Two-flavor simulations of the $\rho(770)$ and the role of the $K\bar{K}$ channel

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A recent lattice QCD study (GWU16) noted that the $\rho$ mass extracted from $N_f=2$ simulations is lighter than its physical value. This is also supported by an independent calculation from the RQCD Collaboration [1] very close to the physical mass.

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**Lang 11** [Lang, Mohler, Prelovsek, Vidmar, PRD (2011)], **CP-PACS07** [S. Aoki et al., PRD (2007) ]

**GWU16** [Guo, Alexandru, Molina, Döring, (2016)], **RQCD16** [Bali et al., PRD (2016)]
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The Contribution of virtual Kaons (simple analogy)

Insertion of a $K\bar{K}$ intermediate state in scattering

Centrifugal barriers times $K\bar{K}$ propagator $G_{K\bar{K}}$
The Contribution of virtual Kaons

Insertion of a $K\bar{K}$ intermediate state in scattering

A maximum contribution may occur close to the $\rho$ mass, effectively shifting it.

Centrifugal barriers times $KK\bar{K}$ propagator $G_{KK\bar{K}}$
The Contribution of virtual Kaons

Insertion of a $K\bar{K}$ intermediate state in scattering

but small inelasticity ($K\bar{K}$ almost decouples above threshold) needs to be checked

A maximum contribution may occur close to the $\rho$ mass, effectively shifting it.

Centrifugal barriers times $K\bar{K}$ propagator $G_{K\bar{K}}$
Coupled channel scattering/Inverse amplitude method

Unitarity in coupled channels

\[ V_{\pi\pi\to\pi\pi} \text{ from NLO CHPT} \]

Propagator \( G_{\pi\pi} \)

\[ V_{\pi\pi\to K\bar{K}} \]

Organized in matrices:

\[
V = \begin{pmatrix}
V_{\pi\pi\to\pi\pi} & V_{\pi\pi\to K\bar{K}} \\
V_{K\bar{K}\to\pi\pi} & V_{K\bar{K}\to K\bar{K}}
\end{pmatrix},
\]

\[
G = \begin{pmatrix}
G_{\pi\pi} & 0 \\
0 & G_{K\bar{K}}
\end{pmatrix}
\]

Scattering equation:

\[
T = V + TGV
\]

Phenomenology:

\[
S = 1 + i \text{ (factors)} T
\]

\[
S = \begin{pmatrix}
\eta e^{2i\delta_{\pi\pi}} & i(1 - \eta^2)^{1/2} e^{i(\delta_{\pi\pi} + \delta_{K\bar{K}})} \\
i(1 - \eta^2)^{1/2} e^{i(\delta_{\pi\pi} + \delta_{K\bar{K}})} & \eta e^{2i\delta_{K\bar{K}}}
\end{pmatrix}
\]

For SU(2) lattice data: remove \( K\bar{K} \) channel and perform SU(2)/SU(3) matching of LECs.

\[\text{1Based on Oller, Oset, Pelaez, PRD (1998).}\]
To fit the lattice phase shifts, the $K\bar{K}$ channel is removed from the coupled-channel $\pi\pi$/$K\bar{K}$ system (including SU(3)/SU(2) matching of LECs).
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The result is extrapolated to the physical pion mass $M_{\pi} = 138$ MeV and then $K\bar{K}$ channel is switched on→post-diction of experimental phaseshifts.
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Unknown LECs taken from fit to experiment.
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Express final results in terms of common notation, Breit-Wigner $m_\rho: \rho$ mass, $g: \rho\pi\pi$ coupling.

Unknown LECs taken from fit to experiment
Results for the $N_f = 2$ simulations

Lattice phase shifts to be included in the fit are chosen in the maximal range around the resonance position, in which the fit passes Pearson's $\chi^2$ test at a 90% upper confidence limit.

