A first look at Staggered Domain Wall Fermions

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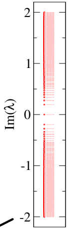
Lattice fermions



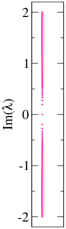
Naive fermions

$D_N = \gamma_\mu D_\mu$

16 species

remove 4-fold degeneracy

add Wilson term



Staggered fermions

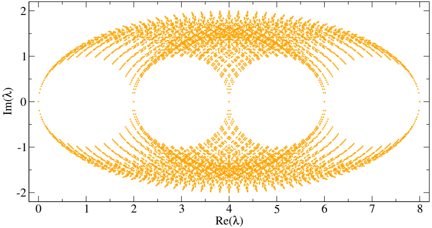
$D_{st} = \eta_\mu D_\mu$

4 species

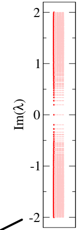
Wilson fermions

$D_W = \gamma_\mu D_\mu + rW$

1+4+6+4+1 species



Lattice fermions



Naive fermions

$D_{\mathrm{N}} = \gamma_\mu D_\mu$

16 species

DW, Overlap fermions

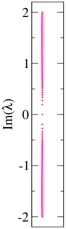
remove 4-fold degeneracy

add Wilson term

Wilson fermions

$D_{\mathrm{W}} = \gamma_\mu D_\mu + rW$

1+4+6+4+1 species



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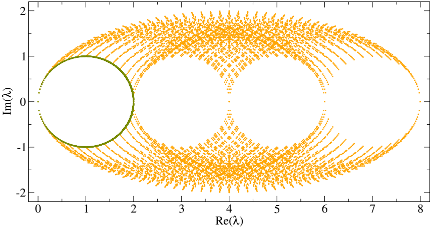


Staggered Wilson term construction

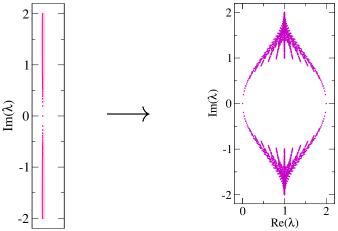
(Golterman, Smit, 1984; Adams, 2010; C.H. 2010; deForcrand et.al. 2010,2012; Durr 2012)

$$A^{\mu_1 \ldots \mu_{2n}} = i^n \eta_{\mu_1} \ldots \eta_{\mu_{2n}} \Gamma_{\mu_1 \ldots \mu_{2n}} \left(C_{\mu_1} \ldots C_{\mu_{2n}}\right)_{\text{sym}}$$

- $\{D_{\text{st}}, \epsilon\} = 0$ and $D_{\text{st}}^\dagger = -D_{\text{st}}$
- Mass term: $[A, \epsilon] = 0$, $A^\dagger = A$
- ➛ $D_{\text{A}} \epsilon = \epsilon D_{\text{A}}^\dagger$
- ➛ $\lambda_i = \lambda_{i*}^*$, real determinant

$$C_\mu := \frac{1}{2}\left(V_\mu + V_\mu^\dagger\right)$$

$$(V_\mu)_{xy} := U_\mu(x) \delta_{x+\hat{\mu}, y}$$



- $\Gamma_{\mu_1 \ldots \mu_{2n}} = \epsilon_{\mu_1 \ldots \mu_{2n}} (-1)^{\sum_i x_{\mu_i}} \sim (\gamma_{\mu_1} \ldots \gamma_{\mu_{2n}} \otimes \xi_{\mu_1} \ldots \xi_{\mu_{2n}})$
- $\eta_\mu = (-1)^{\sum_{\nu<\mu} x_\nu} \sim (\gamma_\mu \otimes 1)$
- $\epsilon = (-1)^{\sum_\mu x_\mu} \sim (\gamma_5 \otimes \xi_5)$
- $\{C_\mu, \epsilon\} = 0$
- $A^{\mu_1 \ldots \mu_{2n}} \sim (1 \otimes \xi_{\mu_1} \ldots \xi_{\mu_{2n}}) + O(a)$

Staggered DW fermions

- Domain wall operator (Kaplan 1992; Shamir 1993; Furman, Shamir 1994)

$$\bar{\Psi} D_{\mathrm{DW}} \Psi = \sum_{s=1}^{N_s} \bar{\Psi}_s \left(D_W^{+} \Psi_s - P_{-} \Psi_{s+1} - P_{+} \Psi_{s+1} \right)$$

with $P_{\pm} = \frac{1}{2}(1 \pm \gamma_5)$, $D_W^{\pm} = D_W(-M_0) \pm 1$

- Boundary conditions with mass term:

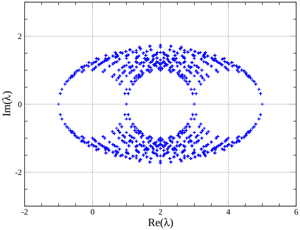
$$P_{+} \left(\Psi_0 + m \Psi_{N_s} \right) = 0 \qquad P_{-} \left(\Psi_{N_s+1} + m \Psi_1 \right) = 0$$

