# Numerical determination of the $\Lambda$-parameter in SU(3) gauge theory from the twisted gradient flow coupling

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4. Summary

# 1. Introduction

- We calculate $\Lambda_{\overline{\mathrm{MS}}}$ from the twisted gradient flow (TGF) coupling.
  → Details are explained later.

$$\Lambda_{\overline{\mathrm{MS}}} = \mu(b_0 g^2_{\overline{\mathrm{MS}}}(\mu))^{-\frac{b_1}{2b_0}} \exp\left(-\frac{1}{2b_0 g^2_{\overline{\mathrm{MS}}}(\mu)}\right) \exp\left(\int_0^{g^2_{\overline{\mathrm{MS}}}(\mu)} \mathrm{d}\bar{g} \left(\frac{1}{\beta(\bar{g})} + \frac{1}{b_0\bar{g}^3} - \frac{b_1}{b_0^2 \bar{g}}\right)\right)$$

  - $b_0 = \frac{11}{3}\frac{N}{16\pi^2}$ and $b_1 = \frac{34}{3}\left(\frac{N}{16\pi^2}\right)^2$ are just constants.

- To relate $\Lambda_{\overline{\mathrm{MS}}}$ with physical quantity, we consider two hadronic scales, $A_{\mathrm{phys}}$, with a mass dimension.
  - Strategy : $\frac{\Lambda_{\overline{\mathrm{MS}}}}{A_{\mathrm{phys}}} = \frac{\Lambda_{\overline{\mathrm{MS}}}}{\Lambda_{\mathrm{SF}}} \cdot \frac{\Lambda_{\mathrm{SF}}}{\Lambda_{\mathrm{TGF}}} \cdot \frac{L_{\max}\Lambda_{\mathrm{TGF}}}{L_{\max}A_{\mathrm{phys}}}$
    - Perturbative calculations with the TGF scheme are quite complicated since we introduce the flow time.
      → Use the SF scheme as an intermediate scheme
      → Evaluate each piece numerically

- Our calculations are for the SU(3) pure gauge theory.

# 1-1. Strategy

1. Calculate $L_{\max}\Lambda_{\mathrm{TGF}}$ by the discrete beta function

$$B_s(g^2) = \frac{g^2(s/L) - g^2(1/L)}{\log(s^2)}$$

3. Calculate $\Lambda_{\mathrm{SF}}/\Lambda_{\mathrm{TGF}}$ numerically in weak coupling regime

Known [S. Sint and R. Sommer, 1996]

$$\frac{\Lambda_{\overline{\mathrm{MS}}}}{A_{\mathrm{phys}}} = \frac{\Lambda_{\overline{\mathrm{MS}}}}{\Lambda_{\mathrm{SF}}} \cdot \frac{\Lambda_{\mathrm{SF}}}{\Lambda_{\mathrm{TGF}}} \cdot \frac{L_{\max}\Lambda_{\mathrm{TGF}}}{L_{\max}A_{\mathrm{phys}}}$$

2. Relate a hadronic scale, $A_{\mathrm{phys}}$, to the scale, $L_{\max}$
   - Use the string tension and the Sommer scale as $A_{\mathrm{phys}}$
   - Data from:
     - [C. Allton, M. Teper and A. Trivini, 2008; A. González-Arroyo, M. Okawa, 2013] for string tension
     - [ALPHA collaboration, 1999; S. Necco, 2003] for Sommer scale

# 1-2. Twisted Gradient Flow coupling

- Ramos proposed to use the twisted b.c. for gradient flow as gauge b.c., and analyzed the SU(2) coupling by using the step scaling function. [A. Ramos, 2014]
- Wilson gauge action with the twisted b.c. on $x$-$y$ plane:

$$S_{\mathrm{W}}(U) = \frac{\beta}{2N} \sum_{n,\mu\neq\nu} Z_{n,\mu,\nu} \mathrm{Tr}\left[U_{n,\mu} U_{n+\mu,\nu} U^{\dagger}_{n+\nu,\mu} U^{\dagger}_{n,\nu}\right], \quad Z_{n=(1,1,n_3,n_4),1,2} = \exp\left(\frac{2\pi i}{N}\right)$$