Experimental data: blue circles from [Estabrooks, Martin, (1974)], squares from [Protopopescu et al., (1973)].
Results for the $N_f = 2$ simulations

$M_\pi = 227\text{MeV}$ GWU16

$M_\pi = 138\text{MeV}$ GWU16

$M_\pi = 240, 250, 390\text{MeV}$ QCDSF08

$M_\pi = 138\text{MeV}$ QCDSF08
Results for the $N_f = 2$ simulations

- $M_\pi = 266$ MeV Lang11
- $M_\pi = 138$ MeV Lang11
- $M_\pi = 315$ MeV GWU16
- $M_\pi = 138$ MeV GWU16
Results for the $N_f = 2$ simulations

- $M_\pi = 290,330 \text{MeV}$ ETMC11
  - $M_\pi = 138 \text{MeV}$ ETMC11

- $M_\pi = 328 \text{MeV}$ CP–PACS07
  - $M_\pi = 138 \text{MeV}$ CP–PACS07
Effects of extrapolation in \((m_\rho, g)\) plane

Lattice result

- 227 MeV GWU16
- 315 MeV GWU16
- 266 MeV Lang11
- 149 MeV RQCD16
- 290, 330 MeV ETMC11
- 328 MeV CP–PACS07
- 240, 250, 390 MeV QCDSF08
Effects of extrapolation in \((m_\rho, g)\) plane

Chiral extrapolation

No chiral extrapolation needed
Effects of extrapolation in \((m_\rho, g)\) plane

Inclusion of \(K\bar{K}\) channel
Effects of extrapolation in \((m_\rho, g)\) plane

Final results with errors
Resonance mass extrapolation

Red curve: an $N_f=2$ extrapolation based on the fit to (GWU16) data. Blue band: inclusion of the $K\bar{K}$ channel (systematic error only).
Inelasticity from $K\bar{K}$ channel

Lang11 is representative for other $N_f = 2$ simulations (not shown).

Left two pictures: $K\bar{K}$ phase shift and elasticity at $M_\pi = 236$ MeV with error band from [Wilson et al. (2015)]. The right picture shows the elasticity at the physical pion mass, together with the elasticity determined from experiment [Protopopescu et al., PRD (1973)] and the $K\bar{K}$ contribution to the inelasticity evaluated in [Niecknig et al. (2012)] from the Roy-Steiner determination of [Büttiker et al. EPJ C (2004)].

Experimental inelasticity [Protopopescu et al., PRD (1973)], Roy-Steiner [Büttiker et al. EPJ C (2004)].
Conclusion

The $K\bar{K}$ channel improves the extrapolations of the mass significantly except when the lattice data have large uncertainties.

Explains the systematically small lattice masses at the physical point after the chiral SU(2) extrapolation (or, directly, Bali result).
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Explain the systematically small lattice masses at the physical point after the chiral SU(2) extrapolation (or, directly, Bali result).

GWU (Guo/Alexandru) test: Including valence strange quark does not change $\rho$ mass.
- If $K\bar{K}$ is really strong, the effect comes from sea quarks.
- $N_f = 2, N_f = 2 + 1$ simulations within same Lattice QCD simulation needed for confirmation.
Thanks
NLO CHPT Lagrangian\(^1\)

\[ \mathcal{L}_2 = \frac{f^2}{4} \left( \partial_\mu U^\dagger \partial^\mu + M(U + U^\dagger) \right) \]

\[ U(\phi) = \exp(i\sqrt{2}\Phi/f) \]

\[ \Phi(x) = \left( \begin{array}{ccc} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & K^0 & -\frac{2}{\sqrt{6}}\eta \end{array} \right)_{\mu} \]