Boriçi's modification (Boriçi, 1999)

$$P_{\pm} \Psi_{s\mp1} \rightarrow -D_W^{-} P_{\pm} \Psi_{s\mp1}$$

Optimal DWF (Chiu, 2002)

$$D_W^{\pm} \rightarrow D_W^{\pm}(s) = \omega_s D_W(-M_0) \pm 1$$



- Staggered versions (Adams, 2011)

$$D_W \rightarrow D_A$$

$$\gamma_5 \rightarrow \epsilon \sim (\gamma_5 \otimes \xi_5)$$

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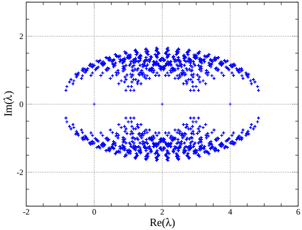

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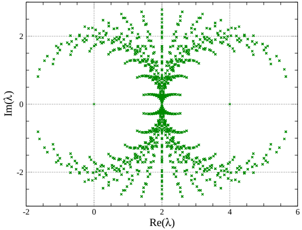

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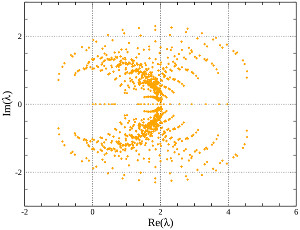

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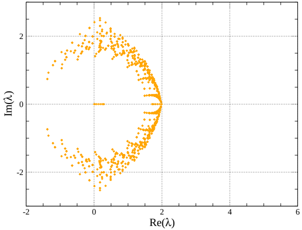

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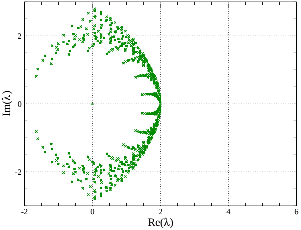

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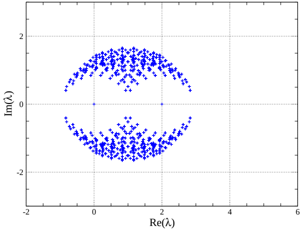

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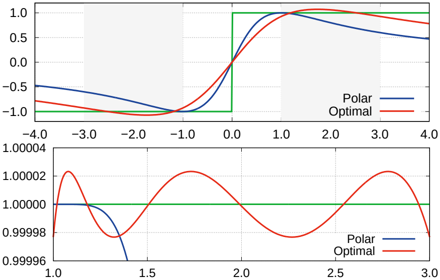
Numerical setup

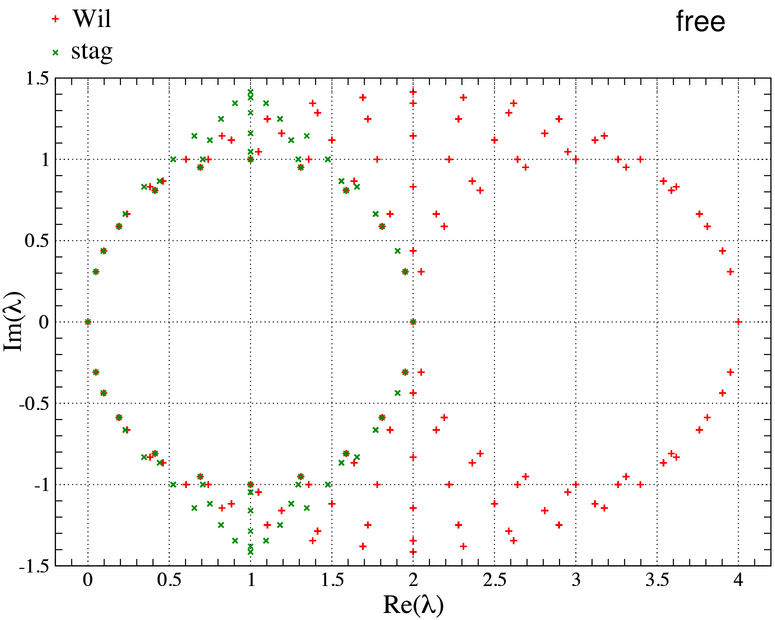
- Schwinger model (QED_2)
- Wilson gauge action
- Unsmeared and 3-APE ($\alpha = 0.5$)

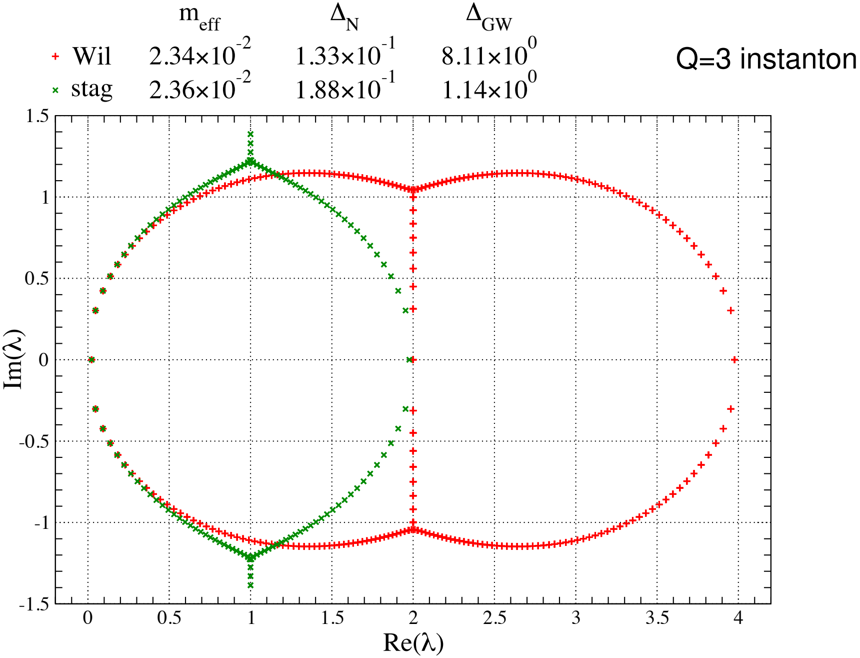
β	$N_x \times N_t$	#conf
0.8	8×8	1000
1.8	12×12	1000
3.2	16×16	1000
5	20×20	1000
7.2	24×24	1000
9.8	28×28	1000
12.8	32×32	1000