  - Imposing the twisted b.c. is equivalent to add the factor $Z_n$ into the action.
- Flow eq.: $$\frac{\mathrm{d}V_{n,\mu}(t)}{\mathrm{d}t} = -\frac{2N}{\beta}\left\{\partial_{n,\mu} S_{\mathrm{W}}(V)\right\} V_{n,\mu}(t), \quad V_{n,\mu}(t=0) = U_{n,\mu}$$
  - We obtain the energy density, $E(t)$, from the solution of this equation.
- TGF coupling: $$g^2_{\mathrm{TGF}}(1/L) = \mathcal{N}_{\mathrm{T}}^{-1}(c, a/L) t^2 \langle E(t)\rangle\big|_{t=c^2L^2/8}$$

$$\mathcal{N}_{\mathrm{T}}^{-1}(c, a/L) = \frac{c^4}{128} \sum_{P}{}' \mathrm{e}^{-\frac{c^2L^2}{4}\hat{P}^2} \frac{\tilde{P}^2 C^2 - (\tilde{P}_{\mu} C_{\mu})^2}{\hat{P}^2}$$

  - Renormalization scale: $$\mu = \frac{1}{\sqrt{8t}} = \frac{1}{cL}$$ [M. Lüscher, 2010]
    - We set $c = 0.3$.

# 2. Lattice Setup

- Wilson gauge action with the twisted b. c.

$$S_{\rm W}(U) = \frac{\beta}{2N} \sum_{n,\mu\neq\nu} Z_{n,\mu,\nu} {\rm Tr}\left[U_{n,\mu}U_{n+\mu,\nu}U^{\dagger}_{n+\nu,\mu}U^{\dagger}_{n,\nu}\right]$$

- Gauge conf. are generated by the heat bath method
- TGF coupling, $g^2_{\rm TGF}(1/L,\beta)$

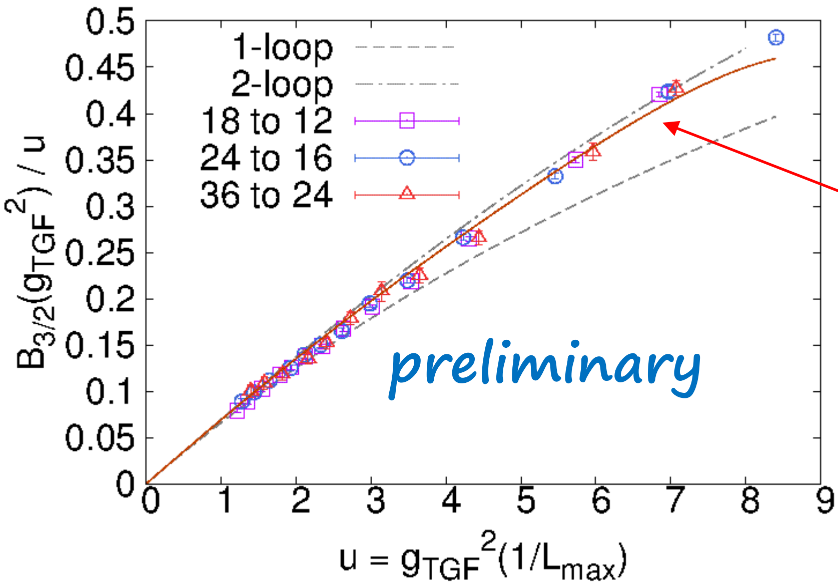
| $L/a$ | 12 | 16 | 18 | 24 | 36 |
|---|---|---|---|---|---|
| **# data points** | 14 | 14 | 15 | 14 | 11 |
| **Largest $\beta$** | 10.0 | 10.0 | 10.0 | 10.0 | 10.0 |
| **Smallest $\beta$** | 6.11 | 6.3 | 6.29 | 6.5 | 6.9 |

- Weak couplings, $g^2_{\rm SF}(1/L,\beta)$ and $g^2_{\rm TGF}(1/L,\beta)$
  - $L/a = 10, 12, 16, 18$ and $\beta = 40, 60, 80$ at each $L/a$.

