Hadron Structure using the Feynman-Hellmann Theorem

Alexander Chambers
The University of Adelaide
QCDSF-UKQCD/CSSM Collaborations

Southampton
July 25, 2016

Outline

Want to demonstrate a Feynman-Hellmann approach to calculating non-forward matrix elements

Particularly form factors of the nucleon and pion

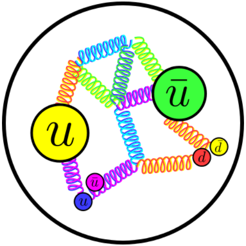
Acknowledgements

- Ross Young (Adelaide)
- James Zanotti (Adelaide)
- Roger Horsley (Edinburgh)
- Paul Rakow (Liverpool)
- Yoshifumi Nakamura (Riken)
- Holger Perlt (Leipzig)
- Arwed Schiller (Leipzig)
- Dirk Pleiter (Jülich)
- Gerrit Schierholz (Hamburg)

Electromagnetic Form Factors

Hadrons are composite particles

How is charge/magnetisation distributed?

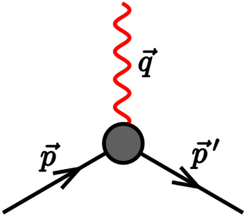


Need to determine vector matrix elements

$$\langle H(\mathbf{p}') \,|\, \mathcal{J}(0) \,|\, H(\mathbf{p}) \rangle = \ldots ?$$

Full analytic form unknown

Parameterise amplitude by form factors

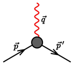


e.g. pion has single form factor

$$\langle \pi(\mathbf{p}') \,|\, \mathcal{J}(0) \,|\, \pi(\mathbf{p}) \rangle \propto F_\pi(Q^2), \quad Q^2 = -(p' - p)^2$$

- Fourier transform of transverse charge density
- Slope at $Q^2 = 0 \to$ charge radius

Electromagnetic Form Factors — Pion



$$\langle\, H(\mathbf{p}') \,|\, \mathcal{J}(0) \,|\, H(\mathbf{p}) \,\rangle = \ldots F(Q^2) + \ldots \qquad Q^2 = (p' - p)^2$$

What can experiment tell us?

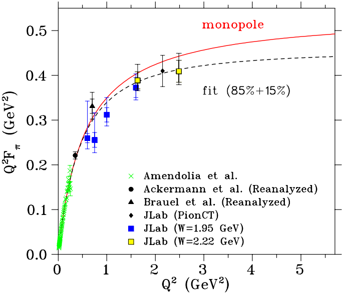
- Low Q^2: π^+ scattering by atomic e^-
- High Q^2: π electroproduction off nucleon

High-Q^2 measurement difficult

Less high-Q^2 data $\Longrightarrow$ less fine-detail information

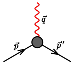
Ongoing experimental efforts

e.g. 12 GeV upgrade at JLab



[Jefferson Lab (PRC 2008)]

Electromagnetic Form Factors — Proton



$$\langle\, H(\mathbf{p}') \,|\, \mathcal{J}(0) \,|\, H(\mathbf{p}) \,\rangle = \ldots F(Q^2) + \ldots \qquad Q^2 = (p' - p)^2$$

Write nucleon matrix element in terms of Sachs EM form factors

$$\langle\, N(\mathbf{p}') \,|\, \mathcal{J}(0) \,|\, N(\mathbf{p}) \,\rangle = \ldots G_E(Q^2) + \ldots G_M(Q^2)$$

Non-relavistically, Fourier Transforms of charge/magnetisation

For the proton

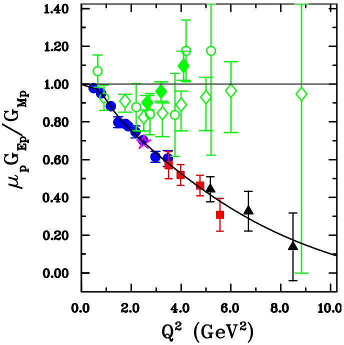
- Early experiments → low G_E sensitivity
- Double polarisation → $\frac{G_E}{G_M}$ directly

Zero crossing in $(G_E/G_M)_p$?