\[ \mathcal{L}_4 = L_1 \langle \partial_\mu U^\dagger \partial^\mu U \rangle^2 + L_2 \langle \partial_\mu U^\dagger \partial_\nu U \rangle \langle \partial^\mu U^\dagger \partial^\nu U \rangle 
+ L_3 \langle \partial_\mu U^\dagger \partial^\mu U \partial_\nu U^\dagger \partial^\nu U \rangle + L_4 \langle \partial U^\dagger \partial^\mu U \rangle \langle U^\dagger M + M^\dagger U \rangle 
+ L_5 \langle \partial_\mu U^\dagger \partial^\mu U(U^\dagger M + M^\dagger U) \rangle + L_6 \langle U^\dagger M + M^\dagger U \rangle^2 
+ L_7 \langle U^\dagger M - M^\dagger U \rangle^2 + L_8 \langle M^\dagger U M^\dagger U + U^\dagger M U^\dagger M \rangle \]

\[ M = \left( \begin{array}{ccc} m_\pi^2 & 0 & 0 \\ 0 & m_\pi^2 & 0 \\ 0 & 0 & 2m_K^2 - m_\pi^2 \end{array} \right) \]

\(^1\text{[Oller, Oset, Pelaez, PRD (1999)].}\)
Inverse amplitude method

\[ S = \begin{pmatrix} \eta e^{2i\delta_1} & i(1 - \eta^2)^{1/2} e^{i(\delta_1 + \delta_2)} \\ i(1 - \eta^2)^{1/2} e^{i(\delta_1 + \delta_2)} & \eta e^{2i\delta_2} \end{pmatrix} \]

\[ (T)_{11} = -\frac{8\pi W}{2ip_1} [(S)_{11} - 1], \quad (T)_{22} = -\frac{8\pi W}{2ip_2} [(S)_{22} - 1] \]

\[ (T)_{12} = (T)_{21} = -\frac{8\pi W}{2i\sqrt{p_1 p_2}} (S)_{12} \]

\[ T = [I - V^{IAM} G]^{-1} V^{IAM} \]

\[ V^{IAM} = V_2 [V_2 - V_4]^{-1} V_2 \]

\[ V = \begin{pmatrix} V_{\pi\pi \rightarrow \pi\pi} & V_{\pi\pi \rightarrow K\bar{K}} \\ V_{K\bar{K} \rightarrow \pi\pi} & V_{K\bar{K} \rightarrow K\bar{K}} \end{pmatrix} \]

\[ ^1 [\text{Oller, Oset, Pelaez, PRD (1999)}]. \]
\[ G_{ii}^{DR}(W) = i \int \frac{d^4 q}{(2\pi)^4} \left( \frac{1}{q^2 - m_1^2 + i\epsilon} \right) \frac{1}{(P - q)^2 - m_2^2 + i\epsilon} \]

\[ = \frac{1}{16\pi^2} \left\{ a(\mu) + \ln \frac{m_1^2}{\mu^2} + \frac{m_2^2 - m_1^2 + W^2}{2W^2} \ln \frac{m_2^2}{m_1^2} \right. \]

\[ + \frac{p_i}{W} \left[ \ln \left( W^2 - (m_1^2 - m_2^2) + 2p_i W \right) + \ln \left( W^2 + (m_1^2 - m_2^2) + 2p_i W \right) \right] \]

\[ \left. - \ln \left( -W^2 + (m_1^2 - m_2^2) + 2p_i W \right) \right\} \]

\[ \bar{p}_i = \sqrt{(W^2 - (m_1 + m_2)^2)(W^2 - (m_1 - m_2)^2)} \]

\[ V_2(E) = - \left( \begin{array}{c} \frac{2p_\pi^2}{3f_\pi^2} \frac{\sqrt{2p_Kp_\pi}}{3f_Kf_\pi} \\ \frac{\sqrt{2p_Kp_\pi}}{3f_Kf_\pi} \frac{p_K^2}{3f_K^2} \end{array} \right) \]

