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Volume reduction through perturbative Wilson loops

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Eguchi & Kawai 82

Eguchi-Kawai volume reduction

$$b = \frac{\beta}{2N^2} = \lambda_L^{-1} \quad \text{fixed}$$

Large N observable on a L^4 lattice

$$O_\infty(b) = \lim_{N\to\infty} \lim_{L\to\infty} O(b, N, L)$$

L^4 lattice

Eguchi-Kawai volume reduction

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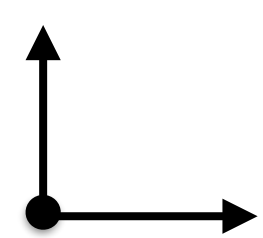
L^4 lattice

Eguchi-Kawai reduction

$$O_\infty(b) = \lim_{N\to\infty} O(b, N, L=1)$$

Thermodynamic limit irrespective of L



$U_\mu \in SU(N)$

one-point lattice

Conditions   Volume independence of single trace observables if

Tr (→) = 0   Center symmetry preserved $Z(N)^d$

Bhanot, Heller & Neuberger

Depends on boundary conditions

For tbc $k, \bar{k} \propto N$ González-Arroyo & Okawa

For pbc $L > L_c$ Narayanan & Neuberger

Depends on matter content

Pbc with adjoint fermions Kotvun, Unsal & Yaffe

Amber, Basar, Cherman, Dorigoni, Hanada, Koren, Poppitz, Sharpe,…

✦ In this talk:

Test volume reduction for Wilson loops in lattice perturbation theory with twisted boundary conditions

$SU(N)$ gauge theory on a L^4 lattice

$$\log W(b, N, L) = -W_1(N, L)\lambda - W_2(N, L)\lambda^2$$

Compare with pbc Heller&Karsch

Compare with infinite volume Weisz, Wetzel & Wohlert

Twisted boundary conditions

L^4 lattice

Twist

$$S = bN \sum_n \sum_{\mu\nu} [N - Z_{\mu\nu}(n) \mathrm{Tr}(U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu^\dagger(n+\hat{\nu}) U_\nu^\dagger(n))]$$

$$Z_{\mu\nu} = \begin{cases} 1 & \\ \exp\left\{2\pi i \frac{k}{\sqrt{N}} \epsilon_{\mu\nu}\right\} & n_\mu = n_\nu = L-1 \end{cases}$$

symmetric twist

k and $\sqrt{N}$ co-prime $\quad k, \bar{k} \propto N$

González-Arroyo & Okawa

$$b = \frac{\beta}{2N^2} = \lambda_L^{-1}$$

$$\lambda = g^2 N$$

't Hooft coupling

Perturbation theory

$$U_\mu(n) = e^{-igA_\mu(n)}\Gamma_\mu(n)$$

Periodic links $U_\mu(n) = U_\mu(n + L\hat{\nu})$

$$\Gamma_\mu(n) = \begin{cases} \mathbb{1} & \text{for } n_\mu \neq L-1 \\ \Gamma_\mu & \text{for } n_\mu = L-1 \end{cases}$$

with

$$\Gamma_\mu \Gamma_\nu = Z_{\nu\mu} \Gamma_\nu \Gamma_\mu$$

twist eaters

Note: zero momentum not compatible with the boundary conditions

Luscher&Weisz, Gonzalez-Arroyo & Korthals-Altes, Snippe

To implement boundary conditions $A_\mu(x + l\,\hat{\nu}) = \Gamma_\nu A_\mu(x)\Gamma_\nu^\dagger$

$$A_\mu(n) = \frac{1}{L^2}\sum_p{}' e^{ip(n+\frac{1}{2})}\hat{A}_\mu(p)\hat{\Gamma}(p)$$