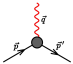
Central negatively charged region?

Require more high-Q^2 data

Can lattice help?



[Punjabi et al. (EPJA 2015)]

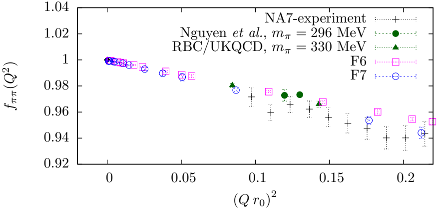


$$\langle H(\mathbf{p}') \,|\, \mathcal{J}(0) \,|\, H(\mathbf{p}) \rangle = \ldots F(Q^2) + \ldots \qquad Q^2 = (p' - p)^2$$

What can lattice tell us?

Low-Mid Q^2: Good progress

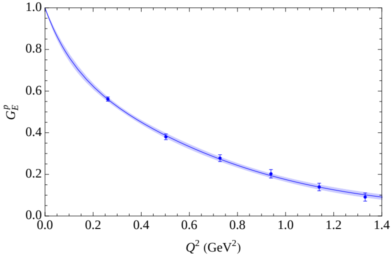
- Clean extraction of form factors



[Brandt et al. (HEP 2013)]

High Q^2: More Difficult

- Low signal/noise ratios



[Shanahan et al. (PRD 2014)]

What can a Feynman-Hellmann approach offer?

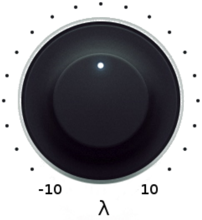
Feynman-Hellmann Recipe (Forward Case)

How to calculate $\langle\, H(\mathbf{p}) \,|\, \mathcal{O}(0) \,|\, H(\mathbf{p}) \,\rangle$?

1. **Add term to Lagrangian**

$$\mathcal{L} \to \mathcal{L} + \lambda \mathcal{O}$$

2. **Measure hadron energy while changing λ**



$$G(\lambda; \mathbf{p}; t) = \int \mathrm{d}x \, e^{-i\mathbf{p}\cdot\mathbf{x}} \langle\, \chi'(x)\chi(0) \,\rangle \overset{\text{large } t}{\propto} e^{-E_H(\lambda,\mathbf{p})t}$$

3. **Calculate matrix element from energy shifts**

$$\left.\frac{\partial E_H(\lambda, \mathbf{p})}{\partial \lambda}\right|_{\lambda=0} = \frac{1}{2E_H(\mathbf{p})} \langle\, H(\mathbf{p}) \,|\, \mathcal{O}(0) \,|\, H(\mathbf{p}) \,\rangle$$

Calculation of matrix element $\to$ hadron spectroscopy

Only need to calculate two-point functions!

Feynman-Hellmann Recipe (Forward Case)

$\mathcal{L} \to \mathcal{L} + \lambda \mathcal{O}$

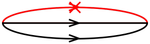
$\partial E_H / \partial \lambda |_{\lambda=0} \propto \langle\, H(\mathbf{p}) \,|\, \mathcal{O}(0) \,|\, H(\mathbf{p}) \,\rangle$

Where should the Lagrangian be modified?

$$\langle\, \mathcal{O} \,\rangle = \frac{1}{N} \sum_{i=1}^{N} \overline{\mathcal{O}}[A_i] \quad \text{where} \quad \begin{array}{l} P(A_i) \propto \det[M] e^{-S_g} \\ S^{ab}_{\mu\nu}(x,y) = {M^{-1}}^{ab}_{\mu\nu}(x,y) \end{array}$$

Modify propagators
$\rightarrow$ connected contribution

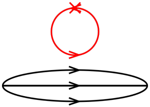
$$\langle\, \chi'(x)\, \mathcal{O}(y)\, \chi(0) \,\rangle$$