Observables:

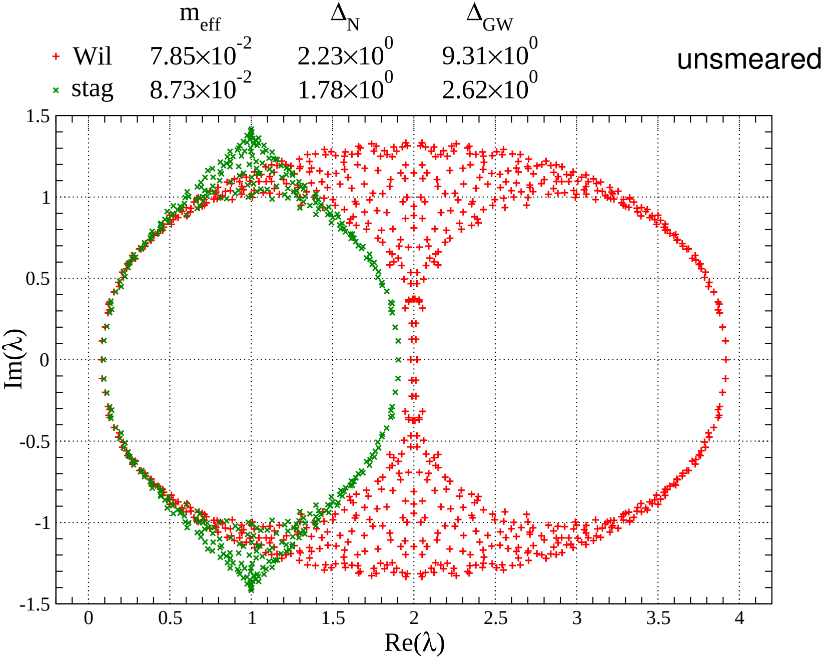
- Normality violation $\Delta_N = ||[D, D^\dagger]||$
- GW violation $\Delta_{GW} = ||\gamma_5 D - D\hat{\gamma}_5||$
- Residual mass $m_{eff} = |\lambda_{min}|$

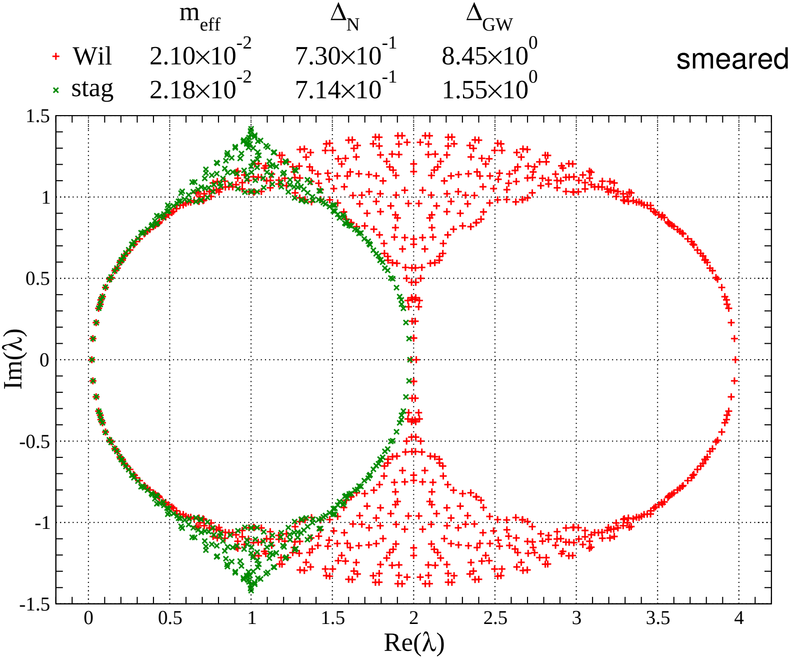



Wil
stag
free
1.5
1
0.5
0
-0.5
-1
-1.5
Im(λ)
0
0.5
1
1.5
2
2.5
3
3.5
4
Re(λ)



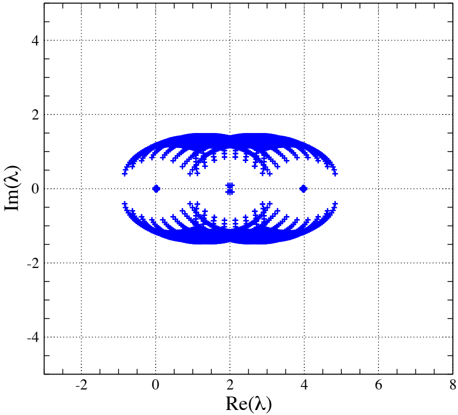
m_eff Δ_N Δ_GW
Wil 2.34×10^-2 1.33×10^-1 8.11×10^0
stag 2.36×10^-2 1.88×10^-1 1.14×10^0
Q=3 instanton
Im(λ)
Re(λ)
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.5
1
1.5
2
2.5
3
3.5
4


m_{eff} Δ_N Δ_{GW}
Wil 7.85×10^{-2} 2.23×10^{0} 9.31×10^{0}
stag 8.73×10^{-2} 1.78×10^{0} 2.62×10^{0}
unsmeared
Im(λ)
Re(λ)
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.5
1
1.5
2
2.5
3
3.5
4