# 3-1. Λ-parameter, $L_{\max}\Lambda_{\rm TGF}$

$$\frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm phys}} = \frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm SF}} \cdot \frac{\Lambda_{\rm SF}}{\Lambda_{\rm TGF}} \cdot \frac{L_{\max}\Lambda_{\rm TGF}}{L_{\max}\Lambda_{\rm phys}}$$

- Calculate the discrete beta function at each lattice and fit them to obtain the continuum discrete beta function




$$B_s(g^2) = \frac{g^2(s/L) - g^2(1/L)}{\log(s^2)}$$

$$\simeq \sigma_0 u^2 + \sigma_1 u^3 - 0.00201(27)u^4$$

$$+ 0.000371(77)u^5 + 0.0000233(57)u^6$$

$$\sigma_0 = b_0, \quad \sigma_1 = b_0 - \frac{\sigma_0^2}{\log(s^2)} \qquad (s = 3/2)$$

Proceed to the stage at which calculate $L_{\max}\Lambda_{\rm TGF}$ by using $B_{s=3/2}(u = g_{\rm TGF}^2)$

# 3-1. Λ-parameter, $L_{\rm max}\Lambda_{\rm TGF}$

$$\frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm phys}} = \frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm SF}} \cdot \frac{\Lambda_{\rm SF}}{\Lambda_{\rm TGF}} \cdot \frac{L_{\rm max}\Lambda_{\rm TGF}}{L_{\rm max}\Lambda_{\rm phys}}$$

- $L_{\rm max}$ is the box size at which hadronic scale is defined.
- Initial value for the running: $g^2_{\rm TGF}(1/L_{\rm max}) = 6.0, 6.1, \ldots, 7.0$
- Changing the scale from $1/L_{\rm max}$ to $s^n/L_{\rm max}$ and running the coupling by using $B_{3/2}(g^2_{\rm TGF})$, we obtain the following table.

| $g^2_{\rm TGF}$ | $L_{\rm max}\Lambda_{\rm TGF}$ |
|---|---|
| 6.0 | 0.576(18) |
| 6.1 | 0.585(18) |
| 6.2 | 0.594(19) |
| 6.3 | 0.602(19) |
| 6.4 | 0.610(19) |
| 6.5 | 0.618(19) |
| 6.6 | 0.626(20) |
| 6.7 | 0.634(20) |
| 6.8 | 0.641(20) |
| 6.9 | 0.649(20) |
| 7.0 | 0.656(21) |

$$u_{j+1} = u_j + B_s(u_j)\log(s^2),$$

$$u_0 = g^2_{\rm TGF}(1/L_{\rm max}), \quad u_j = g^2_{\rm TGF}(s^j/L_{\rm max})$$

$$L_{\rm max}\Lambda_{\rm TGF} \simeq s^n \left(b_0 g^2_{\rm TGF}(s^n/L_{\rm max})\right)^{-\frac{b_1}{2b_0^2}} \exp\left[-\frac{1}{2b_0 g^2_{\rm TGF}(s^n/L_{\rm max})}\right]$$

Now we have the value of $L_{\rm max}\Lambda_{\rm TGF}$ at each value of the TGF coupling.

# 3-2. Hadronic Scale

$$\frac{\Lambda_{\overline{MS}}}{\Lambda_{phys}} = \frac{\Lambda_{\overline{MS}}}{\Lambda_{SF}} \cdot \frac{\Lambda_{SF}}{\Lambda_{TGF}} \cdot \frac{L_{max}\Lambda_{TGF}}{L_{max}\Lambda_{phys}}$$

- String tension, $L_{max}\sqrt{\sigma}$, and Sommer scale, $L_{max}/r_0$
  - We evaluate them by using data from:
    - [C. Allton, M. Teper and A. Trivini, 2008; A. González-Arroyo, M. Okawa, 2013] for the string tension
    - [ALPHA collaboration, 1999; S. Necco, 2003] for the Sommer scale.

| $g^2_{TGF}$ | $L_{max}\sqrt{\sigma}$ | $L_{max}/r_0$ |
|---|---|---|
| 6.0 | 1.9302(80) | 1.7056(88) |
| 6.1 | 1.9589(79) | 1.7209(86) |
| 6.2 | 1.9866(78) | 1.7472(88) |
| 6.3 | 2.0150(79) | 1.7663(87) |
| 6.4 | 2.0470(76) | 1.7905(86) |
| 6.5 | 2.0725(77) | 1.8060(88) |
| 6.6 | 2.0966(78) | 1.8232(87) |
| 6.7 | 2.1200(79) | 1.8374(90) |
| 6.8 | 2.1443(79) | 1.8600(88) |
| 6.9 | 2.1665(80) | 1.8798(86) |
| 7.0 | 2.1911(82) | 1.8962(85) |