- Cheap to implement

Modify gauge fields
$\rightarrow$ disconnected contribution

$$\langle\, \chi'(x)\, \mathcal{O}(y)\, \chi(0) \,\rangle$$



- Require new gauge fields

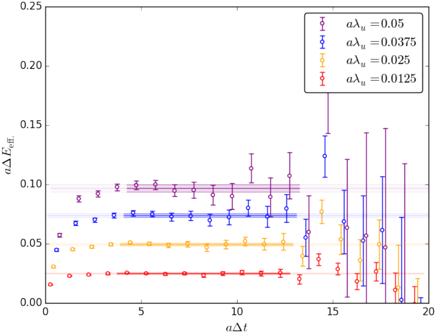
Previously applied to calculation of quark axial charges

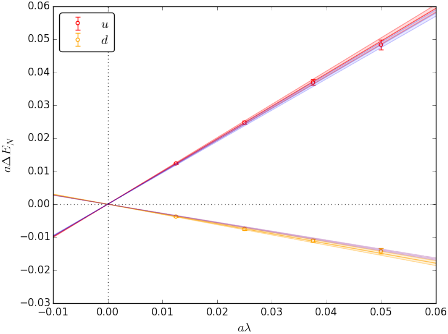
$\mathcal{L} \to \mathcal{L} + \lambda\mathcal{O}$

$\partial E_H/\partial\lambda|_{\lambda=0} \propto \langle\, H(\mathbf{p}) \,|\, \mathcal{O}(0) \,|\, H(\mathbf{p}) \,\rangle$

Want $\langle\, N_s(\mathbf{p}) \,|\, \bar{q}(0)\gamma_\mu\gamma_5 q(0) \,|\, N_s(\mathbf{p}) \,\rangle = 2is_\mu \Delta q \qquad q \in (u, d)$

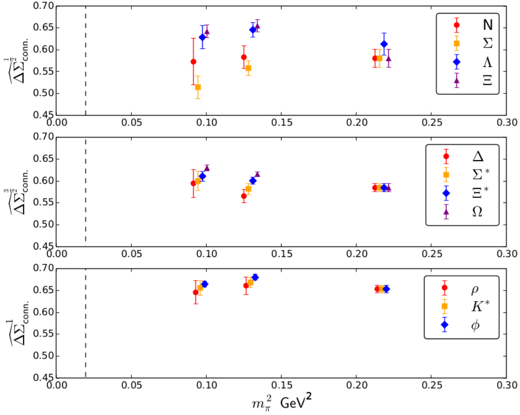
Do $\mathcal{L} \to \mathcal{L} + \lambda\bar{q}(-i\gamma_3\gamma_5)q \implies \left.\frac{\partial E_N(\lambda)}{\partial\lambda}\right|_{\lambda=0}^{\Gamma_\pm} = \pm\Delta q_{\text{conn.}}$





$m_\pi \approx 470$ MeV   350 configurations $32^3 \times 64$

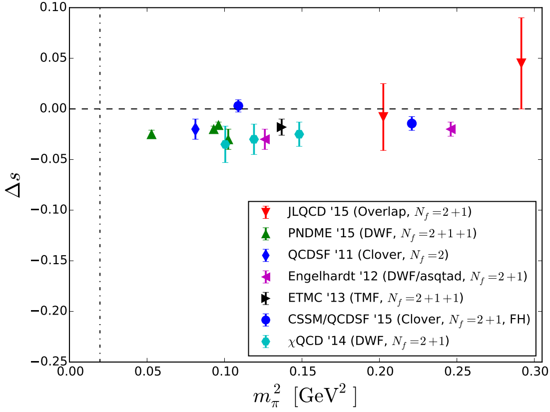
Construct additional hadrons from existing propagators



[Chambers et al. (PRD 2014)]

Quark Axial Charges in the Nucleon (Disconnected)

Disconnected calculation $\implies$ new gauge fields

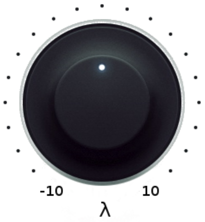


[Chambers et al. (PRD 2015)]

How do we extend method to non-forward matrix elements?

Feynman-Hellmann Recipe (Non-Forward Case)

How to calculate $\langle\, H(\mathbf{p}') \,|\, \mathcal{O}(0) \,|\, H(\mathbf{p}) \,\rangle$?