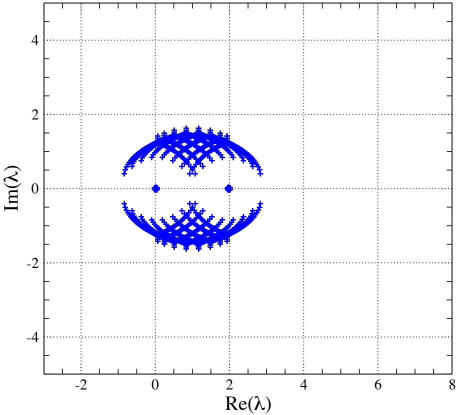

m_eff Δ_N Δ_GW
+ Wil 2.10×10^-2 7.30×10^-1 8.45×10^0
× stag 2.18×10^-2 7.14×10^-1 1.55×10^0
smeared
1.5
1
0.5
0
-0.5
-1
-1.5
Im(λ)
0 0.5 1 1.5 2 2.5 3 3.5 4
Re(λ)

Bulk operators

Wilson kernel



Staggered Wilson kernel



Q=3 instanton

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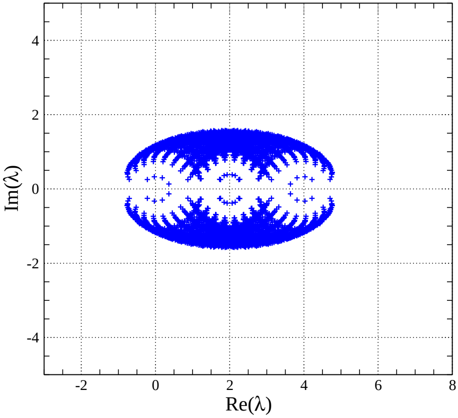

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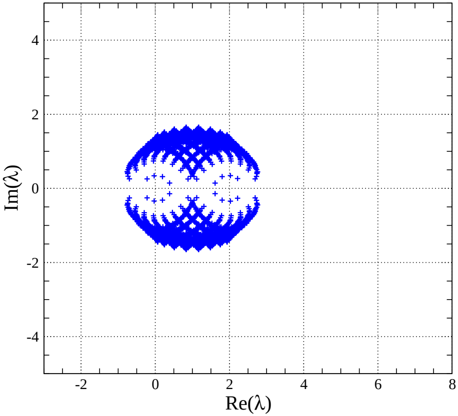

standard DWF

Bulk operators

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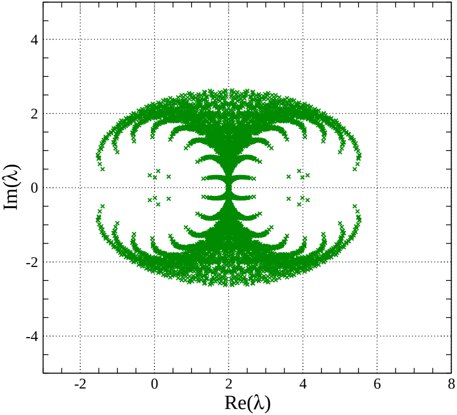

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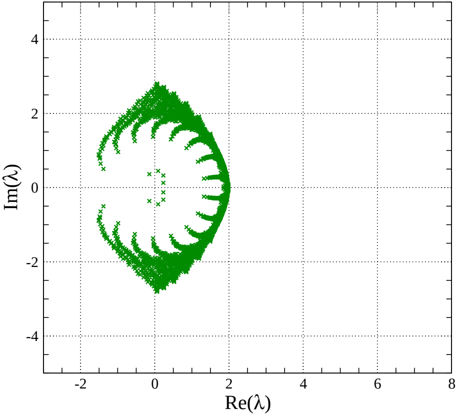

Boriçi

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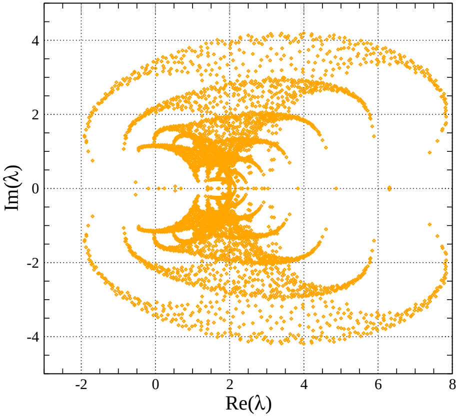