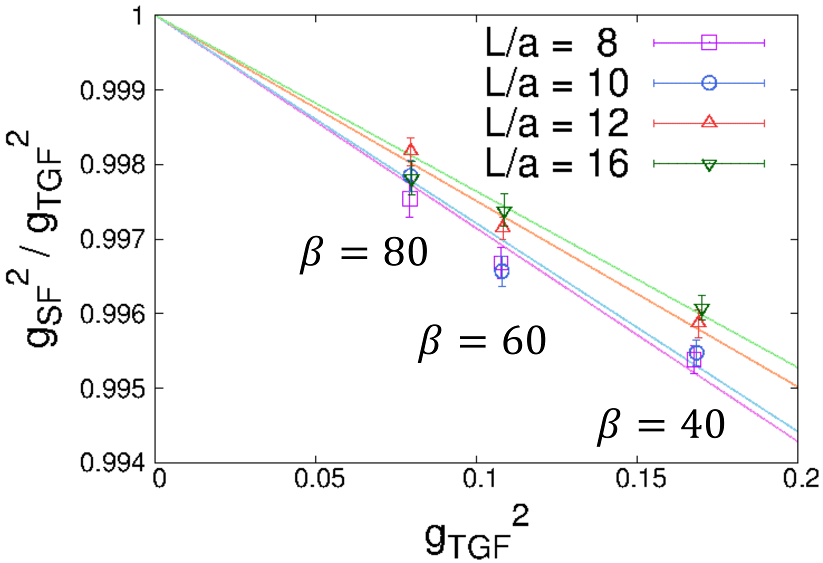
- First column: value of the TGF coupling as the renormalization condition
- Second column: value of the string tension
- Third column: value of the Sommer scale

Now we have the value of $L_{max}\sqrt{\sigma}$ and $L_{max}/r_0$ at each value of the TGF coupling.

# 3-3. Λ-parameter ratio, $\Lambda_{\mathrm{SF}}/\Lambda_{\mathrm{TGF}}$

$$\frac{\Lambda_{\overline{\mathrm{MS}}}}{\Lambda_{\mathrm{phys}}} = \frac{\Lambda_{\overline{\mathrm{MS}}}}{\Lambda_{\mathrm{SF}}} \cdot \frac{\Lambda_{\mathrm{SF}}}{\Lambda_{\mathrm{TGF}}} \cdot \frac{L_{\max}\Lambda_{\mathrm{TGF}}}{L_{\max}\Lambda_{\mathrm{phys}}}$$

- Compute $g^2_{\mathrm{SF}}$ and $g^2_{\mathrm{TGF}}$ and evaluate the ratio of the couplings




$$\frac{g^2_{\mathrm{SF}}(\beta, a/L)}{g^2_{\mathrm{TGF}}(\beta, a/L)} = 1 + c_{\mathrm{g}}(a/L) g^2_{\mathrm{TGF}}(\beta, a/L) + \ldots$$

$$c_{\mathrm{g}}(a/L) = c_{\mathrm{g}}(0) + d\left(\frac{a}{L}\right)^2 + \ldots$$

| $L/a$ | $c_g$ | $\chi^2/\mathrm{DoF}$ |
|---|---|---|
| 8 | −0.02859(92) | 1.42 |
| 10 | −0.02793(85) | 2.76 |
| 12 | −0.02492(82) | 0.98 |
| 16 | −0.02363(84) | 1.11 |

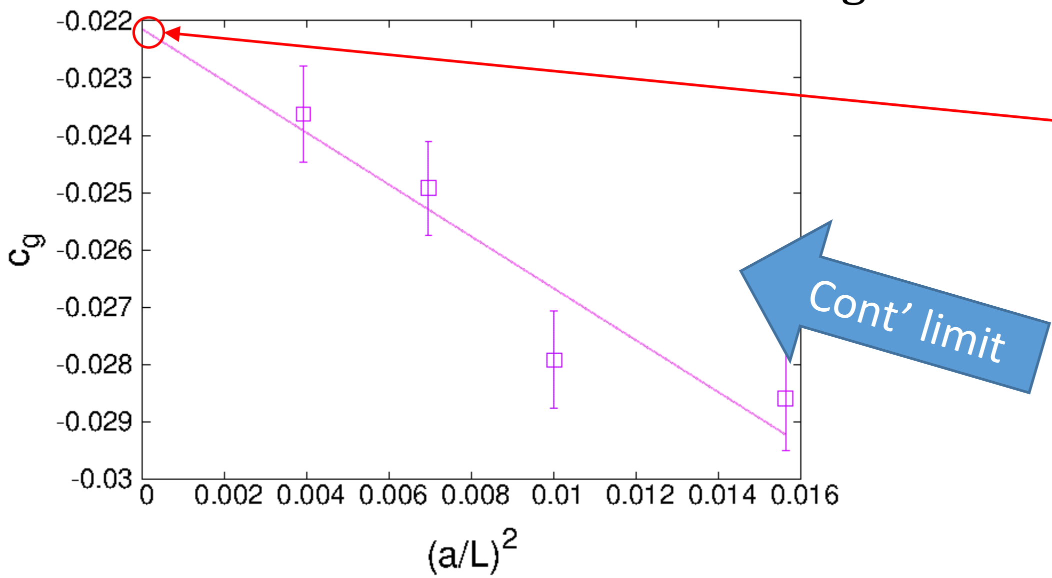
- We have the coefficient $c_{\mathrm{g}}$ at each lattice.
- Taking the continuum limit of $c_{\mathrm{g}}$, we obtain the Λ-parameter ratio.

