New polynomially exact integration rules on $U(N)$ and $SU(N)$

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2016-07-25

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We consider the 0 + 1 dimensional QCD with chemical potential. Dirac operator for a quark of mass m at chemical potential μ:

$$\mathfrak{D}(U) = \begin{pmatrix} m & \frac{e^{\mu}}{2}U_1 & & & \frac{e^{-\mu}}{2}U_n^* \\ -\frac{e^{-\mu}}{2}U_1^* & m & \frac{e^{\mu}}{2}U_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{e^{-\mu}}{2}U_{n-2}^* & m & \frac{e^{\mu}}{2}U_{n-1} \\ -\frac{e^{\mu}}{2}U_n & & & -\frac{e^{-\mu}}{2}U_{n-1}^* & m \end{pmatrix}$$

N_f flavor partition function:

$$Z(m,\mu,G,n) = \int_{G^n} \det \mathfrak{D}(U)^{N_f} \; dh_G^n(U)$$

with $G \in \{SU(N), U(N)\}$ and h_G the (normalized) Haar measure on G.

Choice of Gauge: $U_j = 1$ except $U = U_n$. Then,

$$\det \mathfrak{D} = \det\left(\prod_{j=1}^{n} \tilde{m}_j + 2^{-n} e^{-n\mu} U^* + (-1)^n 2^{-n} e^{n\mu} U\right)$$

with $\tilde{m}_1 := m$,

$$\forall j \in [2, n-1] \cap \mathbb{N}: \ \tilde{m}_j := m + \frac{1}{4\tilde{m}_{j-1}},$$

and

$$\tilde{m}_n := m + \frac{1}{4\tilde{m}_{n-1}} + \sum_{j=1}^{n-1} \frac{(-1)^{j+1} 2^{-2j}}{\tilde{m}_j \prod_{k=1}^{j-1} \tilde{m}_k^2}.$$

0 + 1 dimensional QCD for a quark at non-zero chemical potential

$$\int_G \det\left(\prod_{j=1}^{n} \tilde{m}_j + 2^{-n} e^{-n\mu} U^* + (-1)^n 2^{-n} e^{n\mu} U\right)^{N_f} dh_G(U)$$

is a highly oscillating integral.

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The sign problem is very problematic for m small and $n\mu$ large.

Markov Chain Monte Carlo is unfeasible!

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Idea

Use polynomially exact quadrature rules over $U(N)$, $SU(N)$.

We will consider ($N_f = 1$)

$$Z(m,\mu,G,n) = \int_G \det \mathfrak{D}\ dh_G$$

and the chiral condensate

$$\partial_m \ln Z(m,\mu,G,n) = \frac{\int_G \partial_m \det \mathfrak{D}\ dh_G}{\int_G \det \mathfrak{D}\ dh_G}$$

with

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Solution: Symmetrization?

Bloch, Bruckmann, Wettig (2013)

Center Symmetry of $SU(3)$: Choosing quadrature rules Q satisfying

$$\forall U \in Q: \; U^*, e^{\frac{2\pi i}{3}} U, e^{\frac{4\pi i}{3}} U \in Q$$

works well for $N_f \leq 5$ (but still needs Monte Carlo).

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Can we construct polynomially exact quadrature rules?

We want *completely symmetric* quadrature rules.

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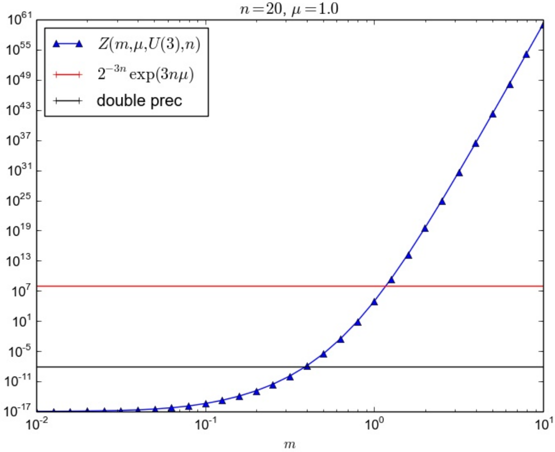
- polynomials in $SU(N)$ of degree $\leq k$ are mapped to “unitary polynomials” of degree $\leq k$ in $\bigtimes_{j=1}^{N-1} S^{2j+1}$

Genz (2003)

(Randomized) Completely symmetric (and thus polynomially exact) quadrature rules on spheres S^n can be constructed explicitly.

computing the partition function $Z(m, \mu, G, n) = \int_G \det \mathfrak{D} \; dh_G$

$$\det \mathfrak{D}(U) = \det\left(\prod_{j=1}^{n} \tilde{m}_j + 2^{-n} e^{-n\mu} U^* + (-1)^n 2^{-n} e^{n\mu} U\right)$$



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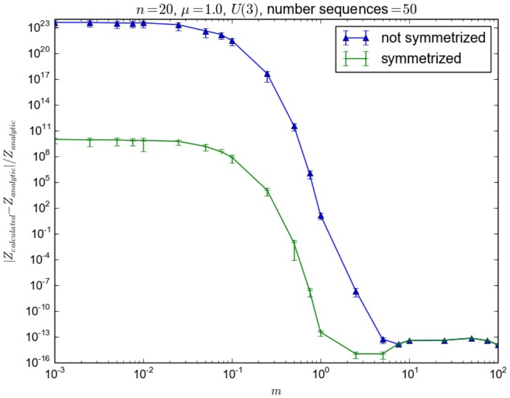

Note that the symmetrized (polynomially exact) quadrature rule yields double precision results for all values of m.