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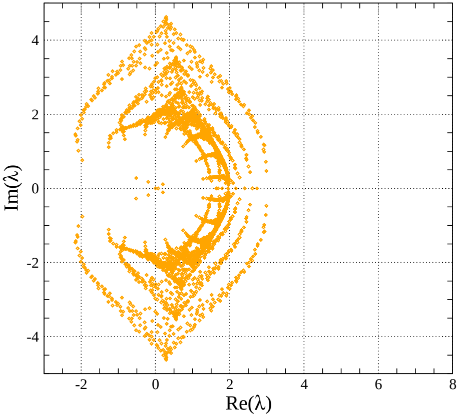
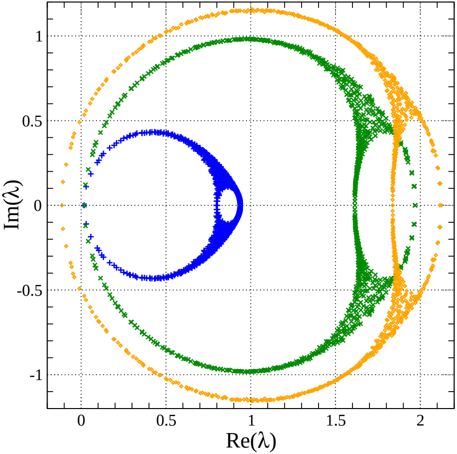


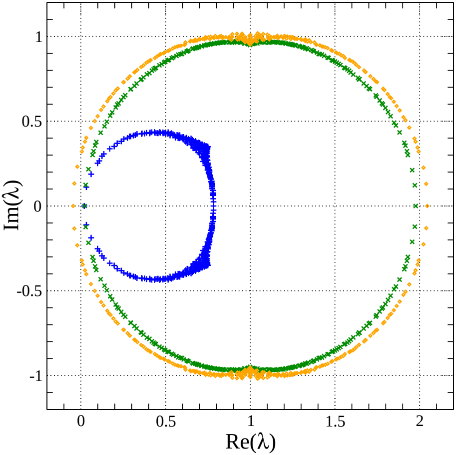
optimal

	m_{eff}	Δ_N	Δ_{GW}
+ std	1.75×10^{-2}	2.38×10^{-1}	8.05×10^{-1}
× Bor	1.95×10^{-2}	8.84×10^{-1}	1.44×10^{0}
◇ opt	1.12×10^{-1}	1.43×10^{0}	1.80×10^{0}



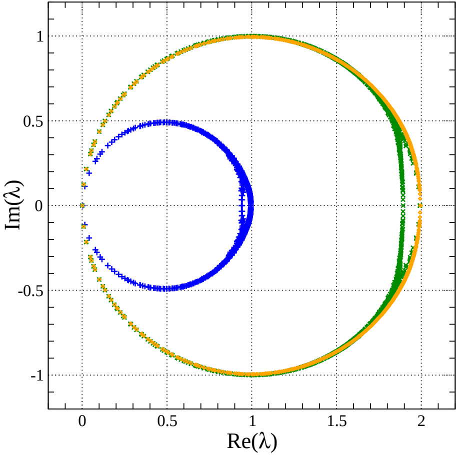
Wilson kernel

	m_{eff}	Δ_N	Δ_{GW}
+ std	2.01×10^{-2}	2.40×10^{-1}	7.74×10^{-1}
× Bor	2.19×10^{-2}	6.68×10^{-1}	7.04×10^{-1}
◇ opt	4.57×10^{-2}	5.59×10^{-1}	5.55×10^{-1}



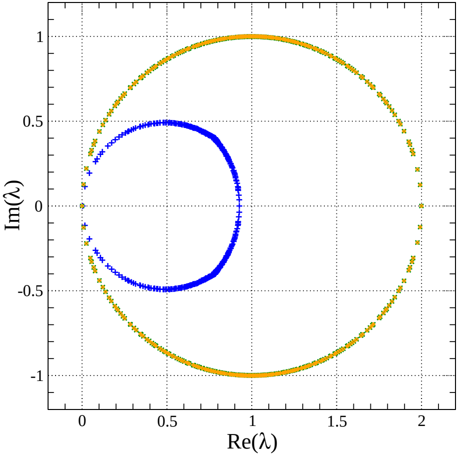
Staggered Wilson kernel

	m_{eff}	Δ_N	Δ_{GW}
std	3.04×10^{-3}	1.72×10^{-1}	5.11×10^{-1}
Bor	6.87×10^{-4}	1.42×10^{-1}	3.30×10^{-1}
opt	1.12×10^{-3}	7.89×10^{-2}	1.12×10^{-1}



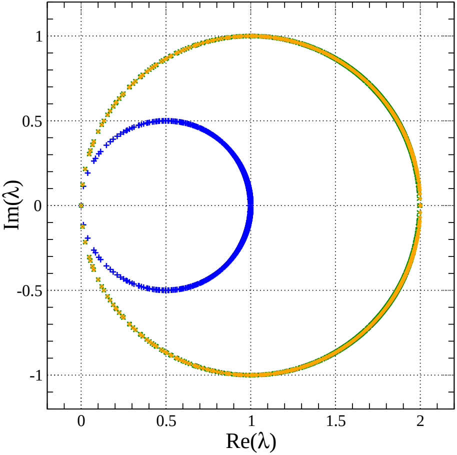
Wilson kernel

	m_{eff}	Δ_N	Δ_{GW}
std	3.71×10^{-3}	1.51×10^{-1}	4.67×10^{-1}
Bor	7.69×10^{-4}	8.73×10^{-2}	8.59×10^{-2}
opt	2.90×10^{-4}	1.17×10^{-2}	1.39×10^{-2}