$$\frac{\Lambda_{\mathrm{SF}}}{\Lambda_{\mathrm{TGF}}} = \exp\left(\frac{c_{\mathrm{g}}(0)}{2b_0}\right)$$

# 3-3. Λ-parameter ratio, $\Lambda_{\rm SF}/\Lambda_{\rm TGF}$

$$\frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm phys}} = \frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm SF}} \cdot \frac{\Lambda_{\rm SF}}{\Lambda_{\rm TGF}} \cdot \frac{L_{\rm max}\Lambda_{\rm TGF}}{L_{\rm max}\Lambda_{\rm phys}}.$$

- Taking the continuum limit of $c_{\rm g}$ and obtain the Λ-parameter ratio




$$c_{\rm g}(0) = -0.00221(99)$$

| $L/a$ | $c_{\rm g}$ | $\chi^2$/DoF |
|---|---|---|
| 8 | −0.02859(92) | 1.42 |
| 10 | −0.02793(85) | 2.76 |
| 12 | −0.02492(82) | 0.98 |
| 16 | −0.02363(84) | 1.11 |

Λ-parameter ratio between the SF and the TGF schemes:

$$\frac{\Lambda_{\rm SF}}{\Lambda_{\rm TGF}} = 0.8530(61)$$

# 3-4. Evaluation of $\Lambda_{\overline{\text{MS}}}$

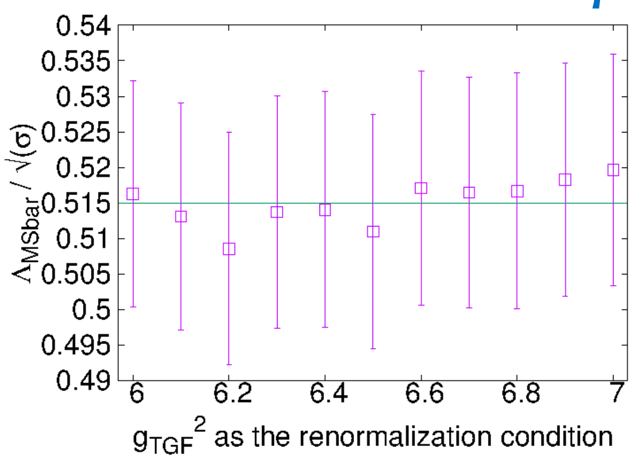
- *Ex.* Calculate $\Lambda_{\overline{\text{MS}}}/\sqrt{\sigma}$ when $g^2_{\text{TGF}} = 6.4$
  - Substituting the result so far to our strategy:
    - $L_{\max}\Lambda_{\text{TGF}} = 0.610(19)$
    - $L_{\max}\sqrt{\sigma} = 2.0392(79)$
    - $\Lambda_{\text{SF}}/\Lambda_{\text{TGF}} = 0.8530(61)$
    - $\Lambda_{\text{SF}}/\Lambda_{\overline{\text{MS}}} = 0.48811(1)$ [S. Sint and R. Sommer, 1996
  - We obtain $\Lambda_{\overline{\text{MS}}}/\sqrt{\sigma} = 0.514(17)$.

