On the accuracy of perturbation theory in QCD

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ALPHA Collaboration

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Motivation

- $\alpha_s(m_Z)$ is a fundamental parameter of the Standard Model;
- Current status & world averages:

$$\alpha_s(m_Z) = \begin{cases} 0.1187\binom{11}{10} & \text{PDG (lattice)} \\ 0.1175(17) & \text{PDG (phenomenology)} \\ 0.1184(12) & \text{FLAG2} \end{cases}$$

- Important input for LHC physics: accuracy $< 1\%$ is required!
- Phenomenological determinations limited by systematic errors!
- Lattice methods: potential for further reduction of the total error below 1% mark.

ALPHA collaboration project

Build on CLS effort [Bruno et al, JHEP 1502 (2015) 043]:

- $N_\mathrm{f} = 2+1$ state of the art lattice QCD simulations
- nonperturbatively $\mathrm{O}(a)$ improved Wilson quarks & Lüscher-Weisz gauge action;
- open boundary conditions (avoids topology freezing)

Use 3 input parameters from experiment, e.g.

$$F_K, m_\pi, m_K \qquad \Rightarrow \qquad m_u = m_d, m_s, g_0$$

$\Rightarrow$ everything else becomes a prediction, for instance

$$\alpha_s^{(N_\mathrm{f}=3)}(1000 \times F_K) \quad \text{(in any renormalization scheme)}$$

Final goal: $\alpha_s^{(N_\mathrm{f}=5)}(m_Z)$ in the $\overline{\mathrm{MS}}$-scheme

- Requires matching to $N_\mathrm{f} = 5$ across the charm and bottom thresholds (not discussed here)

The QCD Λ-parameter and $\alpha_s(\mu) = \bar{g}^2(\mu)/4\pi$

$$\Lambda = \mu\left[b_0\bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} dg\left[\frac{1}{\beta(g)}+\frac{1}{b_0 g^3}-\frac{b_1}{b_0^2 g}\right]\right\}$$

- Continuum relation, exact at any scale μ:
  - require large μ to evaluate integral perturbatively
  - require small μ to match hadronic scale

$\Rightarrow$ problem of large scale differences:

- The scale μ must reach the perturbative regime: $\mu \gg \Lambda_{\mathrm{QCD}}$
- The lattice cutoff must still be larger: $\mu \ll a^{-1}$
- The volume must be large enough to contain pions: $L \gg 1/m_\pi$

$$\Rightarrow \quad L/a \gg \mu L \gg m_\pi L \gg 1 \quad \Rightarrow L/a \simeq \mathrm{O}(10^3)$$

$\Rightarrow$ widely different scales cannot be resolved simultaneously on a single lattice!

Finite volume couplings & Step scaling function

$\Rightarrow$ break calculation up in steps [Lüscher, Weisz, Wolff '91; Jansen et al. '95]:

1. define $\bar{g}^2(L)$ that runs with the space-time volume, i.e. $\mu = 1/L$
2. construct the step-scaling function

$$\sigma(u) = \bar{g}^2(2L)\big|_{u=\bar{g}^2(L)}$$

   for a range of values $u \in [u_{\min}, u_{\max}]$

3. iteratively step up/down in scale by factors of 2:

$$\bar{g}^2(L_{\max}) = u_{\max} \equiv u_0, \quad u_k = \sigma(u_{k+1}) = \bar{g}^2(2^{-k} L_{\max}), \quad k = 0, 1, \ldots$$

4. match to hadronic input at a hadronic scale $L_{\max}$, i.e. $F_K L_{\max} = \mathrm{O}(1)$
5. once arrived in the perturbative regime extract Λ_{QCD}

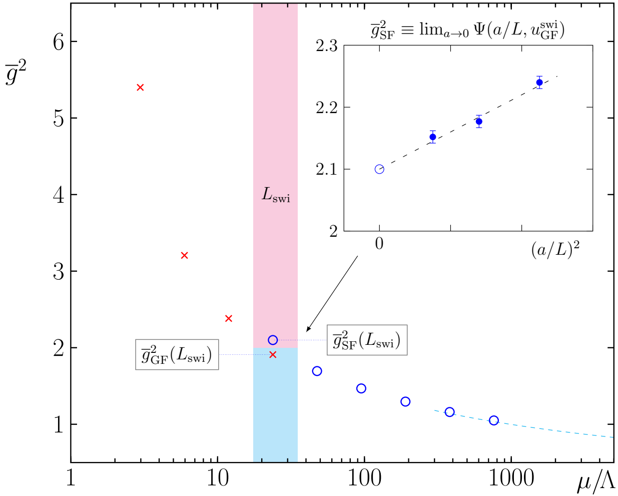
Wanted: renormalized finite volume coupling, which...