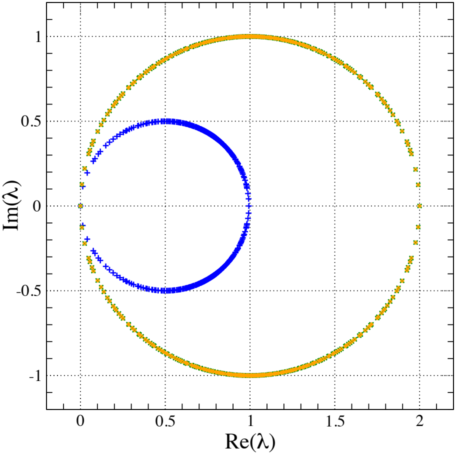
Staggered Wilson kernel

	m_{eff}	Δ_N	Δ_{GW}
std	2.00×10^{-4}	4.05×10^{-2}	1.08×10^{-1}
Bor	4.74×10^{-6}	5.39×10^{-3}	1.92×10^{-2}
opt	3.36×10^{-6}	2.57×10^{-4}	3.39×10^{-4}



Wilson kernel

	m_{eff}	Δ_N	Δ_{GW}
std	2.82×10^{-4}	3.26×10^{-2}	9.12×10^{-2}
Bor	3.55×10^{-6}	1.03×10^{-3}	1.01×10^{-3}
opt	$<3\times10^{-8}$	5.03×10^{-6}	5.71×10^{-6}



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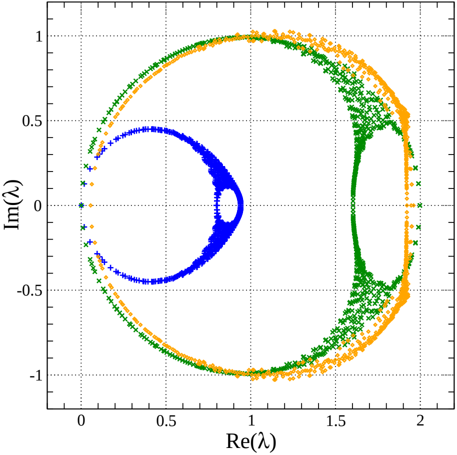
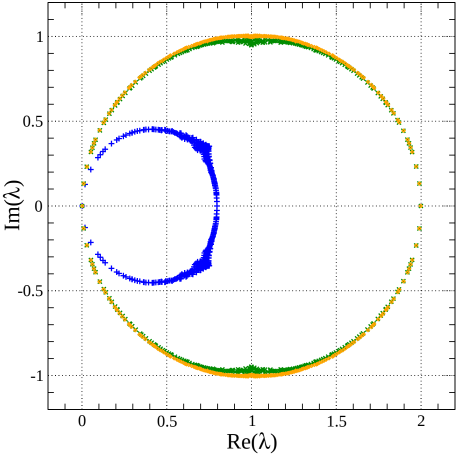
	m_{eff}	Δ_N	Δ_{GW}
+ std	1.50×10^{-3}	7.20×10^{-2}	5.53×10^{-1}
× Bor	1.01×10^{-3}	1.28×10^{-1}	1.03×10^{0}
◇ opt	5.57×10^{-2}	3.32×10^{-1}	8.05×10^{-1}



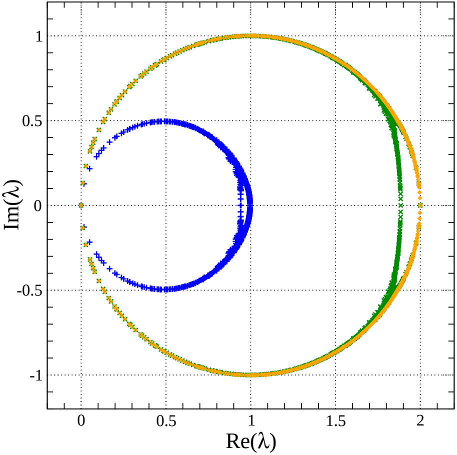
Smeared Wilson kernel

	m_{eff}	Δ_N	Δ_{GW}
+ std	1.55×10^{-3}	7.08×10^{-2}	5.36×10^{-1}
× Bor	1.03×10^{-3}	5.70×10^{-2}	1.79×10^{-1}
◇ opt	1.92×10^{-3}	9.73×10^{-2}	1.38×10^{-1}



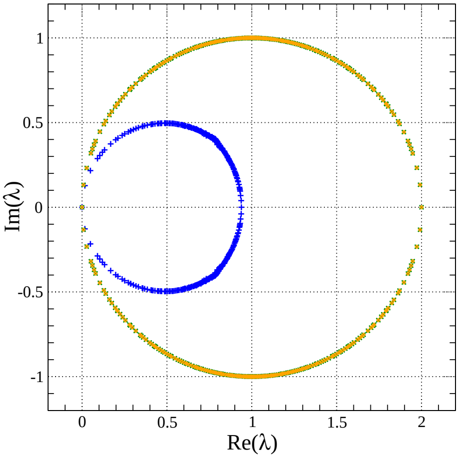
Smeared staggered Wilson kernel

	m_{eff}	Δ_N	Δ_{GW}
std	5.22×10^{-5}	2.41×10^{-2}	2.27×10^{-1}
Bor	4.59×10^{-6}	2.96×10^{-2}	2.78×10^{-1}
opt	1.42×10^{-5}	1.67×10^{-2}	3.04×10^{-2}



