Electromagnetic Form Factors through Parity-Expanded Variational Analysis

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The isolation of excitations of baryons at nonzero momentum is important for the evaluation of baryon form factors and transition moments.
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Existing parity projection techniques are vulnerable to opposite parity contaminations at nonzero momentum.
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Existing parity projection techniques are vulnerable to opposite parity contaminations at nonzero momentum.

We propose the Parity Expanded Variational Analysis (PEVA) technique to resolve this issue.
Nonzero momentum

- Eigenstates of nonzero momentum are not eigenstates of parity
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Categorise states by parity in rest frame:

- Call states that transform positively under parity in their rest frame "positive parity states" ($B^+$).
- Call states that transform negatively under parity in their rest frame "negative parity states" ($B^-$).
Nonzero momentum

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- Categorise states by parity in rest frame
- Call states that transform positively under parity in their rest frame “positive parity states” \((B^+)\)
- Call states that transform negatively under parity in their rest frame “negative parity states” \((B^-)\)
Conventional analysis

- Conventional baryon operators \( \{ \chi^i \} \) couple to states of both parities

\[
\langle \Omega | \chi^i | B^+ \rangle = \lambda_i^{B^+} \sqrt{\frac{m_{B^+}}{E_{B^+}}} u_{B^+}(p, s)
\]

\[
\langle \Omega | \chi^i | B^- \rangle = \lambda_i^{B^-} \sqrt{\frac{m_{B^-}}{E_{B^-}}} \gamma_5 u_{B^-}(p, s)
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Conventional analysis

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Conventional analysis

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\]

\[
\langle \Omega | \chi^i | B^- \rangle = \lambda_i^{B^-} \sqrt{\frac{m_{B^-}}{E_{B^-}}} \gamma_5 u_{B^-}(p, s)
\]

- Form correlation matrix

\[
G_{ij}(p; t) := \sum_x e^{ip \cdot x} \langle \Omega | \chi^i(x) \bar{\chi}^j(0) | \Omega \rangle
\]
Introduce $\Gamma_\pm = (\gamma_4 \pm \mathbb{I})/2$ and define $G_{ij}(\Gamma_\pm; p; t) := \text{tr}(\Gamma_\pm G_{ij}(p; t))$. At zero momentum, projected correlators only contain terms for states of a single parity.
Parity projection

- Introduce $\Gamma_{\pm} = (\gamma_4 \pm \mathbb{I})/2$ and define $G_{ij}(\Gamma; p; t) := \text{tr} (\Gamma \Gamma_{ij}(p; t))$
- At zero momentum, projected correlators only contain terms for states of a single parity
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At zero momentum, projected correlators only contain terms for states of a single parity

\[
G_{ij}(\Gamma_{+}; 0; t) = \sum_{B^+} e^{-m_{B^+} t} \lambda_i^{B^+} \bar{\lambda}_j^{B^+}
\]
Introduce $\Gamma_\pm = (\gamma_4 \pm \mathbb{I})/2$ and define $G_{ij}(\Gamma_\pm; p; t) := \text{tr}(\Gamma_\pm G_{ij}(p; t))$

At zero momentum, projected correlators only contain terms for states of a single parity

\[
G_{ij}(\Gamma_+; 0; t) = \sum_{B^+} e^{-m_{B^+} t} \, \lambda_i^{B^+} \, \lambda_j^{B^+}
\]

\[
G_{ij}(\Gamma_-; 0; t) = \sum_{B^-} e^{-m_{B^-} t} \, \lambda_i^{B^-} \, \lambda_j^{B^-}
\]
Introduce $\Gamma_{\pm} = (\gamma_4 \pm \mathbb{I})/2$ and define $G_{ij}(\Gamma_{\pm}; p; t) := \text{tr}(\Gamma_{\pm} G_{ij}(p; t))$

At zero momentum, projected correlators only contain terms for states of a single parity

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G_{ij}(\Gamma_-; 0; t) = \sum_{B^-} e^{-m_{B^-} t} \lambda_i^{B^-} \bar{\lambda}_j^{B^-}
\]

Can analyse states of each parity independently
Nonzero momentum

- $O(|p|)$ opposite parity contaminations at nonzero momentum
Nonzero momentum

- $O(|p|)$ opposite parity contaminations at nonzero momentum
- Could remove single opposite parity state with

$$\Gamma_{\pm}(p) = \frac{1}{2} \left( \frac{m_{B^{\mp}}}{E_{B^{\mp}}(p)} \gamma_4 \pm \mathbb{I} \right)$$
Nonzero momentum

- $O(|p|)$ opposite parity contaminations at nonzero momentum
- Could remove single opposite parity state with

$$\Gamma_{\pm}(p) = \frac{1}{2} \left( \frac{m_B^{\mp}}{E_{B^{\mp}}(p)} \gamma_4 \pm \gamma \right)$$

