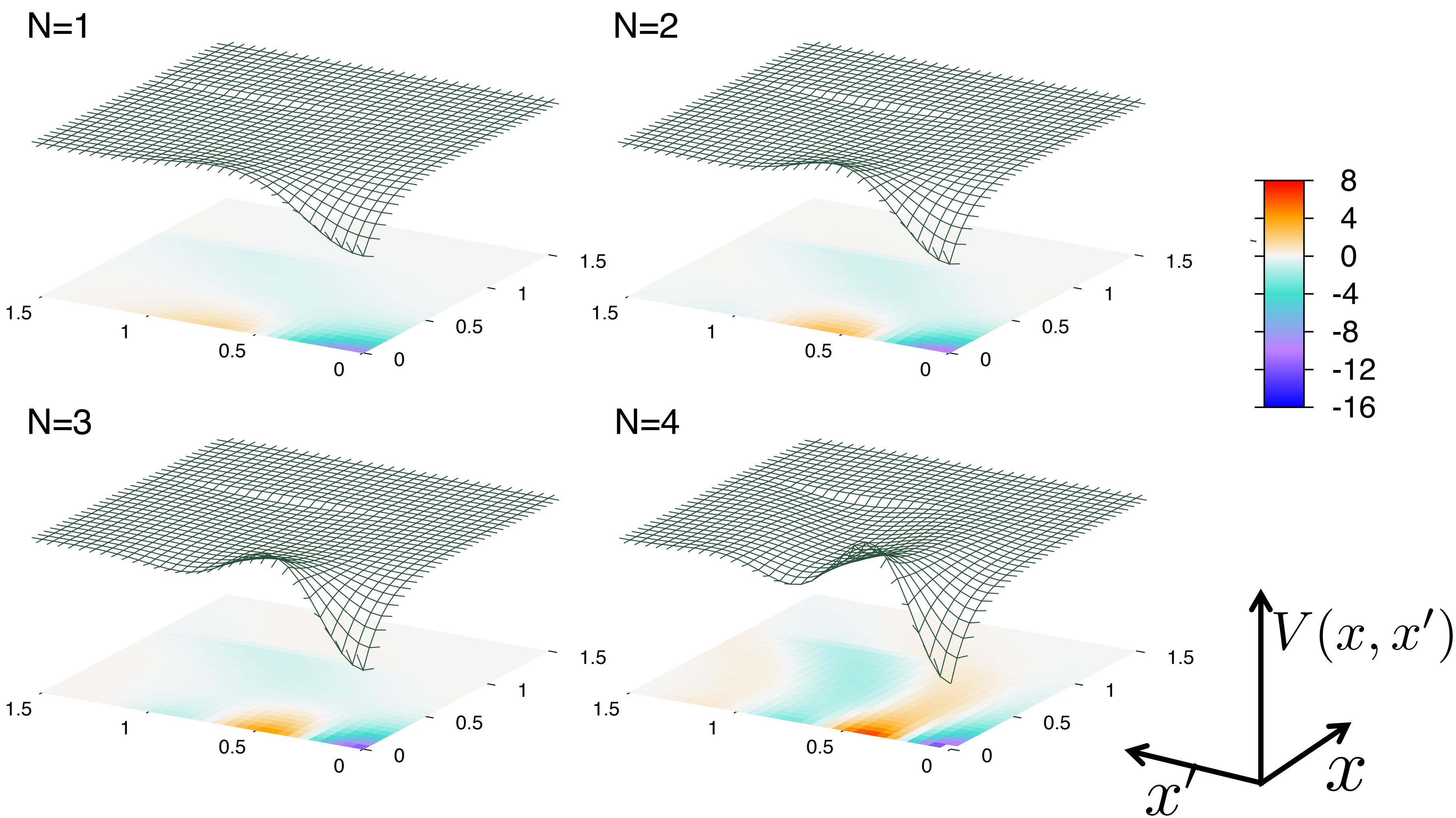
$$V^{(\rho)}(x,x') = \sum_{n=0}^{N} v_n^{(\rho)}(x)\left(\frac{d}{dx}\right)^n \frac{\exp\left\{-(x-x')^2/\rho^2\right\}}{\sqrt{\pi}\rho}$$

Explicit non-locality

Non-local Potentials

$$V^{(\rho)}(x,x') = \sum_{n=0}^{N} v_n^{(\rho)}(x) \left(\frac{d}{dx}\right)^n \frac{\exp\{-(x-x')^2/\rho^2\}}{\sqrt{\pi}\rho}$$

