Determination of latent heat at the finite temperature phase transition of SU(3) gauge theory

Shinji Ejiri (Niigata Univ.)

WHOT-QCD Collaboration
Ryo Iwami (Niigata), Kazuyuki Kanaya (Tsukuba), Masakiyo Kitazawa (Osaka), Yusuke Taniguchi (Tsukuba), Hiroshi Suzuki (Kyushu), Mizuki Shirogane (Niigata), Takashi Umeda (Hiroshima), Naoki Wakabayashi (Niigata)


Introduction

• First order phase transitions are expected in many interesting systems of lattice field theories. e.g. high density QCD, many flavor etc.. important to study first order phase transitions.

• The latent heat (energy gap) the most basic quantity.

• The gap of pressure must be vanish.

Reliability of the calculation can be confirmed.

In this talk,

• We study the equation of state at the first order phase transition of SU(3) gauge theory.

• Gaps of energy density and pressure are measured using the derivative method.
  • Volume dependence is investigated
  • Continuum extrapolation is performed. [H. Suzuki, 2013]

• We tested the gradient flow method for the calculation of EoS.
Thermodynamic quantities by the derivative method

Energy density

\[ \epsilon = -\frac{1}{V} \frac{\partial \ln Z}{\partial T^{-1}} \mid_V \]

Pressure

\[ p = T \frac{\partial \ln Z}{\partial V} \mid_T \]

temperature

\[ \frac{1}{T} = N_t a_t \]

Volume

\[ V = (N_s a_s)^3 \]

\[ Z = \int DU \ e^{-S} \]

For the SU(3) gauge theory,

\[ S = -3 N_{\text{site}} \left( \beta_s P_s + \beta_t P_t \right) \]

\( (P_{s(t)} \) space-like (time-like) plaquette)

\[ \epsilon = -\frac{3 N_t^4 T^4}{\xi^3} \left\{ \left( a_t \frac{\partial \beta_s}{\partial a_t} - \xi \frac{\partial \beta_s}{\partial \xi} \right) (\langle P_s \rangle - \langle P \rangle_0) - \left( a_t \frac{\partial \beta_t}{\partial a_t} - \xi \frac{\partial \beta_t}{\partial \xi} \right) (\langle P_t \rangle - \langle P \rangle_0) \right\} \]

\[ p = \frac{N_t^4 T^4}{\xi^3} \left\{ \frac{\partial \beta_s}{\partial \xi} (\langle P_s \rangle - \langle P \rangle_0) + \frac{\partial \beta_t}{\partial \xi} (\langle P_t \rangle - \langle P \rangle_0) \right\} \]

Independent variables: \( a_t, \ \xi = \frac{a_s}{a_t} \)

For \( \xi = 1 \), the gap of the energy density

\[ \frac{\Delta \epsilon}{T^4} = -3 N_t^4 \left\{ \left( a_t \frac{\partial \beta_s}{\partial a_t} - \frac{\partial \beta_s}{\partial \xi} \right) (\langle P_s \rangle_{\text{hot}} - \langle P_s \rangle_{\text{cold}}) + \left( a_t \frac{\partial \beta_t}{\partial a_t} - \frac{\partial \beta_t}{\partial \xi} \right) (\langle P_t \rangle_{\text{hot}} - \langle P_t \rangle_{\text{cold}}) \right\} \]

These 4 coefficients must be determined.
Determination of the anisotropy coefficients at $\xi = a_s/a_t = 1$

Isotropic lattice ($\beta = \beta_s = \beta_t$): 

$$
\left( a_t \frac{\partial \beta_s}{\partial a_t} \right)_{\xi=1} = \left( a_t \frac{\partial \beta_t}{\partial a_t} \right)_{\xi=1} = a \frac{d\beta}{da}
$$

$a \frac{d\beta}{da}$ is determined by the data of the critical $\beta$ ($\beta_c(N_t)$)

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Data: Francis, kaczmarek, Laine, Neuhaus, Ohno, Phys. Rev. D 91, 096002 (2015) and our data for $N_t = 4 \sim 22$

String tension is independent of $\xi = \frac{a_s}{a_t}$

$$
\left( \frac{\partial \beta_s}{\partial \xi} + \frac{\partial \beta_t}{\partial \xi} \right)_{a_t: \text{fixed, } \xi = 1} = \frac{3}{2} a \frac{d\beta}{da}
$$

Along the phase transition line, $a_t$ is constant because $\frac{1}{T_c} = N_t a_t$.