\[ V_4(E) = -1 \times \left( \begin{array}{c} \frac{8p_\pi^2(2\tilde{l}_1 m_\pi^2 - \tilde{l}_2 E^2)}{3f_\pi^4} \frac{8p_\pi p_K (L_5(m_\pi^2 + m_K^2) - L_3 E^2)}{3\sqrt{2}f_\pi^2 f_K^2} \frac{8p_\pi p_K (L_5(m_\pi^2 + m_K^2) - L_3 E^2)}{3\sqrt{2}f_\pi^2 f_K^2} \frac{4p_K^2(10\tilde{l}_1 m_\pi^2 + 3(L_3 - 2\tilde{l}_2) E^2)}{9f_K^4} \end{array} \right) \]

\[ \Rightarrow \hat{l}_1 = \frac{l_4^r}{4} + 1/8 \nu_K \]

\[ \hat{l}_2 = \frac{l_1^r}{2} - \frac{l_2^r}{4} \]

Meson mass and decay constants\textsuperscript{1}

\[
M_\pi^2 = M_0^2 \pi \left[1 + \mu_\pi - \frac{\mu_\eta}{3} + \frac{16M_0^2}{f_0^2} (2L_6^r - L_4^r) + \frac{8M_0^2}{f_0^2} (2L_6^r + 2L_8^r - L_4^r - L_5^r)\right],
\]

\[
M_K^2 = M_0^2 K \left[1 + \frac{2\mu_\eta}{3} + \frac{8M_0^2}{f_0^2} (2L_6^r - L_4^r) + \frac{8M_0^2}{f_0^2} (4L_6^r + 2L_8^r - 2L_4^r - L_5^r)\right],
\]

\[
M_\eta^2 = M_0^2 \eta \left[1 + 2\mu_K - \frac{4}{3} \mu_\eta + \frac{8M_0^2}{f_0^2} (2L_8^r - L_5^r) + \frac{8}{f_0^2} (2M_0^2 K + M_0^2 \pi) (2L_6^r - L_4^r)\right] + M_0^2 \pi \left[-\mu_\pi + \frac{2}{3} \mu_K + \frac{1}{3} \mu_\eta\right] + \frac{128}{9f_0^2} (M_0^2 K - M_0^2 \pi)^2 (3L_7^r + L_8^r),
\]

with

\[
\mu_P = \frac{M_0^2 P}{32\pi^2 f_0^2} \log \frac{M_0^2 P}{\mu^2}, \quad P = \pi, K, \eta.
\]

\[
f_\pi = f_0 \left[1 - 2\mu_\pi - \mu_K + \frac{4M_0^2}{f_0^2} (L_4^r + L_5^r) + \frac{8M_0^2}{f_0^2} L_4^r\right],
\]

\[
f_K = f_0 \left[1 - \frac{3\mu_\pi}{4} - \frac{3\mu_K}{2} - \frac{3\mu_\eta}{4} + \frac{4M_0^2}{f_0^2} L_4^r + \frac{4M_0^2}{f_0^2} (2L_4^r + L_5^r)\right],
\]

\[
f_\eta = f_0 \left[1 - 3\mu_K + \frac{4L_4^r}{f_0^2} (M_0^2 \pi + 2M_0^2 K) + \frac{4M_0^2}{f_0^2} L_5^r\right].
\]

In the above equations, \(f_0\) is the pion decay constant in the chiral limit, \(4\pi f_0 \approx 1.2\) GeV, \(\mu\) is the regularization scale, commonly fixed at \(\mu = M_\rho\), and \(L_i^r\)’s, with \(i = 1 \sim 8\), are the Low Energy Constants, superscript \(r\) stand for a dependence on the regularization scale \(\mu\). \(L_i^r\)’s for \(f_i\) are obtained from fit to lattice data in [Nebreda, Pelaez]\textsuperscript{1}.

\textsuperscript{1}[Nebreda, Pelaez., PRD (2010)].
The function $\delta_L(E) = g(E)$ returns the phase shift from an energy eigenvalue $E$ measured on the lattice.

Fit function $\delta_{fit}(E) = f(E)$ intersects with $g(E)$ at $E_1$.

$\chi^2$ should be measured along the inclination of the error bar.

$E_1$: energy of the intersect.

$E_0$: energy of the data point.