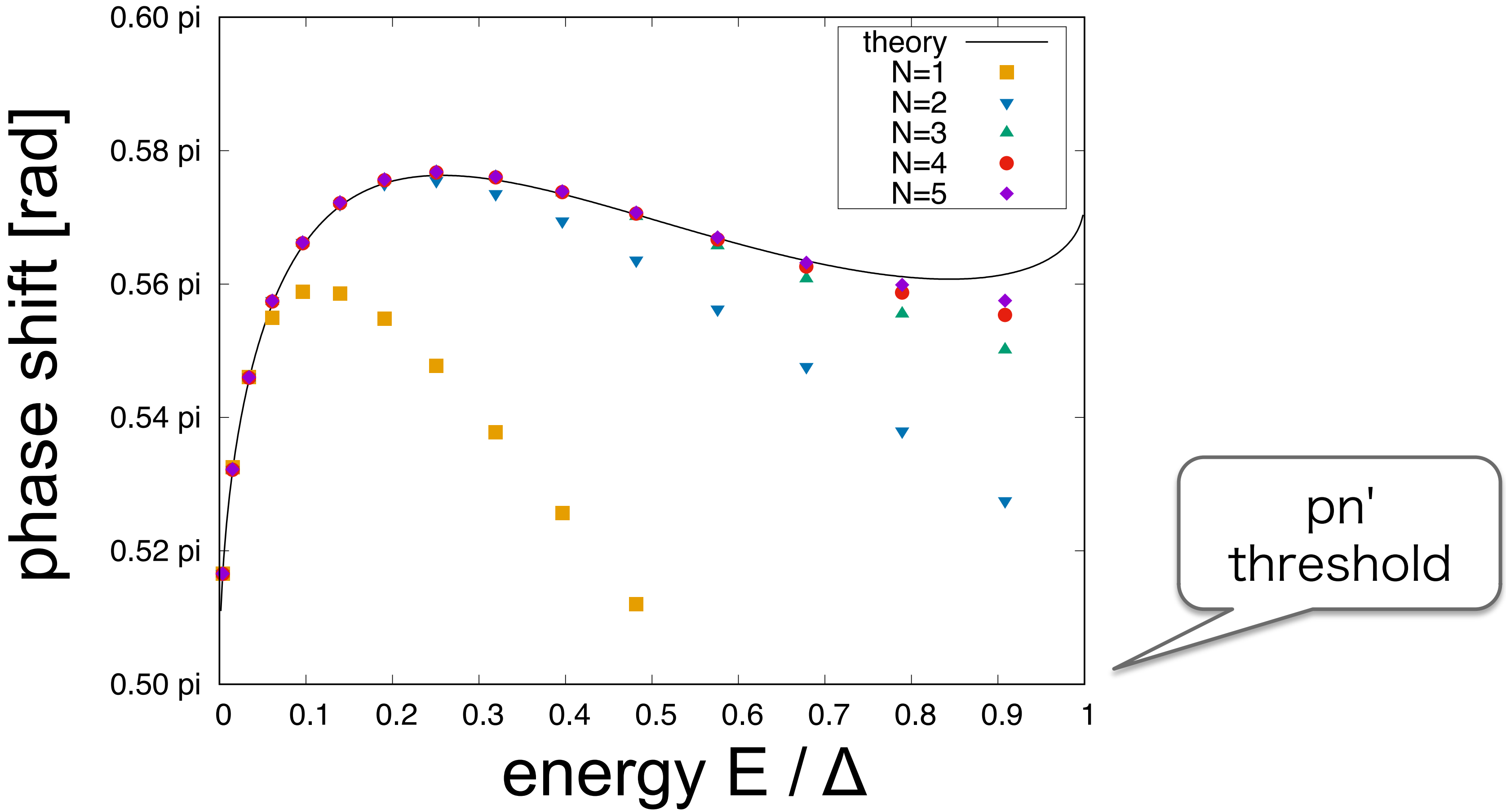
ρ=0.5, q=0.2



Phase Shift and Convergence

$$\psi(x) = \frac{1}{\sqrt{2\pi}} e^{i\delta} \cos(kx + \delta)$$

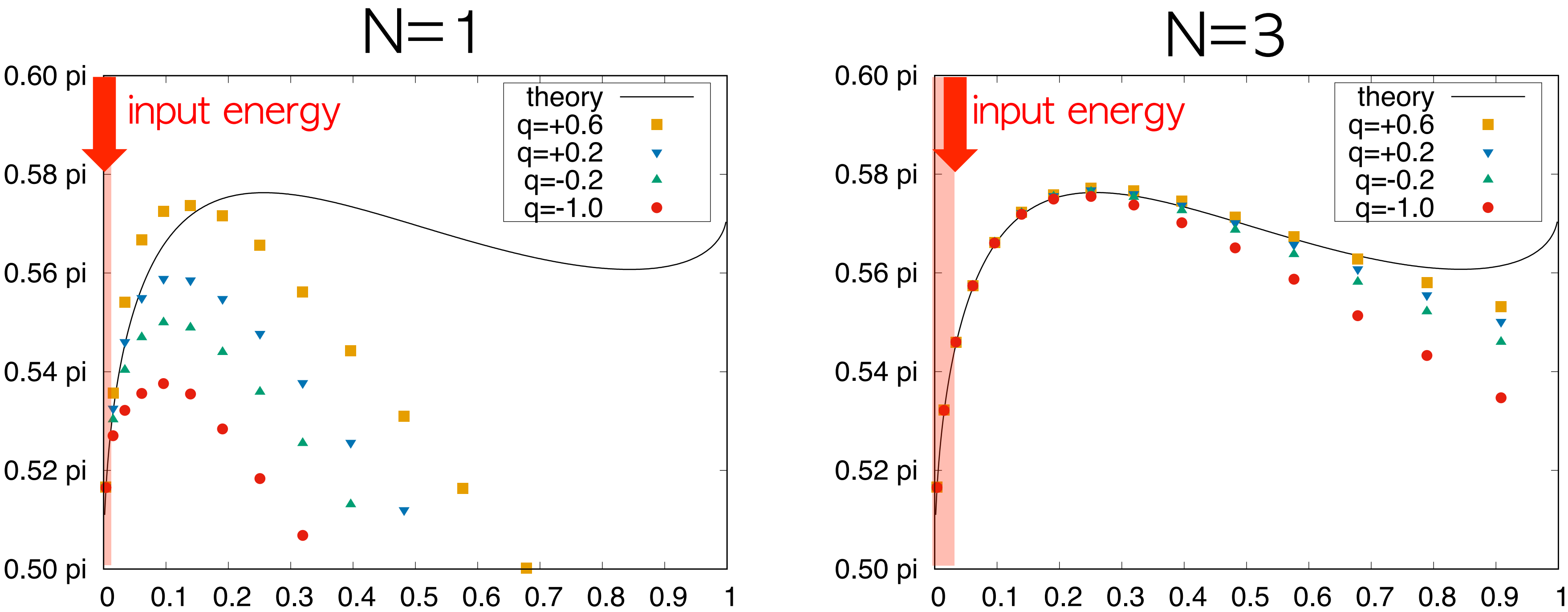
<u>*ρ=0.5, q=0.2*</u>



<u>Larger N is better</u> ➡ Convergence is confirmed

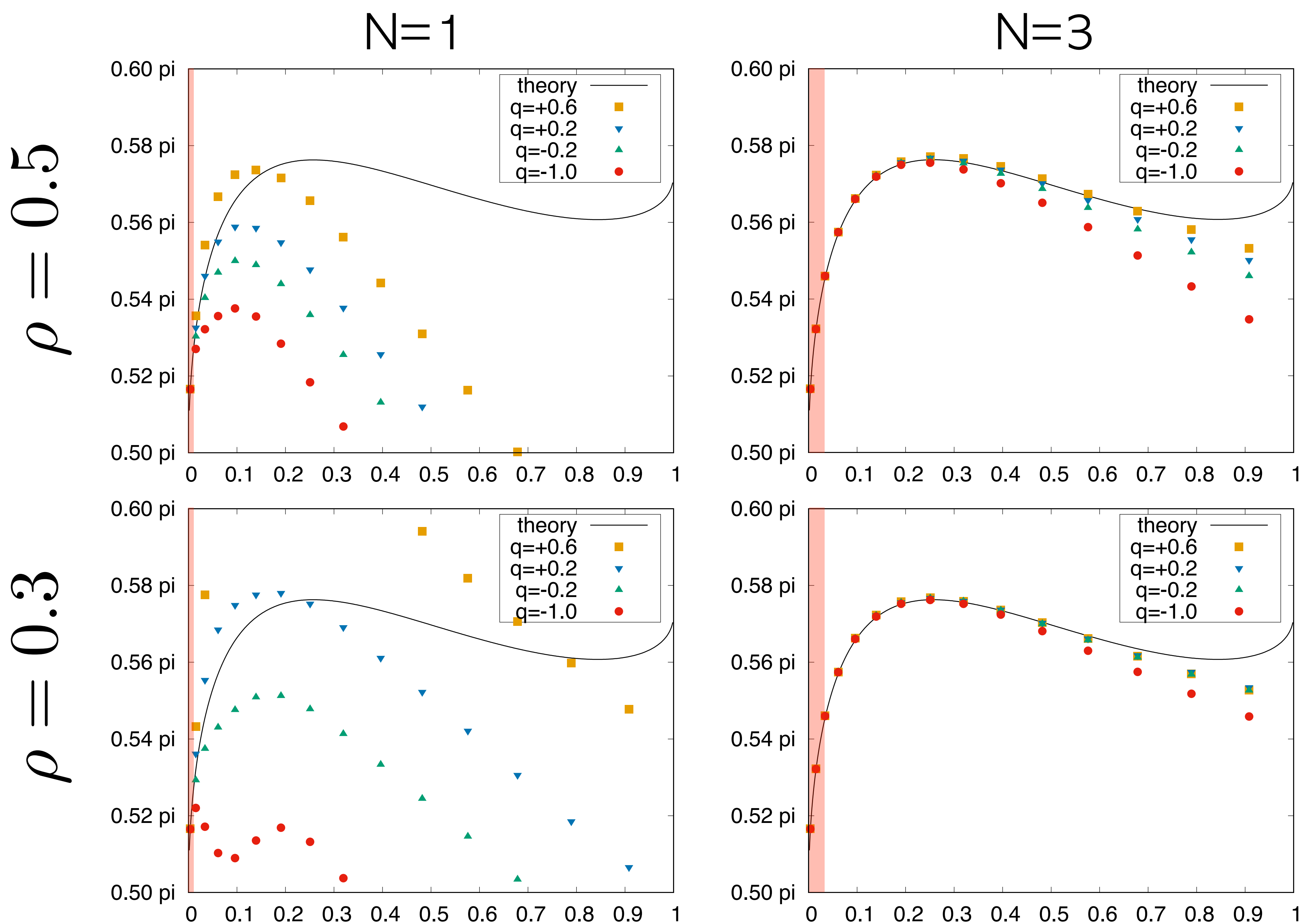
Interpolating Field Dependence

pn′ mixing parameter q is varied while ρ is fixed to 0.5



- Interpolating field dependence can be used to improve the convergence.
- Extrapolation to higher energy is possible.