When one changes $(\beta_s, \beta_t) \rightarrow (\beta_s + d\beta_s, \beta_t + d\beta_t)$,

$$d a_t = \frac{\partial a_t}{\partial \beta_s} d\beta_s + \frac{\partial a_t}{\partial \beta_t} d\beta_t = 0$$

The slope of the phase transition line in the $(\beta_s, \beta_t)$ plane: $r_t$


Using the reweighting method, $(\beta_s, \beta_t)$-dependence of the Polyakov loop susceptibility is measured.
Anisotropy coefficients

From

\[
\left( \frac{\partial \beta_s}{\partial \xi} + \frac{\partial \beta_t}{\partial \xi} \right)_{a_t: \text{ fixed}, \xi=1} = \frac{3}{2} a \frac{d\beta}{da} \quad \text{ and } r_t = \left( \frac{\partial \beta_s}{\partial \xi} \right)_{\xi=1} \left( \frac{\partial \beta_t}{\partial \xi} \right)_{\xi=1}
\]

\[\rightarrow \quad \left( \frac{\partial \beta_s}{\partial \xi} \right)_{\xi=1} = \frac{3r_t}{2(1+r_t)} a \frac{d\beta}{da} \quad \text{ and } \quad \left( \frac{\partial \beta_t}{\partial \xi} \right)_{\xi=1} = \frac{3}{2(1+r_t)} a \frac{d\beta}{da}
\]

Conventional combinations of the energy density and pressure

\[
\frac{\Delta(\epsilon + p)}{T^4} = 3N_t^4 a \frac{d\beta}{da} \frac{r_t - 1}{r_t + 1} \left\{ (\langle P_s \rangle_{\text{hot}} - \langle P_s \rangle_{\text{cold}}) - (\langle P_t \rangle_{\text{hot}} - \langle P_t \rangle_{\text{cold}}) \right\}
\]

\[
\frac{\Delta(\epsilon - 3p)}{T^4} = 3N_t^4 a \frac{d\beta}{da} \left\{ (\langle P_s \rangle_{\text{hot}} - \langle P_s \rangle_{\text{cold}}) - (\langle P_t \rangle_{\text{hot}} - \langle P_t \rangle_{\text{cold}}) \right\}
\]
Simulation details

Pure SU(3) gauge theory

Standard plaquette action is used.

Pseudo-heat bath algorithm + over-relaxation.

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<th>No. of Conf.</th>
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Parts of the results in Ejiri, Iwasaki, Kanaya, Phys.Rev.D 58,094505 (1998) are used in this analysis.

Simulations are performed at 1–6 $\beta$ points near transition point.

High statistics data: $\sim O(10^6)$

The multi-point reweighting method is used for the measurements.

<table>
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Measurement of the slope of the transition line $r_t$

$$r_t = \frac{d\beta_s}{d\beta_t} = \left( \frac{\partial \beta_s}{\partial \xi} \right)_{\xi=1} \left( \frac{\partial \beta_t}{\partial \xi} \right)_{\xi=1}$$

We used the reweighting method. The slope $r_t$ can be determined with sufficient accuracy.

Order parameter: Polyakov loop $\Omega(x, t)$

Transition point:
Peak position of Polyakov loop susceptibility

$$\chi_\Omega(\beta_s, \beta_t) = N_s^3 (\langle \Omega^2 \rangle_{(\beta_s, \beta_t)} - \langle \Omega \rangle_{(\beta_s, \beta_t)}^2)$$

<table>
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<th>$r_t$</th>
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<tr>
<td>$96^3 \times 12$</td>
<td>-1.204(53)</td>
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</table>
Separation of the hot and cold phases

- We identify the phase by the Polyakov loop.
- Two peaks in the histogram.
- Flip-flops between two phases.
- Mixed configurations are rare. (We omit mixed configurations.)
Vanishing pressure gap \( \Delta p = 0 \)

\[
\frac{\Delta p}{T^4} = N_t^4 \left\{ \frac{\partial \beta_s}{\partial \xi} (\langle P_s \rangle_{\text{hot}} - \langle P_s \rangle_{\text{cold}}) + \frac{\partial \beta_t}{\partial \xi} (\langle P_t \rangle_{\text{hot}} - \langle P_t \rangle_{\text{cold}}) \right\} = 0
\]