- is non-perturbatively defined in a finite space-time volume;
- can be expanded in perturbation theory (at least $\leq$ 2-loop) with reasonable effort;
- is gauge invariant;
- is quark mass-independent (defined in the chiral limit).
- can be evaluated by MC simulation with good statistical precision

$\Rightarrow$ not easy to satisfy! Here:

1. impose Schrödinger functional (SF) boundary conditions: periodic in space, Dirichlet in time
2. use 2 definitions of the coupling
   - traditional SF coupling [Narayanan et al. '92]
   - gradient flow coupling & SF b.c.'s [Fritzsch & Ramos '13]

Overview of the strategy



(courtesy Patrick Fritzsch (Lattice'14))

A family of SF couplings I

- Dirichlet b.c.'s in Euclidean time, Abelian, spatially constant boundary values C_k, C'_k [Narayanan et al. '92]:

$$A_k(x)|_{x_0=0} = C_k(\eta,\nu), \qquad A_k(x)|_{x_0=L} = C'_k(\eta,\nu)$$

$\Rightarrow$ induce family of abelian, spatially constant background fields B_μ with parameters η, ν ($\rightarrow$ 2 abelian generators of SU(3)):

$$B_k(x) = C_k(\eta,\nu) + \frac{x_0}{L}\left(C'_k(\eta,\nu) - C_k(\eta,\nu)\right), \qquad B_0 = 0.$$

- Absolute minimum of the action, unique up to gauge equiv.

$$\mathrm{e}^{-\Gamma[B]} = \int D[A,\psi,\overline{\psi}]\mathrm{e}^{-S[A,\psi,\overline{\psi}]}, \quad \Gamma[B] = \tfrac{1}{g_0^2}\Gamma_0[B] + \Gamma_1[B] + \mathrm{O}(g_0^2)$$

- Define

$$\frac{1}{\bar{g}_\nu^2(L)} = \left.\frac{\partial_\eta \Gamma[B]}{\partial_\eta \Gamma_0[B]}\right|_{\eta=0} = \left.\frac{\langle \partial_\eta S\rangle}{\partial_\eta \Gamma_0[B]}\right|_{\eta=0}$$

$\Rightarrow$ 1-parameter family of SF couplings as response of the system to a change of a colour electric background field.

A family of SF couplings II

- ν-dependence is explicit, obtained by computing $\bar{g}^2 \equiv \bar{g}^2_{\nu=0}$ and $\bar{v}$ at $\nu = 0$:
$$\frac{1}{\bar{g}^2_\nu} = \frac{1}{\bar{g}^2} - \nu \bar{v}$$
- relation between couplings at ν and $\nu = 0$ gives exact ratio:
$$r_\nu = \Lambda/\Lambda_\nu = \exp(-\nu \times 1.25516)$$
- The β-function is known to 3-loops:
$$(4\pi)^3 \times b_{2,\nu} = -0.06(3) - \nu \times 1.26$$
N.B.: values ν of O(1) look perfectly fine!
- infrared cutoff (finite volume) $\Rightarrow$ no renormalons
- secondary minimum B^* of the action with $\Delta S = S[B^*] - S[B] = 10\pi^2/(3g^2)$:
$$\exp(-\Delta S) = \exp(-2.62/\alpha) \simeq (\Lambda/\mu)^{3.8}$$

$\Rightarrow$ evaluates to $O(10^{-6})$ for $\alpha = 0.2$. Instanton contributions are even smaller.