ρ Dependence



Summary

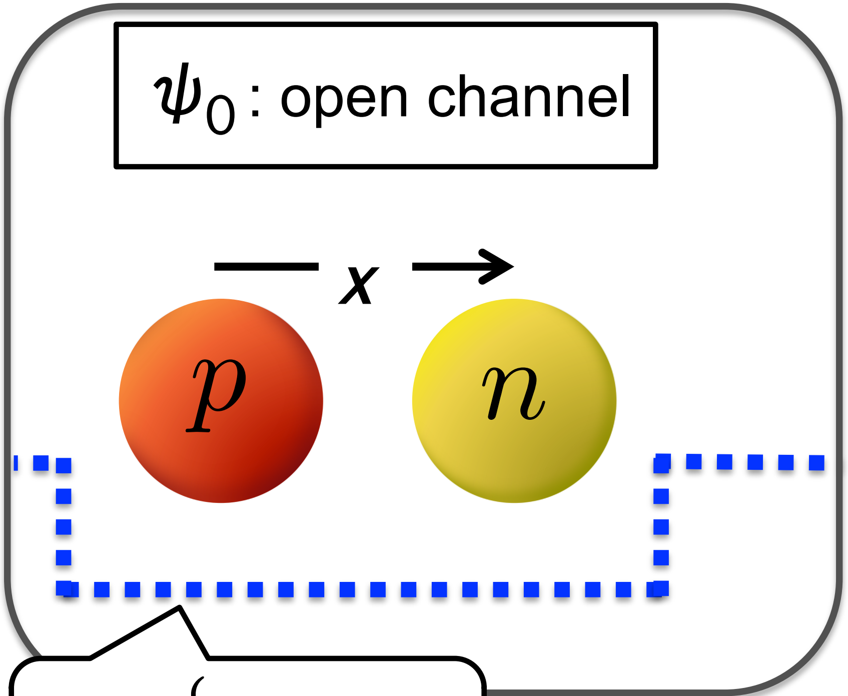
- We have carried out the generalized derivative expansion in an explicit manner using a 1+1 dimensional toy model.
- The convergence of the expansion has been confirmed as the phase shift improves order by order.
- We can improve the convergence by changing
  1. the choice of interpolating fields
  2. the Gaussian width in the generalized expansion
- The findings will be applied to a coupled-channel system, where a small-momentum assumption is not promising.