$A^a_\mu(p)T_a$

$$\hat{\Gamma}(p) \propto \Gamma_1^{s_1}\Gamma_2^{s_2}\cdots\Gamma_d^{s_d}$$

momentum dependent basis for the SU(N) Lie algebra

$$p_\mu = \frac{2\pi m_\mu}{L_{\text{eff}}}$$

To satisfy b.c. momentum is quantised in units of

Effective box - size $L_{\text{eff}} = L\sqrt{N}$

TEK $L = 1$

$$l_{\text{eff}} = a\sqrt{N} \xrightarrow[N\to\infty,\ a \text{ fixed}]{} l_{\text{eff}} = \infty$$

thermodynamic limit

Perturbation theory

- Momentum quantized in units of $L_{\rm eff}$
- Free propagator identical that on a finite lattice $L_{\rm eff}$
- Group structure constants $\Gamma(p)$

$$F(p,q,-p-q) = -\sqrt{\frac{2}{N}}\sin\left(\frac{\theta_{\mu\nu}}{2}p_\mu q_\nu\right)$$

Momentum dependent phases in the vertices

$$\theta_{\mu\nu} = \frac{L_{\rm eff}^2}{4\pi^2} \times \tilde{\epsilon}_{\mu\nu}\,\tilde{\theta}$$

$$\tilde{\theta} = \frac{2\pi\bar{k}}{\sqrt{N}}$$

$$\bar{k}k = 1\,(\mathrm{mod}\,\sqrt{N})$$

Links to non-commutative gauge theories

González-Arroyo, Korthals Altes, Okawa

Volume independence

Vertices ∝ $\sqrt{\frac{2\lambda}{V_{\mathrm{eff}}}} \sin\left(\frac{\theta_{\mu\nu}}{2} p_\mu q_\nu\right)$

In perturbation theory,

$$\tilde{\theta},\ \lambda,\ L_{\mathrm{eff}}$$

For fixed $\tilde{\theta}$, volume and N dependence encoded in the effective size

Comment

- Certain momenta excluded by the twist in SU(N)

$$A_\mu(n) = \frac{1}{L^2} \sum_p{}' e^{ip(n+\frac{1}{2})} \hat{A}_\mu(p) \hat{\Gamma}(p)$$

$$\mathrm{Tr}\, \hat{\Gamma}(p) = 0$$

$$p_\mu = \frac{2\pi n_\mu}{L_{\rm eff}}$$

Lattice of momenta $\Lambda_{L_{\rm eff}}$

Exclude

$$n_\mu = 0 \,(\mathrm{mod}\, \sqrt{N})\, \forall \mu$$

$$p_\mu = \frac{2\pi n_\mu}{L}, \ \forall \mu$$

Reintroduces N dependence - gives correct number of degrees of freedom

$p \in \Lambda_{L_{\rm eff}} \backslash \Lambda_L$ degrees of freedom $L_{\rm eff}^4 - L^4 = L^4(N^2 - 1)$

✦ The Wilson loop at $\mathcal{O}(\lambda)$

$$\tilde{W}_1^{(R\times T)}(N,L,k) = \frac{1}{4V_{\rm eff}} \sum_q{}' \frac{\sin^2(Rq_\mu/2)\sin^2(Tq_\nu/2)}{\widehat{q}_\mu^2\,\widehat{q}_\nu^2} \frac{\widehat{q}_\mu^2+\widehat{q}_\nu^2}{\widehat{q}^2}$$

The same as with pbc but with different set of momenta

$$\frac{1}{V_{\rm eff}} \sum_q{}' \longrightarrow \frac{1}{L_{\rm eff}^4} \sum_{q\in\Lambda'_{L_{\rm eff}}} - \frac{1}{N^2L^4} \sum_{q\in\Lambda'_L}$$

Exclude momenta in	Momenta in	Momenta in
Λ_L	$\Lambda'_{L_{\rm eff}}$	Λ'_L
	$L_{\rm eff} = L\sqrt{N}$	L

Zero momentum excluded in all cases

The Wilson loop at $\mathcal{O}(\lambda)$

For TBC

Effective size $\quad \frac{1}{N^2}$ correction

$$W_1(N,L) = F_1(L\sqrt{N}) - \frac{1}{N^2} F_1(L) \underset{N\to\infty}{\longrightarrow} F_1(\infty)$$

Volume independence

MGP, González-Arroyo & Okawa

For PBC

$$W_1(N,L) = F_1(L)\,\frac{N^2-1}{N^2} \underset{N\to\infty}{\longrightarrow} F_1(L)$$

retains L dependence

Heller&Karsch

✦ The Wilson loop at $\mathcal{O}(\lambda^2)$

With periodic boundary conditions

Heller&Karsch

$$W_2^{\text{pbc}}(L,N,k=0) = (1-\frac{1}{N^2})F_2(L) + (1-\frac{1}{N^2})^2 F_W(L)$$