1. **Add term to Lagrangian**

$$\mathcal{L}(x) \to \mathcal{L}(x) + \lambda\left(e^{i\mathbf{q}\cdot\mathbf{x}} + e^{-i\mathbf{q}\cdot\mathbf{x}}\right)\mathcal{O}(x)$$

2. **Measure hadron energy while changing λ**

$$G(\lambda; \mathbf{p}'; t) \overset{\text{large } t}{\propto} e^{-E_H(\lambda, \mathbf{p}')t}$$

3. **Calculate matrix element from energy shifts**

$$\left.\frac{\partial E_H(\lambda, \mathbf{p}')}{\partial \lambda}\right|_{\lambda=0} = \frac{1}{2E(\mathbf{p}')} \langle\, H(\mathbf{p}') \,|\, \mathcal{O}(0) \,|\, H(\mathbf{p}) \,\rangle$$

Additional Requirement → Breit frame kinematics only

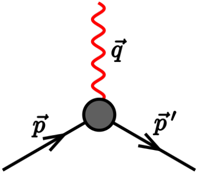
Turns out that $\mathrm{p}' = -\mathrm{p}$ is a good idea

Electromagnetic Form Factors — Pion

$\mathcal{L} \to \mathcal{L} + \lambda 2 \cos(\mathbf{q}\cdot\mathbf{x})\,\mathcal{O}$ 

$\partial E_H/\partial\lambda|_{\lambda=0} \propto \langle\, H(\mathbf{p}') \,|\, \mathcal{O}(0) \,|\, H(\mathbf{p}) \,\rangle$

Want to calculate pion form factor



Flavour contributions to vector matrix element

$$\langle\, \pi(\mathbf{p}') \,|\, \bar{q}(0)\gamma_\mu q(0) \,|\, \pi(\mathbf{p}) \,\rangle = (p'_\mu + p_\mu) F_\pi^q(Q^2)$$

Add vector operator to Lagrangian

$$\mathcal{L}(x) \to \mathcal{L}(x) + 2\lambda \cos(\mathbf{q}\cdot\mathbf{x})\, \bar{q}(x)\, \gamma_\mu\, q(x)$$

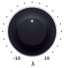
For temporal current insertion

$$\left.\frac{\partial E_\pi(\lambda,\mathbf{p})}{\partial\lambda}\right|_{\lambda=0} \overset{\mathbf{p}'=-\mathbf{p}}{=} F_\pi^q(Q^2)$$

For spatial current insertion

$$\left.\frac{\partial E_\pi(\lambda,\mathbf{p})}{\partial\lambda}\right|_{\lambda=0} \overset{\mathbf{p}'=-\mathbf{p}}{=} 0$$

Choose this

$\mathcal{L} \to \mathcal{L} + \lambda 2\cos(\mathbf{q}\cdot\mathbf{x})\,\mathcal{O}$ 

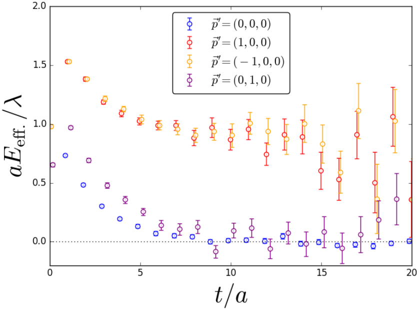
$\partial E_H/\partial\lambda|_{\lambda=0} \propto \langle\, H(\mathbf{p}') \,|\, \mathcal{O}(0) \,|\, H(\mathbf{p}) \,\rangle$

$m_\pi \approx 470$ MeV $\quad N_{\text{conf}} = 750 \quad 32^3 \times 64 \quad \mathbf{q} = (2, 0, 0)$