Backup Slides

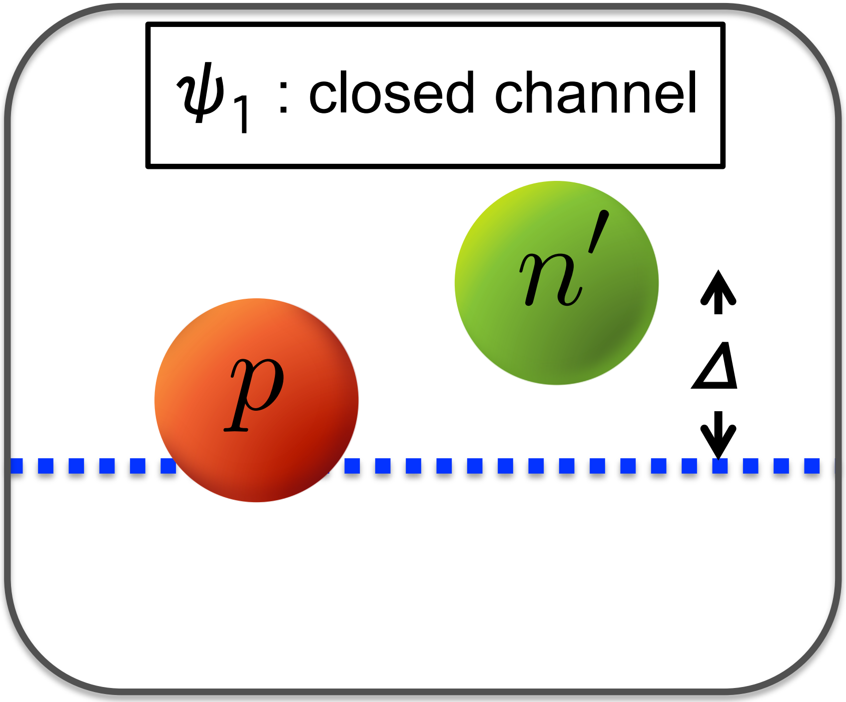
Backup: Schematic Picture of the Birse Model

$$\left(\frac{1}{M}\frac{d^2}{dx^2}+E\right)\begin{bmatrix}\psi_0(x)\\ \psi_1(x)\end{bmatrix}=\begin{pmatrix}V(x) & 2g\delta(x)\\ 2g\delta(x) & \Delta\end{pmatrix}\begin{bmatrix}\psi_0(x)\\ \psi_1(x)\end{bmatrix}$$

[M.Birse, arXiv:1208.4807]



$$V(x)=\begin{cases}-V_0 & |x|<R\\ 0 & |x|>R\end{cases}$$



1 bound state at E = - 33.7

(M=1, Δ=6, g=6, R=1)

Backup: Energy Spectrum

Parameter set

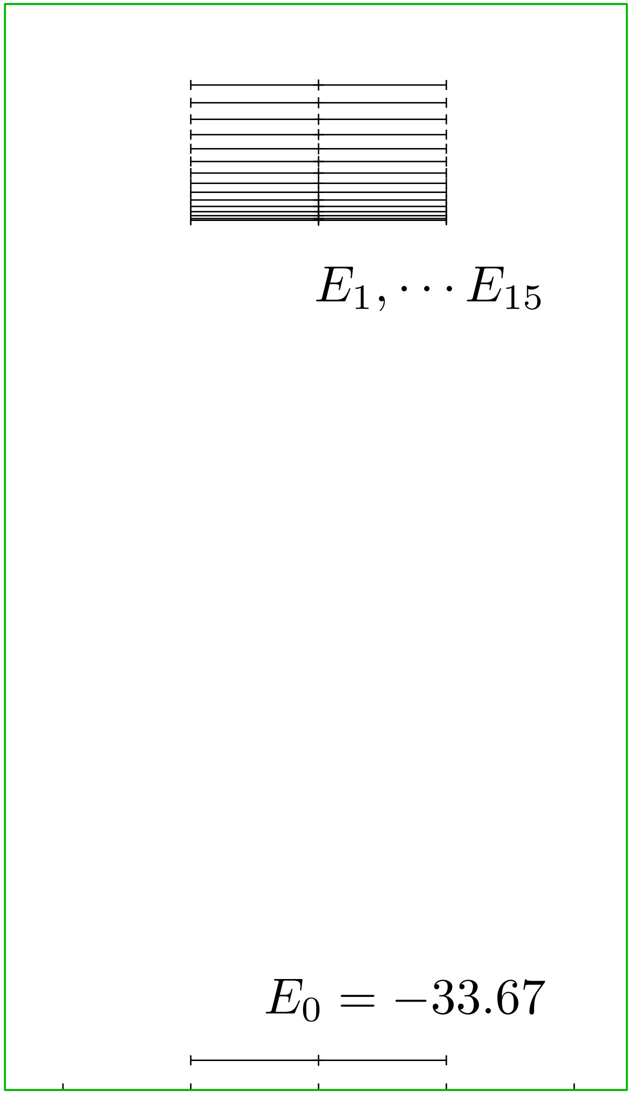
$$MV_0 = 1/R^2,\ M\Delta = 6/R^2,\ Mg = 6/R$$
$$R = 1$$