Tadpole

$$F_W(L) = \frac{1}{8}\left(1-\frac{1}{V}\right)F_1(L)$$

For $N \to \infty$

$$W_2^{\text{pbc}}(L,N=\infty,k=0) = \boxed{F_2(L) + F_W(L)}$$

retains L dependence

The Wilson loop at $\mathcal{O}(\lambda^2)$ with tbc

Non-abelian terms containing the structure constant

$$N F^2(p, q, -p-q) = 1 - \cos(\theta_{\mu\nu} p_\mu q_\nu)$$

It is zero for momenta in Λ_L

$$\frac{1}{NL^4} \sum_q{}' F^2 \longrightarrow \boxed{\frac{1}{L_{\rm eff}^4} \sum_{q \in \Lambda'_{L_{\rm eff}}}} - \boxed{\frac{1}{L_{\rm eff}^4} \sum_{q \in \Lambda'_{L_{\rm eff}}} \cos(\theta_{\mu\nu} p_\mu p_\nu)}$$

Planar diagrams

The same structure as pbc

Non-planar diagrams

Contain all the dependence on the twist

With twisted boundary conditions

Effective size $\qquad \dfrac{1}{N^2}$ corrections

$$W_2^{\rm tbc}(L,N,k) = \left(F_2(L\sqrt{N}) - \frac{1}{N^2}F_2(L)\right)$$

$$+\frac{1}{8}\left(1-\frac{1}{N^2}\right)\left(F_1(L\sqrt{N}) - \frac{F_1(L)}{N^2}\right)$$

Planar diagrams

$$-\frac{1}{N^2}F_2^{NA}(L)$$

$$+F_{2T}(L,N,k)$$

Non-planar diagrams

$$+\frac{1}{N^2}F_2^{\rm NA}(L)$$

twist dependence $\quad \tilde{\theta} = \dfrac{2\pi\bar{k}}{\sqrt{N}} \qquad \bar{k}k = 1\,(\mathrm{mod}\,\sqrt{N})$