Step scaling function for $\nu = 0$

$$\Sigma(u, a/L) = \bar{g}^2(2L)|_{\bar{g}^2(L)=u}, \qquad \sigma(u) = \lim_{a/L \to 0} \Sigma(u, a/L)$$

- Non-perturbatively O(a) improved action with perturbative boundary O(a) improvement ($c_\mathrm{t}, \tilde{c}_\mathrm{t}$)
- Simulate for u-values $\in [1, 2.012]$, $L/a = 4, 6, 8, 12$.
- Double lattice size and measure $\Sigma(u, a/L) = \bar{g}^2(2L)$
- reduce cutoff effects perturbatively up to 2-loop order [Bode, Weisz & Wolff '99]

$$\delta(u, a/L) = \frac{\Sigma(u, a/L) - \sigma(u)}{\sigma(u)} = \delta_1(L/a)u + \delta_2(L/a)u^2 + \mathrm{O}(u^3)$$

$\Rightarrow$ cutoff effects in

$$\Sigma^{(2)}(u, a/L) = \frac{\Sigma(u, a/L)}{1 + \delta_1(L/a)u + \delta_2(L/a)u^2}$$

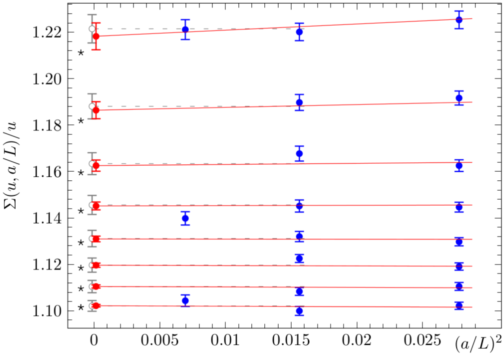
start at order u^4!

Obtaining the SSF in the continuum

Example for global fit ansatz:

$$\Sigma^{(2)}(u, a/L) \quad = \quad u + s_0 u^2 + s_1 u^3 + c_1 u^4 + c_2 u^5 + \rho_1 u^4 \frac{a^2}{L^2} + \rho_2 u^5 \frac{a^2}{L^2}$$

- s_0, s_1 fixed to perturbative values:
  $s_0 = 2b_0 \ln 2, \; s_1 = s_0^2 + 2b_1 \ln 2$
- 4 parameters: c_1, c_2, ρ_1, ρ_2; 19 data points, $\chi^2/\text{d.o.f.} \approx 1$



Remnant O(a) boundary effects as systematic error

- O(a) effects, if still present, seem to be very small.

$\Rightarrow$ continuum extrapolations with leading O(a^2) justified

- As a safeguard we include a systematic error due to incomplete cancellation of O(a) effects:
  - Estimate the derivative $\frac{\partial \Sigma}{\partial c_\mathrm{t}}$ combining perturbation theory with simulations at the larger couplings:

$$\frac{\partial \Sigma(u, a/L)}{\partial c_\mathrm{t}} = -\frac{a}{L} u \times \delta_b(u), \qquad \delta_b(u) = -(1 + 0.57(3)u)$$

  - In the expansion $c_\mathrm{t}(g_0) = 1 + c_\mathrm{t}^{(1)} g_0^2 + c_\mathrm{t}^{(2)} g_0^4 + \ldots$ we use the last known term at the corresponding $\beta = 6/g_0^2$ to estimate

$$\Delta^{\mathrm{sys}} \Sigma(u, a/L) = \left| c_\mathrm{t}^{(2)} g_0^4 \frac{\partial \Sigma(u, a/L)}{\partial c_\mathrm{t}} \right|$$

  - Add systematic error in quadrature.
  - Similarly for $\tilde{c}_\mathrm{t}$ (error is 3-4 times smaller than for c_t)
- Error estimate is conservative and subdominant

Computation of $L_0\Lambda$

- Define L_0 implicitly by

$$\bar{g}^2(L_0) = 2.012 = u_0$$

- Use the non-perturbative continuum SSF $\sigma(u)$:

$$u_{n-1} = \sigma(u_n), \quad n = 1, \ldots, \qquad \Rightarrow \qquad u_n = \bar{g}^2\left(L_0/2^n\right)$$

- At $L_n = L_0/2^n$ obtain $L_0\Lambda$ using the perturbative β-function:

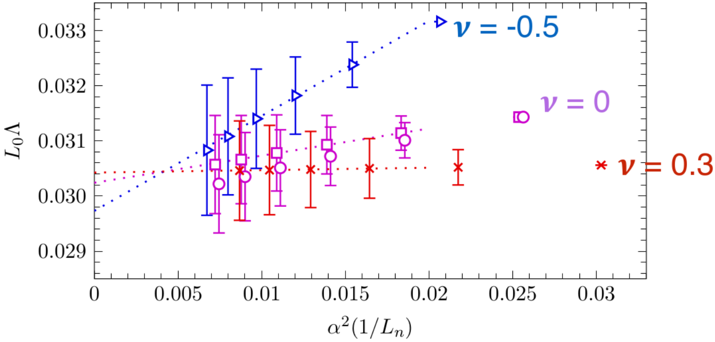
$$\begin{aligned} L_0\Lambda \;&=\; 2^n\left[b_0\bar{g}^2(L_0/2^n)\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2(L_0/2^n)}} \\ &\quad\times \exp\left\{-\int_0^{\bar{g}(L_0/2^n)} dg \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\} \end{aligned}$$

- Repeat for schemes $\nu \neq 0$ using the continuum relation:

$$\frac{1}{\bar{g}_\nu^2(L_0)} = \frac{1}{2.012} - \nu \times 0.1199(10)$$

$\Rightarrow$ check accuracy of perturbation theory: $L_0\Lambda$ must be independent of n and ν!

Result for $L_0\Lambda$

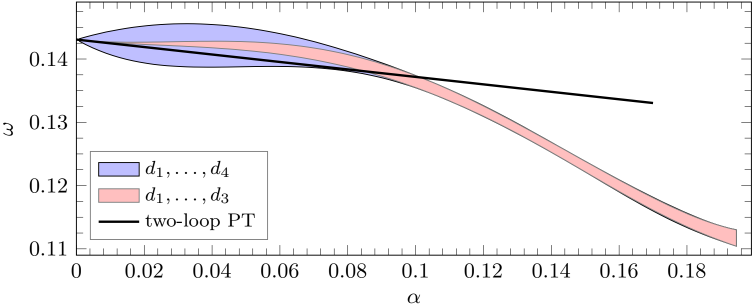


- All results agree around $\alpha = 0.1$:

$$L_0\Lambda = 0.0303(8) \qquad \text{error} < 3\% \,!$$

- $\nu = 0.3$: this result could be inferred from larger values of α
- $\nu = -0.5$: large coefficient $\propto \alpha^2$, requires data for $\alpha \approx 0.1$.

Continuum results for $\bar{v} = \omega(u)$



- Continuum extrapolations analogous to step scaling function
- The 2 fits perfectly agree where the data is ($\alpha > 0.08$)
- Observe large deviation from perturbation theory at $\alpha = 0.19$:

$$\left(\omega(\bar{g}^2) - v_1 - v_2\bar{g}^2\right)/v_1 = -3.7(2)\alpha^2$$

- At L_0 we find $\omega(2.012) = 0.1199(10)$

$\Rightarrow$ determines $\bar{g}_\nu^2(L_0) = 2.012/\left\{1 - \nu \times 0.1199(10) \times 2.012\right\}$

Conclusions

- Step-scaling techniques allow us to track the SF coupling between $2L_0$ and $L_0/32$; covering the range $0.08 < \alpha < 0.2$
- Contact with PT established $\Rightarrow$ use PT from high scale to extract Λ-parameter.

$$L_0\Lambda = 0.0303(8) \quad \Rightarrow \quad L_0\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3} = 0.0791(21)$$

- < 3 percent accuracy for Λ can be achieved provided $\alpha = 0.1$ is reached!
- Scheme dependence: data around $\alpha = 0.2$ can be both perfectly fine ($\nu = 0.3$) and clearly not sufficient ($\nu = -0.5$)

$\Rightarrow$ seems impossible to know beforehand!

- The reference scale L_0, defined by $\bar{g}^2(L_0) = 2.012$, corresponds to $1/L_0 \approx 4$ GeV.
- For connection to even lower energies and matching to pion and kaon data cf. talks by A. Ramos & R. Sommer.

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Science Foundation Ireland sfi For what's next