- Better to use variational analysis to remove all contaminating states simultaneously
Terms in unprojected correlation matrix have Dirac structure

\[
\begin{pmatrix}
E_{B^\pm(p)} \pm m_{B^\pm} & -\sigma_k p_k \\
\sigma_k p_k & - (E_{B^\pm(p)} \mp m_{B^\pm})
\end{pmatrix}
\]
Terms in unprojected correlation matrix have Dirac structure

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Parity-Expanded Variational Analysis (PEVA)

- Terms in unprojected correlation matrix have Dirac structure

\[
\begin{pmatrix}
E_{B\pm}(p) \pm m_{B\pm} & -\sigma_k p_k \\
\sigma_k p_k & -(E_{B\pm}(p) \mp m_{B\pm})
\end{pmatrix}
\]

- Define PEVA projector

\[
\Gamma_p = \frac{1}{4}(\mathbb{1} + \gamma_4)(\mathbb{1} - i\gamma_5 \gamma_k \hat{p}_k)
\]
Terms in unprojected correlation matrix have Dirac structure

\[
\begin{pmatrix}
    E_{B\pm}(p) \pm m_{B\pm} & -\sigma_k p_k \\
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Define PEVA projector

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\Gamma_p = \frac{1}{4} (\mathbb{I} + \gamma_4)(\mathbb{I} - i\gamma_5 \gamma_k \hat{p}_k)
\]

\[
\chi_p^i := \Gamma_p \chi^i \text{ couples to positive parity states at zero momentum}
\]
Parity-Expanded Variational Analysis (PEVA)

- Terms in unprojected correlation matrix have Dirac structure

\[
\begin{pmatrix}
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- Define PEVA projector

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\Gamma_p = \frac{1}{4} (\mathbb{1} + \gamma_4)(\mathbb{1} - i\gamma_5 \gamma_k \hat{p}_k)
\]

- \(\chi_p^i := \Gamma_p \chi^i\) couples to positive parity states at zero momentum

- \(\chi_p^i' := \Gamma_p \gamma_5 \chi^i\) couples to negative parity states at zero momentum
Terms in unprojected correlation matrix have Dirac structure

\[
\begin{pmatrix}
E_{B\pm}(p) \pm m_{B\pm} & -\sigma_k p_k \\
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Define PEVA projector

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\Gamma_p = \frac{1}{4}(\mathbb{I} + \gamma_4)(\mathbb{I} - i\gamma_5\gamma_k\hat{p}_k)
\]

\(\chi^i_p := \Gamma_p \chi^i\) couples to positive parity states at zero momentum

\(\chi^{i'}_p := \Gamma_p \gamma_5 \chi^i\) couples to negative parity states at zero momentum

Both couple to states with consistent Dirac structure \(\Gamma_p u_B(p, s)\)
Lattice results

- Second lightest PACS-CS (2 + 1)-flavour full-QCD ensemble
  - $32^3 \times 64$ lattices
  - $a = 0.0951(14)$ fm by Sommer parameter
  - $\kappa_{u,d} = 0.1377$, corresponding to $m_\pi = 280(5)$ MeV
Lattice results

- Second lightest PACS-CS (2 + 1)-flavour full-QCD ensemble
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- Using conventional spin-$\frac{1}{2}$ nucleon operators

$$\chi_1 = \epsilon^{abc} [u^a \top (C \gamma_5) d^b] u^c$$
$$\chi_2 = \epsilon^{abc} [u^a \top (C) d^b] \gamma_5 u^c$$

- Apply 16, 35, 100 and 200 sweeps of gauge invariant gaussian smearing in creating the propagators
Eigenvector components

Ground state

\( \chi^+ \) Interpolators

\( \chi^- \) Interpolators

- \( \chi_1^+ = \Gamma_p \chi_1 \)
- \( \chi_1^- = \Gamma_p \gamma_5 \chi_1 \)
- \( \chi_2^+ = \Gamma_p \chi_2 \)
- \( \chi_2^- = \Gamma_p \gamma_5 \chi_2 \)

- 16 sweeps
- 35 sweeps
- 100 sweeps
- 200 sweeps

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Parity-Expanded Variational Analysis
Lattice 2016
Effective energy
Ground state - $p^2 \simeq 0.166 \text{GeV}^2$

![Graph showing effective energy $E_{\alpha}(t)$ as a function of $t/a$. The title 'Ground State' is visible at the top of the graph. The x-axis is labeled $t/a$ ranging from 16 to 34, and the y-axis is labeled $E_{\alpha}(t)$ (GeV) ranging from 0.0 to 3.0. The graph includes two curves: one representing the conventional method with $\chi^2$/dof = 0.598, and the other representing the PEVA method with $\chi^2$/dof = 0.367. The graph also includes data points with error bars for both methods.]