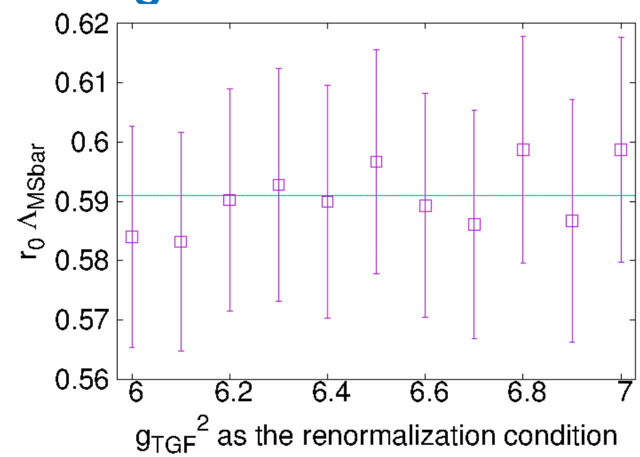
$$\frac{\Lambda_{\overline{\text{MS}}}}{\sqrt{\sigma}} = \frac{\Lambda_{\overline{\text{MS}}}}{\Lambda_{\text{SF}}} \cdot \frac{\Lambda_{\text{SF}}}{\Lambda_{\text{TGF}}} \cdot \frac{L_{\max}\Lambda_{\text{TGF}}}{L_{\max}\sqrt{\sigma}}.$$

$$r_0\Lambda_{\overline{\text{MS}}} = \frac{\Lambda_{\overline{\text{MS}}}}{\Lambda_{\text{SF}}} \cdot \frac{\Lambda_{\text{SF}}}{\Lambda_{\text{TGF}}} \cdot \frac{L_{\max}\Lambda_{\text{TGF}}}{L_{\max}/r_0}.$$

- Results of $\Lambda_{\overline{\text{MS}}}/\sqrt{\sigma}$ (left figure) and $r_0\Lambda_{\overline{\text{MS}}}$ (right figure) at each renormalization scale:

*preliminary*



# 3-4. Evaluation of $\Lambda_{\overline{\mathrm{MS}}}$

- Finally we obtain the following results (preliminary):

$$\frac{\Lambda_{\overline{\mathrm{MS}}}}{\sqrt{\sigma}} = 0.515(17)_{\mathrm{stat.}}\left({}^{+5}_{-6}\right)_{\mathrm{syst.}},$$

$$r_0\Lambda_{\overline{\mathrm{MS}}} = 0.591(19)_{\mathrm{stat.}}(7)_{\mathrm{syst.}}.$$

  - *cf.* [G. S. Bali and K. Schilling, 1993]

$$\frac{\Lambda_{\overline{\mathrm{MS}}}}{\sqrt{\sigma}} = 0.555\left({}^{+19}_{-17}\right),$$

  - *cf.* [ALPHA collaboration, 1999]

$$r_0\Lambda_{\overline{\mathrm{MS}}} = 0.602(48).$$

Our results are consistent with known values in $1.6\sigma$ for the string tension, $\Lambda_{\overline{\mathrm{MS}}}/\sqrt{\sigma}$.

# 4. Summary

- In this study

*All results are preliminary.*

  - We computed $g^2_{\mathrm{TGF}}$ in SU(3) pure gauge theory by the lattice simulation.
    - In poster session on Tuesday, Mr. E. Ibanez Bribian talks about the perturbative calculation.
  - We calculated $\Lambda_{\mathrm{SF}}/\Lambda_{\mathrm{TGF}}$ by lattice simulation:
    - $\frac{\Lambda_{\mathrm{SF}}}{\Lambda_{\mathrm{TGF}}} = 0.8530(61).$
  - We evaluated $\Lambda_{\overline{\mathrm{MS}}}/\sqrt{\sigma}$ and $r_0\Lambda_{\overline{\mathrm{MS}}}$:
    - $\frac{\Lambda_{\overline{\mathrm{MS}}}}{\sqrt{\sigma}} = 0.515(17)_{\mathrm{stat.}}\left({}^{+5}_{-6}\right)_{\mathrm{syst.}}, r_0\Lambda_{\overline{\mathrm{MS}}} = 0.591(19)_{\mathrm{stat.}}(7)_{\mathrm{syst.}}.$
    - This results are consist of $\Lambda_{\mathrm{SF}}/\Lambda_{\mathrm{TGF}}$ computation, i.e. they support the validity of the $\Lambda_{\mathrm{SF}}/\Lambda_{\mathrm{TGF}}$ from our lattice simulation.
- We conclude that the twisted gradient flow method actually works as one of the renormalization scheme in pure QCD.