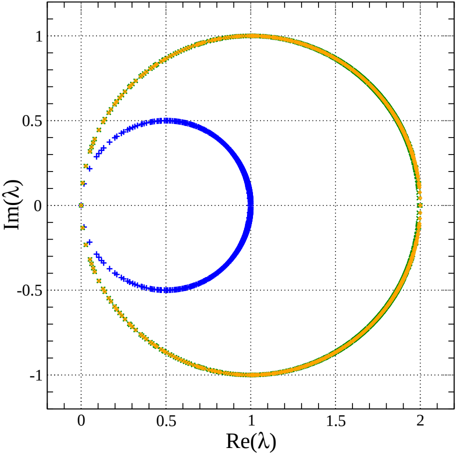
Smeared Wilson kernel

	m_{eff}	Δ_N	Δ_{GW}
std	5.19×10^{-5}	2.29×10^{-2}	1.97×10^{-1}
Bor	1.16×10^{-6}	1.12×10^{-3}	5.97×10^{-3}
opt	1.47×10^{-5}	4.40×10^{-4}	6.06×10^{-4}



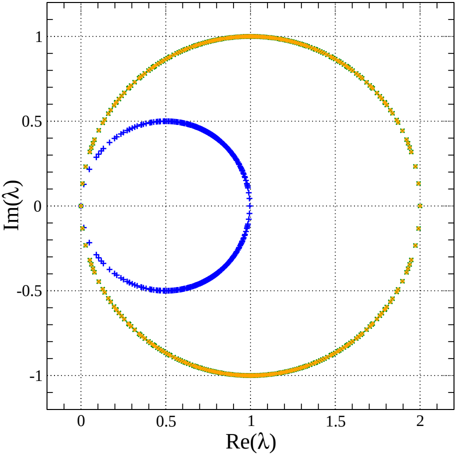
Smeared staggered Wilson kernel

	m_{eff}	Δ_N	Δ_{GW}
+ std	3.16×10^{-7}	1.89×10^{-3}	1.90×10^{-2}
× Bor	1.47×10^{-7}	2.03×10^{-3}	1.79×10^{-2}
◆ opt	1.88×10^{-7}	1.18×10^{-5}	1.88×10^{-5}



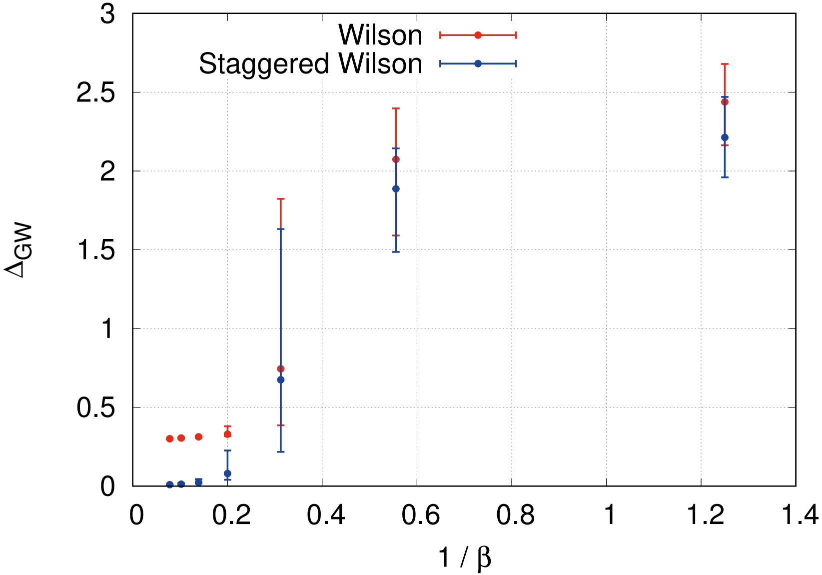
Smeared Wilson kernel

	m_{eff}	Δ_N	Δ_{GW}
+ std	3.31×10^{-7}	1.96×10^{-3}	1.66×10^{-2}
× Bor	1.60×10^{-8}	8.01×10^{-7}	6.48×10^{-6}
◆ opt	8.29×10^{-9}	5.67×10^{-9}	7.07×10^{-9}