Twisted Boundary Condition

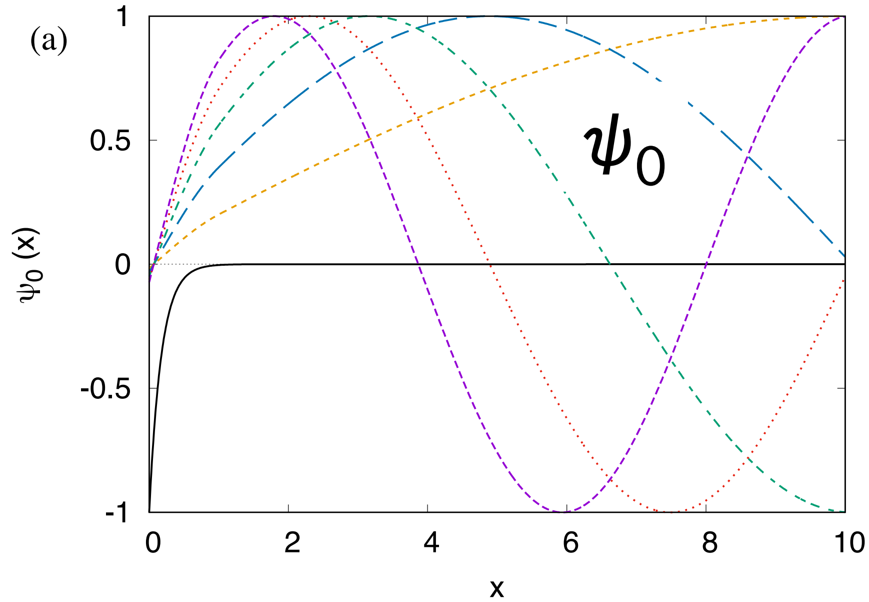
$$\psi_i(x+2L) = e^{i\theta}\psi_i(x)$$
$$\psi_i^*(x) = \psi_i(-x) \qquad i = 0, 1$$