$f'(E_0)$: the numerical derivative of fit function.

$g'(E_0)$ can be reconstructed from error bars in $E$ and $\delta(E)$, and determining the sign of the slope of $\delta_L$ at each measured energy.

\[
\begin{align*}
\text{Inclined error bar fit} \\
\delta(E) & \quad \delta_{fit} = f(E) \\
\delta_L = g(E) & \quad \text{LüScher function} \\
\text{fit function} & \quad \text{Inclined error bar} \\
\text{reconstructed} & \quad E_0 \quad E_1 \\
\end{align*}
\]

\[
\begin{align*}
f(E_1) &= g(E_1) \\
f(E_0) + f'(E_0)(E_1 - E_0) &= g(E_0) + g'(E_0)(E_1 - E_0) \\
f'(E_0)(E_1 - E_0) - g'(E_0)(E_1 - E_0) &= g(E_0) - f(E_0) \\
E_1[f'(E_0) - g'(E_0)] - E_0[f'(E_0) - g'(E_0)] &= g(E_0) - f(E_0) \\
\Rightarrow E_1 &= \frac{g(E_0) - f(E_0)}{f'(E_0) - g'(E_0)} + E_0 \\
\Rightarrow \chi^2 &= \frac{(E_1 - E_0)^2}{\Delta E^2} = \frac{(g(E_0) - f(E_0))^2}{\Delta E^2} \frac{1}{(f'(E_0) - g'(E_0))^2}
\end{align*}
\]
Effects of extrapolation in \((m_\rho, g)\) plane
68% confidence ellipses in $l_1, l_2$

- Linear combinations of LECs:
  $\hat{l}_1 = 2L_4 + L_5,$
  $\hat{l}_2 = 2L_1 - L_2 + L_3.$
- The error ellipses from RQCD, GWU ($M_\pi = 227$ MeV and $M_\pi = 315$ MeV), Lang et al., and CP-PACS all have a common overlap.
- The ellipse from QCDSF is very slightly off, while the one from ETMC is clearly incompatible.
- The contribution of kaon tadpole (Gasser, Leutwyler, Nucl. Phys. B, 1985) is very tiny.
# Breit-Wigner $\rho$ masses of the $\rho$ meson

<table>
<thead>
<tr>
<th></th>
<th>$M_\pi$</th>
<th>BW</th>
<th>BW,converted</th>
<th>$N_f=2$ extrapolated</th>
<th>$N_f=2+1$ extrapolated</th>
</tr>
</thead>
<tbody>
<tr>
<td>GWU16[2]</td>
<td>227</td>
<td>749.2(1.6)(15)</td>
<td>749</td>
<td>721</td>
<td>776(3)(10)</td>
</tr>
<tr>
<td></td>
<td>315</td>
<td>795.5(0.7)(16)</td>
<td>795</td>
<td>724</td>
<td>778(4)(11)</td>
</tr>
<tr>
<td>QCDSF08[5]</td>
<td>240</td>
<td>770(10)</td>
<td>776</td>
<td>730</td>
<td>779(7)(6)</td>
</tr>
<tr>
<td></td>
<td>250</td>
<td>784(10)</td>
<td>781</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>390</td>
<td>846(10)</td>
<td>844</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ETMC11[6]</td>
<td>290</td>
<td>980(31)</td>
<td>983</td>
<td>821</td>
<td>827(46)(0.4)</td>
</tr>
<tr>
<td></td>
<td>330</td>
<td>1033(31)</td>
<td>1031</td>
<td></td>
<td></td>
</tr>
<tr>
<td>CP-PACS07[7]</td>
<td>328</td>
<td>808(24)(25)</td>
<td>833</td>
<td>750</td>
<td>786(&gt;100)(&gt;100)</td>
</tr>
</tbody>
</table>