For TBC

large N limit

Volume independence

$$W_2^{\rm tbc}(L, N=\infty, k) = F_2(\infty) + F_W(\infty)$$

Correct thermodynamic limit

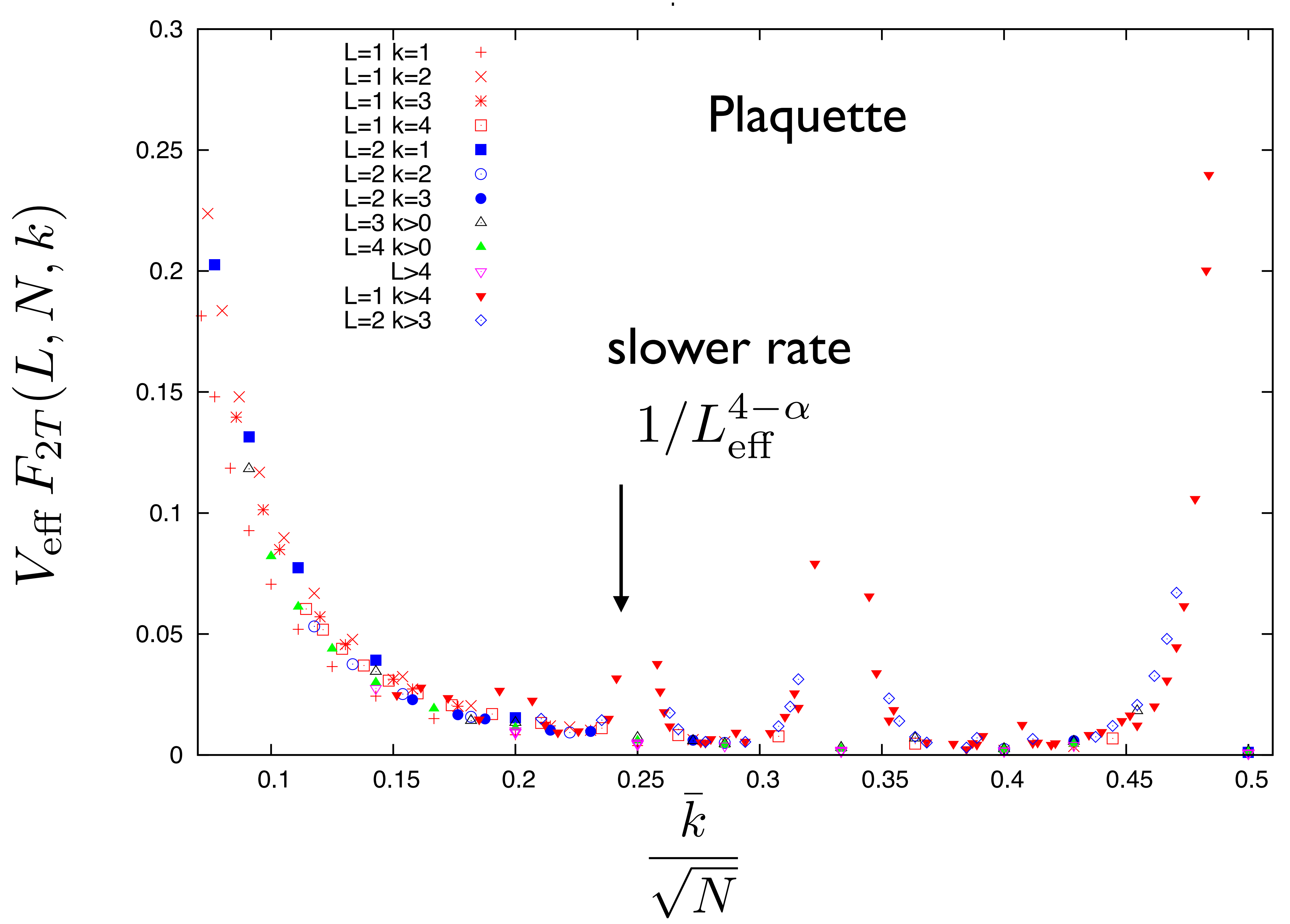
$$+ F_{2T}(L, N=\infty, k)$$

For volume independence to hold it is essential that

$$\lim_{N\to\infty} F_{2T} = 0$$

Non-planar diagrams Suppressed as $1/V_{\rm eff}$

Goes to zero both for N infinity or L infinity



For TBC   large V limit

$$W_2^{\rm tbc}(L=\infty, N, k) = \left(1-\frac{1}{N^2}\right)F_2(\infty) + \left(1-\frac{1}{N^2}\right)^2 F_W(\infty)$$