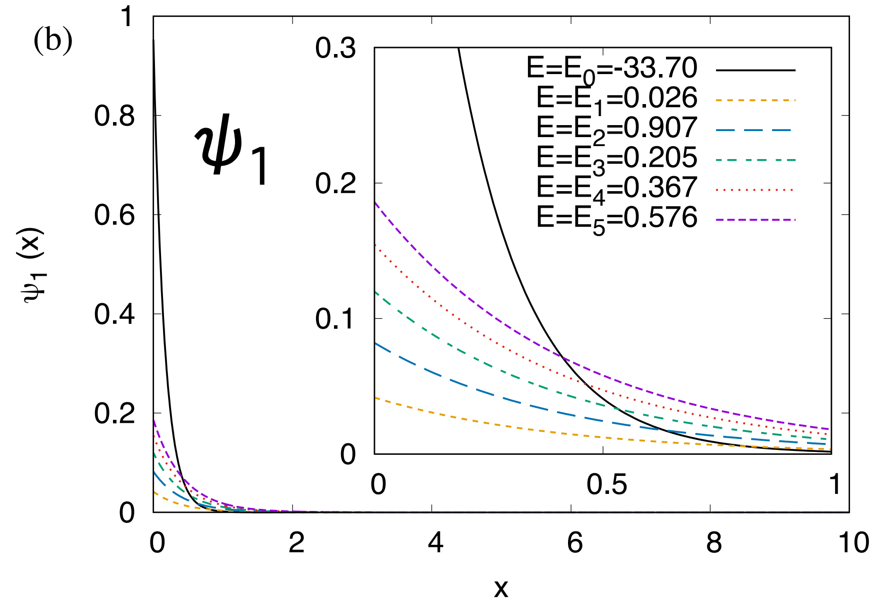
with $L = 10,\ \theta = \pi/2$

A deeply bound state exists

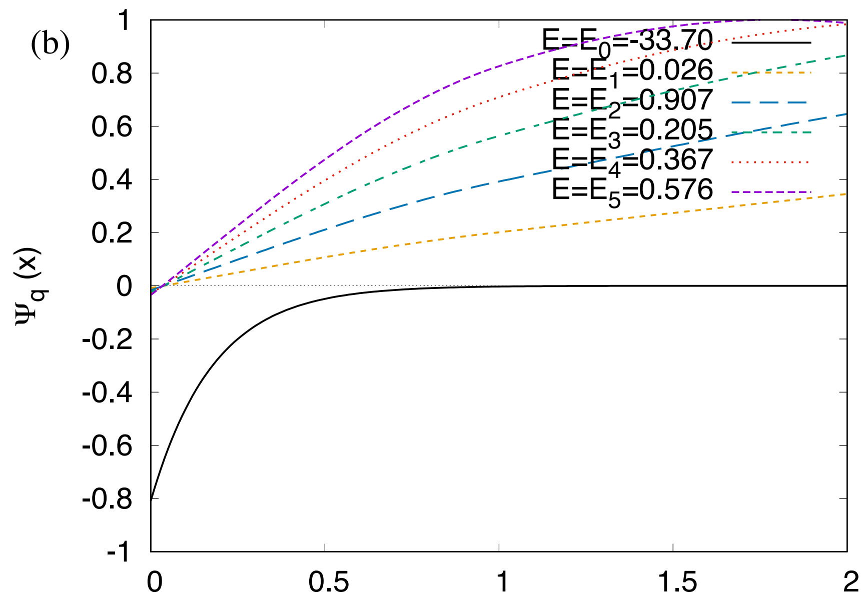


Backup : Birse Model Wave Functions





$$\Psi_{q=0.2}(x)$$
$$= \psi_0(x) + 0.2\,\psi_1(x)$$



Solutions with PBC at L=±10

Backup: Potential Determination (regular part)

$$V^{(\rho)}(x,x') = \sum_{n=0}^{N} v_n(x) \left(\frac{d}{dx^2}\right)^n \frac{\exp\left\{-(x-x')^2/\rho^2\right\}}{\sqrt{\pi}\rho}$$

determination

$$(-H_0 + E_i)\Psi_q(x;E_i) = \int dx' V(x,x')\Psi_q(x;E_i)$$

$$\begin{pmatrix} (-H_0+E_0)\Psi_q(x;E_0) \\ (-H_0+E_1)\Psi_q(x;E_1) \\ \vdots \\ (-H_0+E_N)\Psi_q(x;E_N) \end{pmatrix} = \begin{pmatrix} \varphi(x;E_0) & D\varphi(x;E_0) & \cdots & D^N\varphi(x;E_0) \\ \varphi(x;E_1) & D\varphi(x;E_1) & \cdots & D^N\varphi(x;E_1) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(x;E_N) & D\varphi(x;E_N) & \cdots & D^N\varphi(x;E_N) \end{pmatrix} \begin{pmatrix} v_0(x) \\ v_1(x) \\ \vdots \\ v_N(x) \end{pmatrix}$$

$$\varphi^{(\rho,\,q)}(x) \equiv \int dx' \frac{\exp\left\{-(x-x')^2/\rho^2\right\}}{\sqrt{\pi}\rho}\Psi_q(x')$$

Backup: Potential Determination (delta part)

potential / δ function

- delta function potential at the origin

$$v_n(x) = \widetilde{v_n}(x) + g_n \delta(x) \qquad \widetilde{v_n}(x) \text{ : regular part}$$

determine gn to make it energy-independent

$$(E - H_0)\Psi(x) = \sum_{n=0}^{N} \left(\widetilde{v_n}(x) + g_n \delta(x)\right) D^n \varphi(x)$$ Schrodinger eq.

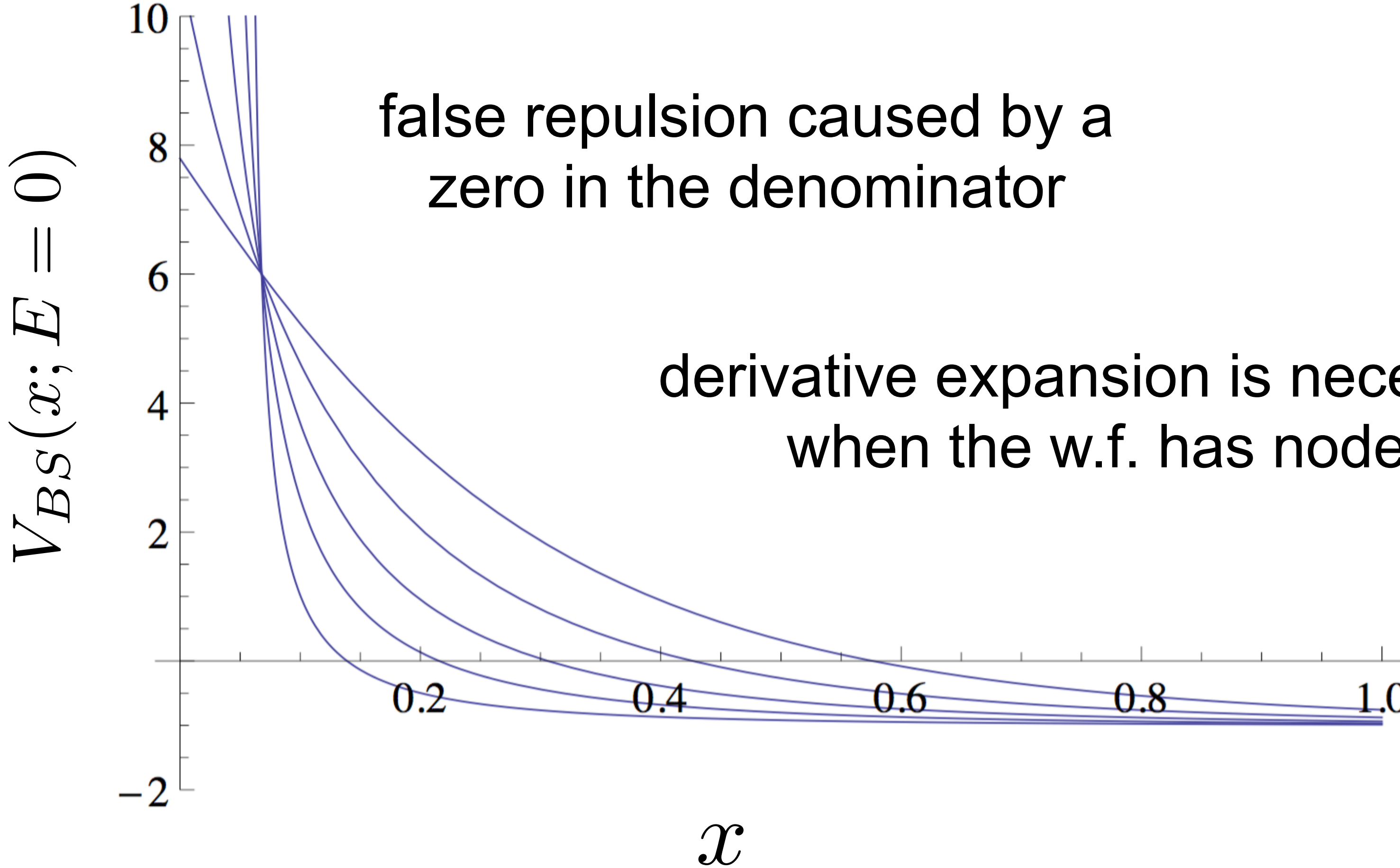
$$\frac{2}{M}\Psi'(+0) = \sum_n g_n \left[D^n \varphi(x)\right]_{x=0}$$ integrate for [–ε,ε]