Second column: Pion masses of respective studies [MeV]. Third column: $\rho$ masses as quoted or extracted from pictures in the respective publications. Fourth column: Breit-Wigner $\rho$ masses as reconstructed from our UCHPT solutions at unphysical pion masses. Fifth column: SU(2) extrapolated fits at physical pion mass. Sixth column: Final results after including the $K\bar{K}$ channel. Uncertainties (statistical, systematic) only quoted for these cases.
### Scattering lengths and effective ranges

<table>
<thead>
<tr>
<th></th>
<th>Exp./CHPT</th>
<th>RQCD16</th>
<th>GWU16</th>
<th>QCDSF08</th>
<th>Lang11</th>
<th>ETMC11</th>
<th>CP-PACS07</th>
</tr>
</thead>
<tbody>
<tr>
<td>10a₁</td>
<td>0.38 ± 0.02 (exp.)</td>
<td>0.37 ± 0.31</td>
<td>0.349 ± 0.003</td>
<td>0.350 ± 0.001</td>
<td>0.345 ± 0.003</td>
<td>0.348 ± 0.005</td>
<td>0.367 ± 0.003</td>
</tr>
<tr>
<td>100b₁</td>
<td>0.48 [O(p⁴)] 0.79 [O(p⁶)]</td>
<td>1.14 ± 2 9.1</td>
<td>0.683 ± 0.065</td>
<td>0.627 ± 0.023</td>
<td>0.695 ± 0.113</td>
<td>0.656 ± 0.136</td>
<td>0.911 ± 0.092</td>
</tr>
</tbody>
</table>

Scattering lengths $a_1$ and effective ranges $b_1$. The experimental value, the $O(p^4)$, and the $O(p^6)$ results are taken from Ref. [12].

Ref. [12][Bijnens, Colangelo, Ecker, Gasser, Sainio, Nucl. Phys. B (1997)].
Estimate systematic uncertainties

\[
V_2(E) = - \left( \begin{array}{cc}
\frac{2p^2_\pi}{3f^2_\pi} & \frac{\sqrt{2}p_\pi p_\pi}{3f_K f_\pi} \\
\frac{\sqrt{2}p_\pi p_\pi}{3f_K f_\pi} & \frac{p_\pi^2}{3f^2_K}
\end{array} \right)
\]

\[
V_4(E) = -1 \times \left( \begin{array}{cc}
\frac{8p^2_\pi (2\tilde{l}_1 m^2_\pi - \tilde{l}_2 E^2)}{3f^4_\pi} & \frac{8p_\pi p_K (L_5 (m^2_K + m^2_\pi) - L_3 E^2)}{3\sqrt{2}f^2_\pi f^2_K} \\
\frac{8p_\pi p_K (L_5 (m^2_K + m^2_\pi) - L_3 E^2)}{3\sqrt{2}f^2_\pi f^2_K} & \frac{4p^2_K (10\tilde{l}_1 m^2_K + 3(L_3 - 2\tilde{l}_2) E^2)}{9f^4_K}
\end{array} \right)
\]

\(\tilde{l}_1\) and \(\tilde{l}_2\) in purple circles:
1. from fit to experimental phase shifts.
2. from fit to \(N_f = 2\) lattice phase shifts.

Do both, the difference is estimated as systematic/model error.

\(L_3\) and \(L_5\) in red circles:
from fit to experimental phase shifts.
Further tests

Results of different subtractions
We test the dependence of the results on the value of the subtraction constant, changing it from the default value $a = -1.28$ to $a = -0.8$ and $a = -1.7$. The global fits to experimental phase shifts visibly deteriorate for these extreme values, e.g., for $\pi K$ scattering, but barely change in the $\rho$ channel (less than 10 MeV in $m_\rho$ and less than 0.08 in $g$) as experimental phase-shift data are more precise.

Effects of the kaon tadpoles
If one goes from SU(2) to SU(3), actually the $\pi\pi \rightarrow \pi\pi$ transition changes by a tiny bit, given by so-called kaon tadpoles$^2$ we have checked that they indeed change nothing (resonance positions by < 1 MeV).

References