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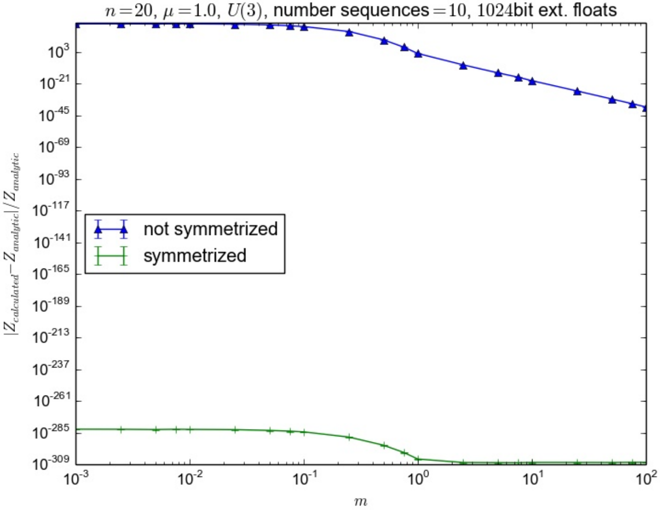
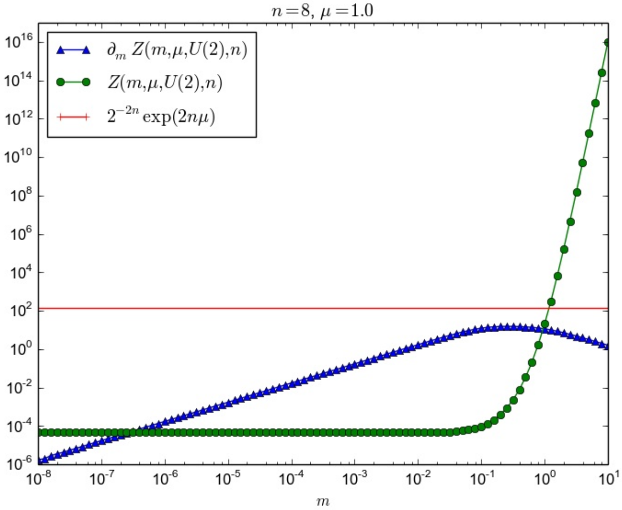


computing the chiral condensate $\partial_m \ln Z(m,\mu,G,n) = \partial_m Z/Z = \int_G \partial_m \det \mathfrak{D}\ dh_G \ / \ \int_G \det \mathfrak{D}\ dh_G$



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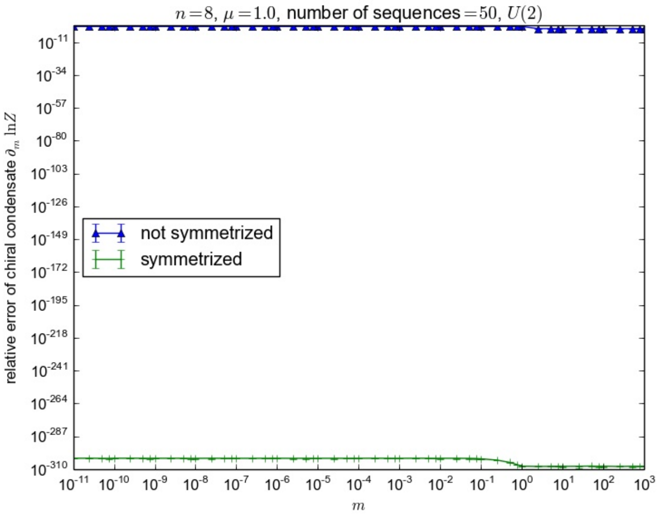

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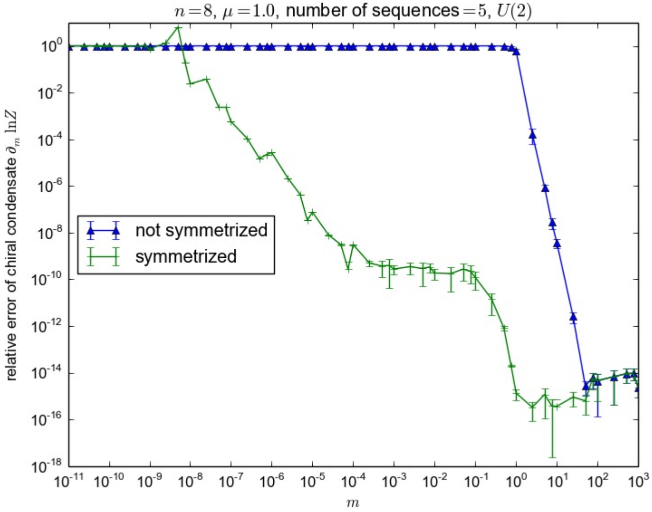
arXiv:1607.05027 [hep-lat]


$n = 8$, $\mu = 1.0$, number of sequences $= 5$, $U(2)$
relative error of chiral condensate $\partial_m \ln Z$
m
not symmetrized
symmetrized
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-11} 10^{-10} 10^{-9} 10^{-8} 10^{-7} 10^{-6} 10^{-5} 10^{-4} 10^{-3} 10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3}