$$\begin{pmatrix} \frac{2}{M}\Psi'(x;E_0) \\ \frac{2}{M}\Psi'(x;E_1) \\ \frac{2}{M}\Psi'(x;E_2) \end{pmatrix} = \begin{pmatrix} [\varphi(x;E_0)]_{x=0} & [D\varphi(x;E_0)]_{x=0} & \left[D^2\varphi(x;E_0)\right]_{x=0} \\ [\varphi(x;E_1)]_{x=0} & [D\varphi(x;E_1)]_{x=0} & \left[D^2\varphi(x;E_0)\right]_{x=1} \\ [\varphi(x;E_2)]_{x=0} & [D\varphi(x;E_2)]_{x=0} & \left[D^2\varphi(x;E_2)\right]_{x=0} \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}$$

Backup: False Repulsion of Birse

$$V_{BS}(x;E) = \frac{1}{M}\frac{\Psi_q''(x;E)}{\Psi_q(x;E)} + E$$

[M.Birse, 1208.4807]



false repulsion caused by a zero in the denominator

derivative expansion is necessary when the w.f. has nodes

Backup: Derivation of GDE

We consider the potential in the momentum space and in the coordinate space as

$$\begin{aligned} \langle p_1 \,|\, V \,|\, p_2 \rangle &\equiv \tilde{V}(k, P), \\ \langle x_1 \,|\, V \,|\, x_2 \rangle &\equiv V(R, r), \end{aligned} \tag{A1}$$

respectively, where k, P, r, and R are defined by

$$\begin{aligned} k &= p_1 - p_2, \quad P = (p_1 + p_2)/2, \\ r &= x_1 - x_2, \quad R = (x_1 + x_2)/2. \end{aligned} \tag{A2}$$

They are related to each other by the Fourier transformation

$$V(R, r) = \iint \frac{dk dP}{2\pi} e^{ikR} \, \tilde{V}(k, P) \, e^{iPr}. \tag{A3}$$

Backup: Derivation of GDE

Now, we attempt to keep a part of the P dependence explicitly: we divide $\tilde{V}(k,P)$ into two parts as

$$\tilde{V}(k,P) \equiv \tilde{U}(k,P)\exp\left\{-\frac{1}{4}\rho^2 P^2\right\}, \qquad \text{(A6)}$$

using an arbitrary parameter ρ. This time, P in $\tilde{U}(k,P)$ is replaced by the derivative, while that in the Gaussian is holded. The result is given as

$$V(R,r) = \int dk\, e^{ikR}\, \tilde{U}(k, -i\partial/\partial r)\frac{\exp\{-r^2/\rho^2\}}{\sqrt{\pi}\rho}, \qquad \text{(A7)}$$

$$\equiv U_l(R, -i\partial/\partial r)\frac{\exp\{-r^2/\rho^2\}}{\sqrt{\pi}\rho}, \qquad \text{(A8)}$$