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Lattice 2016
Effective energy

Ground state

![Graph showing the relationship between \( E_{\text{eff}} \) (GeV) and \( p^2 \) (GeV\(^2\)). The graph compares different models, including Conv. \( \Gamma^+ \) 8 × 8 and PEVA 16 × 16.]
Eigenvector components
First negative parity excitation

\[ \chi^+ \text{ Interpolators} \]

\[ \chi^- \text{ Interpolators} \]

\[ \chi^+_1 = \Gamma_p \chi_1 \quad \chi^-_1 = \Gamma_p \gamma_5 \chi_1 \]
\[ \chi^+_2 = \Gamma_p \chi_2 \quad \chi^-_2 = \Gamma_p \gamma_5 \chi_2 \]

16 sweeps
35 sweeps
100 sweeps
200 sweeps

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Parity-Expanded Variational Analysis
Effective energy

First negative parity excitation - $p^2 \approx 0.166 \text{ GeV}^2$
Effective energy
First negative parity excitation

First Negative Parity Excitation

$E_{\text{eff}}(\alpha)$ (GeV)

$p^2$ (GeV$^2$)

Conv. $\Gamma^-$ 8 × 8
PEVA 16 × 16

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Eigenvector components
Second negative parity excitation

\[ \chi^+ \text{ Interpolators} \]

\[ \chi^- \text{ Interpolators} \]

\[ \chi^+_1 = \Gamma_p \chi_1 \]
\[ \chi^-_1 = \Gamma_p \gamma_5 \chi_1 \]
\[ \chi^+_2 = \Gamma_p \chi_2 \]
\[ \chi^-_2 = \Gamma_p \gamma_5 \chi_2 \]

\[ p^2 (\text{GeV}^2) \]

-1.0 0.0 0.5 1.0

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Effective energy
Second negative parity excitation

Second Negative Parity Excitation

Conv. □ − 8 × 8
PEVA 16 × 16

\( E_{\text{eff}}^\alpha \) (GeV)

\( p^2 \) (GeV\(^2\))

Conv. \( \Gamma^- 8 \times 8 \)
PEVA 16 \times 16
Eigenvector components

First positive parity excitation

\( p^2 (\text{GeV}^2) \)

\( \chi^+ \) Interpolators

\( \chi^- \) Interpolators

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Parity-Expanded Variational Analysis

Lattice 2016
Effective energy
First positive parity excitation

First Positive Parity Excitation
Conv. □ + 8 × 8
PEVA 16 × 16

\( E_{\text{eff}}^\alpha (\text{GeV}) \)

\( p^2 (\text{GeV}^2) \)

Conv. \( \Gamma^+ 8 \times 8 \)
PEVA \( 16 \times 16 \)
Effective energy
Nucleon spectrum

Nucleon Spectrum
Conv. □ + 8 × 8
Conv. □ − 8 × 8
PEVA 16 × 16

\begin{align*}
\text{Finn M. Stokes (CSSM)}
\end{align*}
Construct three point correlator for a single energy eigenstate

\[ G_{ij}^\mu(p', p; \Gamma; t_2, t_1) := \text{tr} \left[ \Gamma \sum_{x,y} e^{ip \cdot x} e^{i(p' - p) \cdot y} \langle \Omega | \chi^i(x) J^\mu(y) \bar{\chi}^j(0) | \Omega \rangle \right] \]

\[ G_{\alpha}^\mu(p', p; \Gamma; t_2, t_1) := v_{i}^\alpha(p') G_{ij}^\mu(p', p; \Gamma; t_2, t_1) u_{j}^\alpha(p) \]
Construct three point correlator for a single energy eigenstate

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G_{\mu}^{ij}(p', p; \Gamma; t_2, t_1) := \text{tr} \left[ \Gamma \sum_{x,y} e^{ip \cdot x} e^{i(p' - p) \cdot y} \langle \Omega | \chi^i(x) J^\mu(y) \overline{\chi}^j(0) | \Omega \rangle \right]
\]

\[
G_{\alpha}^{\mu}(p', p; \Gamma; t_2, t_1) := v_{i}^{\alpha}(p') G_{ij}^{\mu}(p', p; \Gamma; t_2, t_1) u_{j}^{\alpha}(p)
\]

Define ratio

\[
R_{\alpha}^{\mu}(p', p; t_1) := \sqrt{\frac{G_{\alpha}^{\mu}(p', p; \Gamma; t_2, t_1) G_{\alpha}^{\mu}(p, p'; \Gamma; t_2, t_1)}{G_{\alpha}(p'; t_2) G_{\alpha}(p; t_2)}}
\]
Construct three point correlator for a single energy eigenstate

\[ G_{ij}^\mu(p', p; \Gamma; t_2, t_1) := \text{tr} \left[ \Gamma \sum_{x,y} e^{ip\cdot x} e^{i(p' - p)\cdot y} \langle \Omega | \chi^i(x) J^\mu(y) \bar{\chi}^j(0) | \Omega \rangle \right] \]

\[ G_\alpha^\mu(p', p; \Gamma; t_2, t_1) := v_i^\alpha(p') G_{ij}^\mu(p', p; \Gamma; t_2, t_1) u_j^\alpha(p) \]