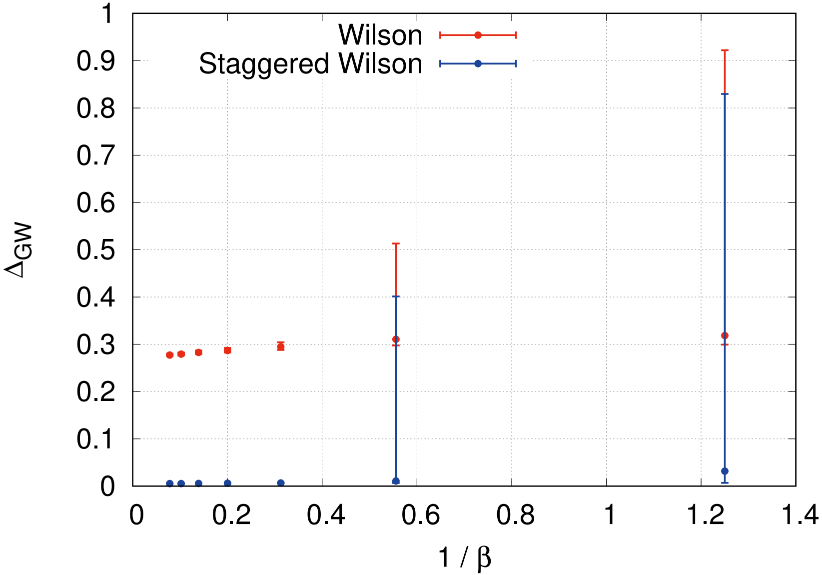


Smeared staggered Wilson kernel

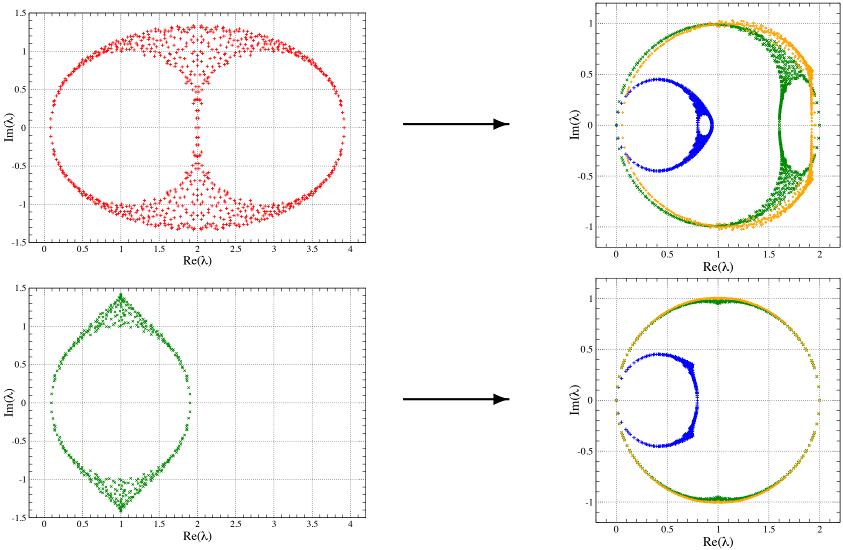
Continuum behaviour



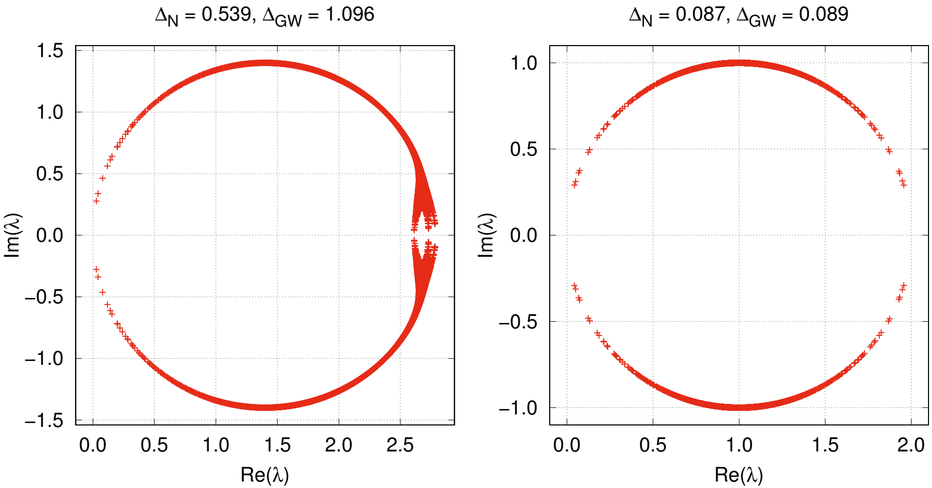
Smeared continuum behaviour



Why the good chiral properties?

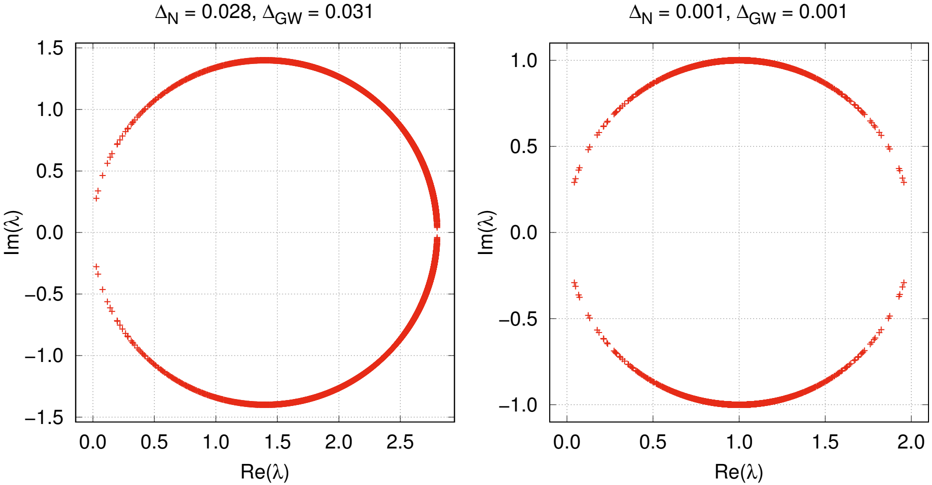


QCD, $6^4 \times 8$, $\beta = 6$, APE smeared



Boriçi Wilson Boriçi staggered

QCD, $6^4 \times 8$, $\beta = 6$, APE smeared



Optimal Wilson

Optimal staggered

Could staggered DWF be useful?

- ✓ Construction is straightforward
- ✓ Good chiral symmetry properties
- ✓ Easily parallelizable
  - ➢ Bulk/chiral properties of $N_f = 2$ QCD?

- ✗ Counterterms for $N_f = 1$ (Sharpe 2012)
- ✗ Staggered spin structure