The formula reproduces the correct infinite volume limit

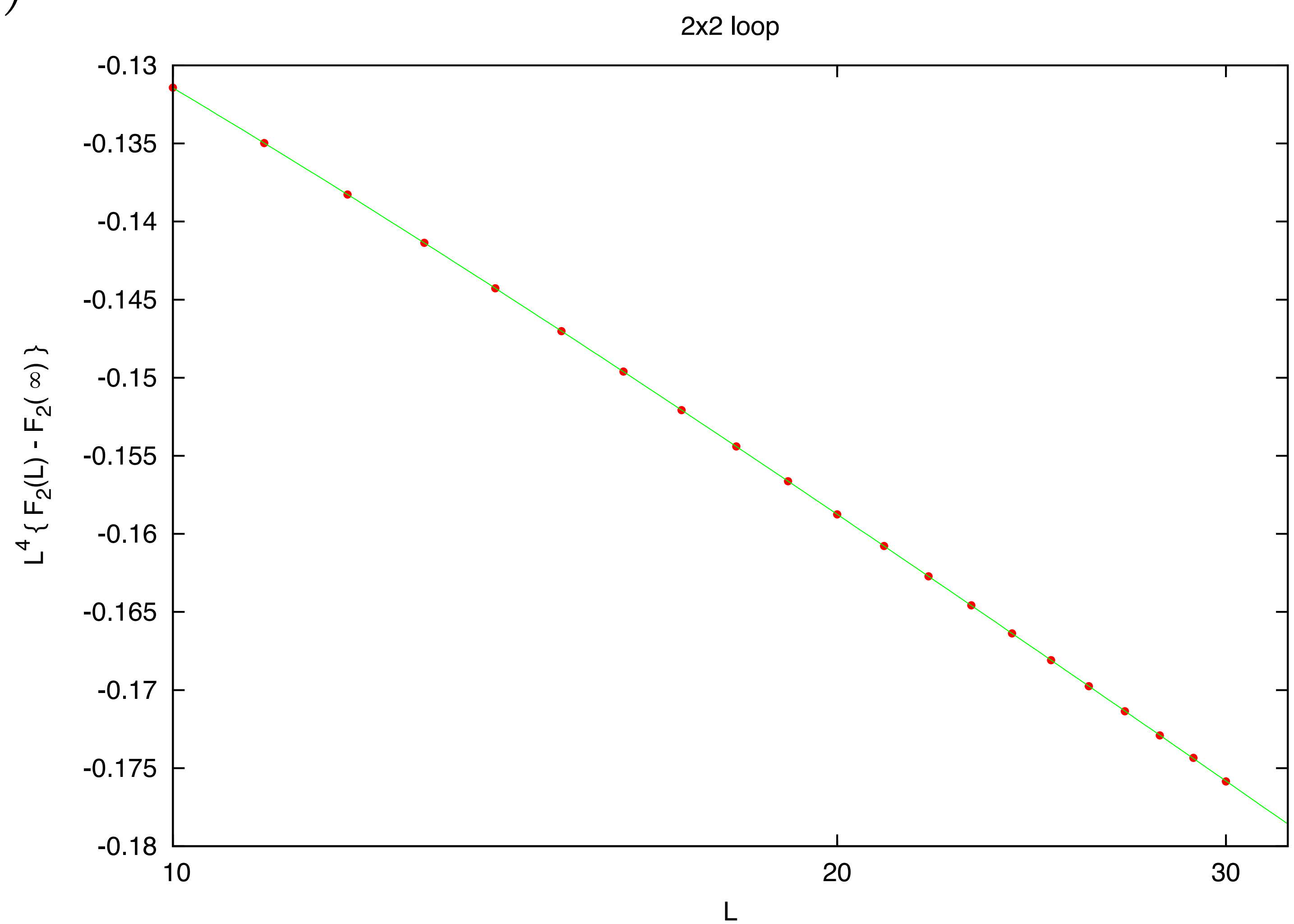
We have used that F_{2T} goes to zero in the thermodynamic limit

LOOP	$F_1(\infty)$	$F_2(\infty)$	$\tilde{W}_2(\infty,\infty)$
1×1	0.125	-0.0027055703(3)	0.0129194297(3)
2×2	0.34232788379	-0.00101077(1)	0.04178022(1)
3×3	0.57629826424	0.00295130(2)	0.07498858(2)
4×4	0.81537096352	0.0076217(1)	0.1095431(1)

Numerically evaluated

Consistent with B. Alles e.a

$$F_2(L) = F_2^{NA}(L) + F_{\rm meas}(L)$$



$$F_2(L) = F_2(\infty) - \frac{R^2T^2(\gamma_2 + \gamma_2' \log(L))}{L^4} + \dots$$

Bali e.a.

For Twisted Eguchi-Kawai L=1

Simplification $F_i(L=1)=0$

$$W_1^{\rm tbc}(L=1,N,k) = W_1^{\rm pbc}(L=\sqrt{N},\infty,0)$$

$$W_2^{\rm tbc}(L=1,N,k) = W_2^{\rm pbc}(L=\sqrt{N},\infty,0) + F_{2T}(L=1,N,k)$$

Effective size $L=\sqrt{N}$

Effective colour $N=\infty$

Non-planar contribution

$$\lim_{N\to\infty} F_{2T} = 0$$

Summary

- We have analysed the PT expansion of Wilson loops with tbc

- The expansion is expressed in terms of 3 functions:

$$F_1(L), \ F_2(L), \ F_{2T}(L, N.k)$$

- Volume independence holds as far as

$$\lim_{N \to \infty} F_{2T} = 0$$

- Our analysis shows that this holds, also for TEK on the one-site lattice

- The code developed can be applied to other twists and number of dimensions