Require Breit frame kinematics

$$\mathbf{q} = (2, 0, 0) \implies \mathbf{p}' = (\pm 1, 0, 0)$$

Otherwise no signal at $\mathcal{O}(\lambda)$



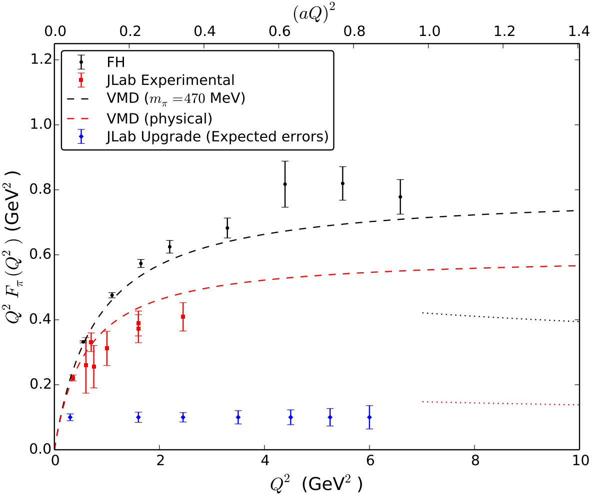
Choose $\mathbf{q}^2$ points allowing $\mathbf{p}' = -\mathbf{p}$

$$\mathbf{q}^2 = (4n)\frac{2\pi}{L} \quad n \in \mathbb{Z}^+$$

Minimises source/sink momentum for particular $\mathbf{q}^2$ $\to$ minimises noise

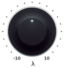
Electromagnetic Form Factors — Pion

$m_\pi \approx 470$ MeV $\approx 1000 - 1500$ configurations $32^3 \times 64$



Electromagnetic Form Factors — Nucleon

$$\mathcal{L} \to \mathcal{L} + \lambda 2 \cos(\mathbf{q}\cdot\mathbf{x})\,\mathcal{O}$$



$$\partial E_H/\partial\lambda|_{\lambda=0} \propto \langle\, H(\mathbf{p}') \,|\, \mathcal{O}(0) \,|\, H(\mathbf{p}) \,\rangle$$

In the Breit frame, individual flavour contributions

$$\langle\, N_s(\mathbf{p}') \,|\, \bar{q}(0)\gamma_\mu q(0) \,|\, N_s(\mathbf{p}) \,\rangle = \bar{u}_s(\mathbf{p}') \left[\gamma_\mu F_1^q(Q^2) + \frac{\sigma_{\mu\nu}q_\nu}{2M} F_2^q(Q^2)\right] u_s(\mathbf{p})$$

Add vector operator to Lagrangian (identical to pion calculation)

$$\mathcal{L}(x) \to \mathcal{L}(x) + 2\lambda\cos(\mathbf{q}\cdot\mathbf{x})\,\bar{q}(x)\,\gamma_\mu\, q(x)$$

For **Temporal Current** project unpolarised states

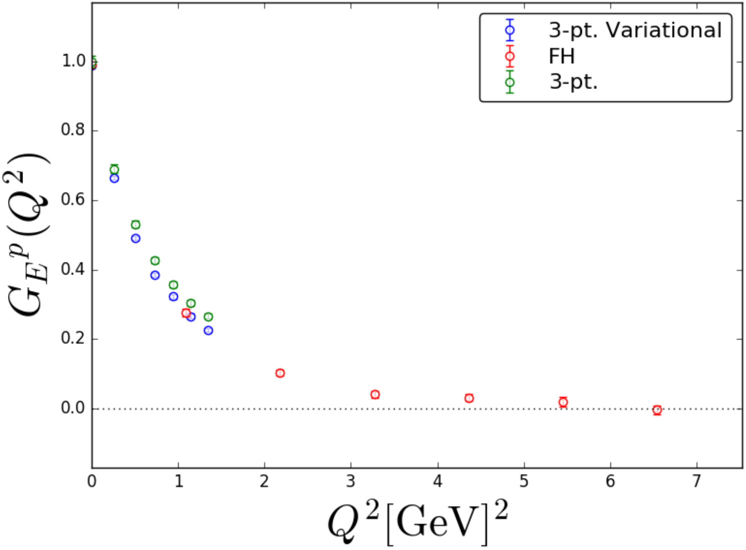
$$\left.\frac{\partial E_N(\lambda,\mathbf{p})}{\partial\lambda}\right|_{\lambda=0}^{\Gamma_{\text{unpol.}}} \overset{\mathbf{p}'=-\mathbf{p}}{=} \frac{M}{E} G_E^q(Q^2) \qquad G_E^q(Q^2) = F_1^q(Q^2) - \frac{Q^2}{4M^2} F_2^q(Q^2)$$