Condition for \( \Delta p = 0 \)

\[
\frac{\partial \beta_s}{\partial \xi} = r_t = -\frac{\langle P_t \rangle_{\text{hot}} - \langle P_t \rangle_{\text{cold}}}{\langle P_s \rangle_{\text{hot}} - \langle P_s \rangle_{\text{cold}}}
\]

<table>
<thead>
<tr>
<th>lattice</th>
<th>( r_t )</th>
<th>( \frac{\langle P_t \rangle_{\text{hot}} - \langle P_t \rangle_{\text{cold}}}{\langle P_s \rangle_{\text{hot}} - \langle P_s \rangle_{\text{cold}}} )</th>
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<td>-1.204(53)</td>
<td>1.283(53)</td>
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</table>

The pressure gap is zero on each finite lattice.
Because the correlation length is finite at the first order transition, $\Delta\epsilon$ should be constant for large volume ($> \text{correlation length}$). We did a constant fit as a function of $N_s/N_t$ for each $N_t$. 
Continuum extrapolation of the latent heat

Fit the data at $N_t = 6, 8, 12$ with a linear function of $1/N_t^2$ assuming $O(a^2)$ error.

$1/N_t^2 = \alpha^2 T_c^2$

$\Delta(\epsilon-3p)/T^4 = 0.75 \pm 0.17$

$\Delta\epsilon/T^4 = 0.623 \pm 0.056$
EoS by the Gradient Flow (H. Suzuki, 2013)


- Smeared field strength: \( F_{\mu\nu} \xrightarrow{\text{Gradient Flow}} G_{\mu\nu} \)

- Dim. 4 operators:
  \[
  E(t, x) = \frac{1}{4} \delta_{\mu\nu} G_{\rho\sigma}(t, x) G_{\rho\sigma}(t, x)
  \]
  \[
  U_{\mu\nu}(t, x) = G_{\mu\rho}(t, x) G_{\nu\sigma}(t, x) - \frac{1}{4} \delta_{\mu\nu} G_{\rho\sigma}(t, x) G_{\rho\sigma}(t, x)
  \]

- Energy momentum tensor
  \[
  T^{R}_{\mu\nu} = \lim_{t \to 0} \left\{ \frac{1}{\alpha_{U}(t)} U_{\mu\nu}(t, x) + \frac{\delta_{\mu\nu}}{4\alpha_{E}(t)} \left[ E(t, x) - \langle E(t, x) \rangle_0 \right] \right\}
  \]
  \[
  \alpha_{U}(t) = g^2 \left[ 1 + 2b_0 s_1 g^2 + \cdots \right] \quad \alpha_{E}(t) = \frac{1}{2b_0} \left[ 1 + 2b_0 s_2 g^2 + \cdots \right]
  \]
  \[g: \text{running coupling constant with } \overline{\text{MS}} \text{ scheme based on the 4-loop beta function.}\]

- Energy density and Pressure
  \[
  \epsilon = \langle T_{00} \rangle \quad p = \frac{1}{3} \sum_{i} \langle T_{ii} \rangle
  \]
EoS by gradient flow method (preliminary)

- $\Delta\varepsilon / T^4$ becomes constant at long flow time.
- As flow time proceeds, $\Delta p$ vanishes. ($\Delta\varepsilon - 3\Delta p \approx \Delta\varepsilon + \Delta p$)

$G_{\mu\nu}^2$ defined by plaquette

$G_{\mu\nu}^2$ defined by clover tem
In the \( t=0 \) limit, the results by the gradient flow method and the derivative method are consistent within the error.

are the results by the derivative method on \( 96^3 \times 12 \) lattice.
Conclusions and Outlook

• We study the equation of state at the first order phase transition of SU(3) gauge theory.

• Gaps of energy density and pressure are measured using the derivative method.
  • We confirmed that the pressure gap is zero on each finite lattice.
  • The result of the latent heat in the continuum limit is
  \[
  \frac{\Delta \varepsilon}{T^4} = 0.75 \pm 0.17 \quad \frac{\Delta (\varepsilon - 3p)}{T^4} = 0.623 \pm 0.056
  \]

• We tested the gradient flow method for the calculation of EoS.
  • As flow time increases, $\Delta \varepsilon / T^4$ becomes constant and $\Delta p$ vanishes.
  • In the $t=0$ limit, the latent heat is consistent with that by the derivative method on the finite lattice.