Define ratio

\[ R_\alpha^\mu(p', p; t_1) := \sqrt{\frac{G_\alpha^\mu(p', p; \Gamma; t_2, t_1) G_\alpha^\mu(p, p'; \Gamma; t_2, t_1)}{G_\alpha(p'; t_2) G_\alpha(p; t_2)}} \]

Extract \( G_E(Q^2) \) and \( G_M(Q^2) \) from ratio
Form Factors

Fits to ground state form factors

Ground State Form Factors

$G_E (u_p)$

$G_E (d_p)$

$G_M (u_p)$

$G_M (d_p)$

$t/a$
$G_E(Q^2 = 0.15(1) \text{GeV}^2)$ for ground state
Form Factors

\( G_M(Q^2 = 0.15(1) \text{GeV}^2) \) for ground state
Form Factors

Fits to $G_E$ for first negative parity excitation

$G_E$ for First Negative Parity Excitation

- PEVA ($u_p$)
- Conv. ($u_p$)
- PEVA ($d_p$)
- Conv. ($d_p$)
Form Factors

$G_E(Q^2 = 0.15(1)\text{GeV}^2)$ for first negative parity excitation
Form Factors

\( G_E(Q^2 = 0.15(1) \text{ GeV}^2) \) for second negative parity excitation

<table>
<thead>
<tr>
<th>( m_{\pi}^2 ) (GeV(^2))</th>
<th>( G_E )</th>
<th>Grnd State (( u_p ))</th>
<th>2nd −ve Par. Exc. (( u_p ))</th>
<th>Grnd State (( d_p ))</th>
<th>2nd −ve Par. Exc. (( d_p ))</th>
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</table>
Form Factors

Fits to $G_M$ for first negative parity excitation
Form Factors

$G_M(Q^2 = 0.15(1) \text{ GeV}^2)$ for first negative parity excitation

**Graph:**

- **Y-axis:** $G_M(\mu N)$
- **X-axis:** $m^2_{\pi} (\text{GeV}^2)$

- **Data Points:**
  - Green dots: Proton
  - Red dots: Neutron

**Legend:**

- Proton*
- Neutron*
Form Factors

$G_M(Q^2 = 0.15(1) \text{ GeV}^2)$ for second negative parity excitation
Form Factors

$G_M(Q^2 = 0.15(1) \text{GeV}^2)$ for second negative parity excitation

![Graph showing $G_M$ of Individual Quark Sectors]

- **Ground State ($u_p$)**
- **2nd $-ve$ Par. Exc. ($u_p$)**
- **Ground State ($d_p$)**
- **2nd $-ve$ Par. Exc. ($d_p$)**
Positive Parity Nucelon Spectrum

\[ m_\pi^2 (\text{GeV}^2) \]

\[ E_{\text{eff}}^\alpha (\text{GeV}) \]

- Ground State
- First Positive Parity Excitation

Parity-Expanded Variational Analysis

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Form Factors

$G_E(Q^2 = 0.15(1)\text{ GeV}^2)$ for first positive parity excitation

![Graph showing $G_E$ for First Positive Parity Excitation](image-url)
$G_M(Q^2 = 0.15(1) \text{ GeV}^2)$ for first positive parity excitation

![Graph showing $G_M$ for First Positive Parity Excitation

Proton* Neutron*](Image)

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Conventional baryon spectroscopy at nonzero momentum contaminated by opposite parity states
Conclusion

- Conventional baryon spectroscopy at nonzero momentum contaminated by opposite parity states
- The PEVA technique can effectively remove these opposite parity contaminations
Conclusion

- Conventional baryon spectroscopy at nonzero momentum contaminated by opposite parity states
- The PEVA technique can effectively remove these opposite parity contaminations
- Clear effect on two point function for lowest lying negative parity excitation

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Conventional baryon spectroscopy at nonzero momentum contaminated by opposite parity states

The PEVA technique can effectively remove these opposite parity contaminations

Clear effect on two point function for lowest lying negative parity excitation

Has significant effects on three point functions for excited states
Conventional baryon spectroscopy at nonzero momentum contaminated by opposite parity states
The PEVA technique can effectively remove these opposite parity contaminations
Clear effect on two point function for lowest lying negative parity excitation
Has significant effects on three point functions for excited states
This is an important step towards making contact with experiment through calculations such as baryon transition moments
“Parity-expanded variational analysis for nonzero momentum”
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