For **Spatial Current** project spin-up/down states

$$\left.\frac{\partial E_N(\lambda,\mathbf{p})}{\partial\lambda}\right|_{\lambda=0}^{\Gamma_\pm} \overset{\mathbf{p}'=-\mathbf{p}}{=} \pm\frac{\mathbf{q}\times\hat{\mathbf{e}}}{2E} G_M^q(Q^2) \qquad G_M^q(Q^2) = F_1^q(Q^2) + F_2^q(Q^2)$$

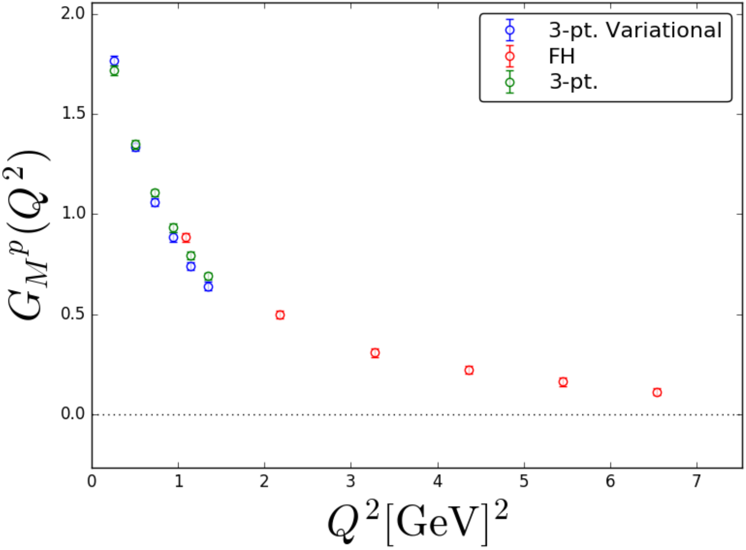
Electromagnetic Form Factors — Nucleon

$m_\pi \approx 470$ MeV $\approx 1000 - 1500$ configurations $32^3 \times 64$



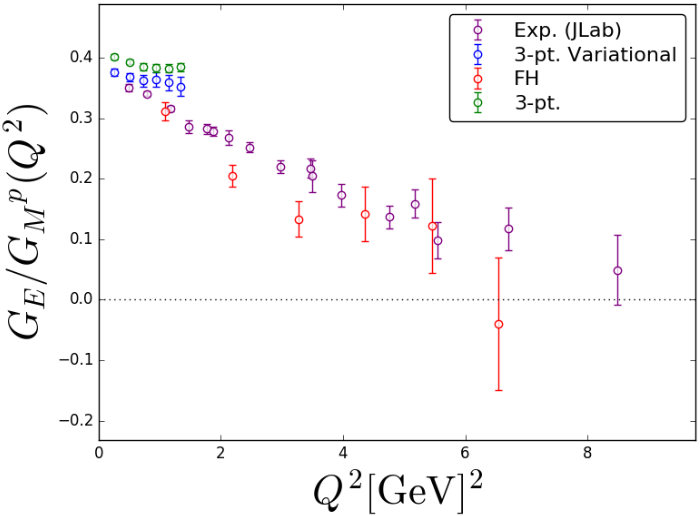
Electromagnetic Form Factors — Nucleon

$m_\pi \approx 470$ MeV $\approx 1000 - 1500$ configurations $32^3 \times 64$



Electromagnetic Form Factors — Proton

$m_\pi \approx 470$ MeV $\approx 1000 - 1500$ configurations $32^3 \times 64$



Summary

Exciting results from application of Feynman-Hellmann technique to non-forward matrix elements

Able to access much higher momentum transfers

Electromagnetic Form Factors — Proton

$m_\pi \approx 470$ MeV $\approx 1000 - 1500$ configurations $32^3 \times 64$