Lattice QCD on Non-Orientable Manifolds Part II

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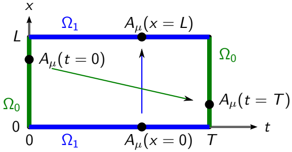
Outline

Topology of $SU(3)$-bundle in 4 dimensions

- Space reflection at $t = T$

$$p_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1), \qquad \tau \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} L-x \\ L-y \\ L-z \end{pmatrix}$$



$$\begin{aligned}
A_\mu(x = L, y, z, t) &= [\Omega_1(y, z, t)]\, A_\mu(x = 0, y, z, t) \\
A_\mu(x, y = L, z, t) &= [\Omega_2(x, z, t)]\, A_\mu(x, y = 0, z, t) \\
A_\mu(x, y, z = L, t) &= [\Omega_3(x, y, t)]\, A_\mu(x, y, z = 0, t) \\
A_\mu(x, y, z, t = T) &= [\Omega_0(\underline{\mathbf{x}})]\, p_{\mu\nu} A_\nu(\tau\underline{\mathbf{x}}, t = 0)
\end{aligned}$$

- Ω_μ: transition functions – gauge transformations we need to apply when crossing the boundary in direction μ.
- Cocycle conditions

$$\begin{aligned}
\Omega_1(y, z, t = T)\, \Omega_0(x = 0, y, z) &= \Omega_0(x = L, y, z)\, \Omega_1(\tau y, \tau z, t = 0) \\
\Omega_2(x, z, t = T)\, \Omega_0(x, y = 0, z) &= \Omega_0(x, y = L, z)\, \Omega_2(\tau x, \tau z, t = 0) \\
\Omega_3(x, y, t = T)\, \Omega_0(x, y, z = 0) &= \Omega_0(x, y, z = L)\, \Omega_3(\tau x, \tau y, t = 0) \\
\Omega_1(y = L, z, t)\, \Omega_2(x = 0, z, t) &= \Omega_2(x = L, z, t)\, \Omega_1(y = 0, z, t) \\
\Omega_1(y, z = L, t)\, \Omega_3(x = 0, y, t) &= \Omega_3(x = L, y, t)\, \Omega_1(y, z = 0, t) \\
\Omega_2(x, z = L, t)\, \Omega_3(x, y = 0, t) &= \Omega_3(x, y = L, t)\, \Omega_2(x, z = 0, t)
\end{aligned}$$

Sectors

- Gauge transformation

$$
\begin{aligned}
A'_\mu(x) &= g(x)\, A_\mu(x)\, g^\dagger(x) + i\, g(x)\, \partial_\mu g^\dagger(x) \\
\Omega'_1(y,z,t) &= g(x=L,y,z,t)\ \Omega_1(y,z,t)\ g^\dagger(x=0,y,z,t) \\
\Omega'_2(x,z,t) &= g(x,y=L,z,t)\ \Omega_2(x,z,t)\ g^\dagger(x,y=0,z,t) \\
\Omega'_3(x,y,t) &= g(x,y,z=L,t)\ \Omega_3(x,y,t)\ g^\dagger(x,y,z=0,t) \\
\Omega'_0(\underline{\mathbf{x}}) &= g(\underline{\mathbf{x}}, t=T)\ \Omega_0(\underline{\mathbf{x}})\ g^\dagger(\tau\underline{\mathbf{x}}, t=0)
\end{aligned}
$$

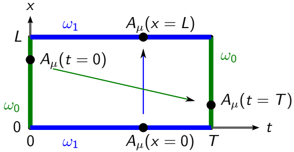
- Choosing suitable g: $\Omega_{1,2,3} = 1$ achievable
- In this choice of gauge

$$
Q = -\frac{i}{24\pi^2}\,\epsilon_{ijk}\int \mathrm{d}^3\underline{\mathbf{x}}\ \mathrm{Tr}\left[\left(\Omega_0^\dagger\partial_i\Omega_0\right)\left(\Omega_0^\dagger\partial_j\Omega_0\right)\left(\Omega_0^\dagger\partial_k\Omega_0\right)\right]
$$

- $Q \in \mathbb{Z}$
- non-orientability $\longrightarrow$ via g $Q \to Q \pm 2$ possible.
- 2 distinct topological sectors, characterized by $Q \pmod 2$.

$U(1)$ in 2d: Continuum

- $\Omega_\mu(x) = \exp\left(i\,\omega_\mu(x)\right)$



$$
\begin{aligned}
A_0(x, t=T) &= A_0(L-x, t=0) \\
A_1(x, t=T) &= -A_1(L-x, t=0) + \partial_x \omega_0(x) \\
A_0(x=L, t) &= A_0(x=0, t) + \partial_t \omega_1(t) \\
A_1(x=L, t) &= A_1(x=0, t)
\end{aligned}
$$

$$
Q = \frac{1}{2\pi}\left[\int_0^L \partial_x \omega_0(x)\,\mathrm{d}x - \int_0^T \partial_t \omega_1(t)\,\mathrm{d}t - 2\,\omega_1(t=0)\right] \pmod 2
$$

- Cocycle condition

$$
\omega_1(t=T) + \omega_0(x=0) = \omega_0(x=L) + \omega_1(t=0) + 2k\pi, \quad k \in \mathbb{Z}
$$

- Gauge transformation

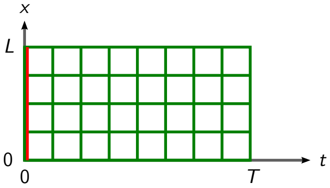
$$
\begin{aligned}
A'_\mu(x,t) &= A_\mu(x,t) + \partial_\mu \alpha(x,t) \\
\omega'_1(t) &= \omega_1(t) + \alpha(x=L,t) - \alpha(x=0,t) \\
\omega'_0(x) &= \omega_0(x) + \alpha(x,t=T) - \alpha(L-x,t=0)
\end{aligned}
$$

$U(1)$ in 2d: Lattice

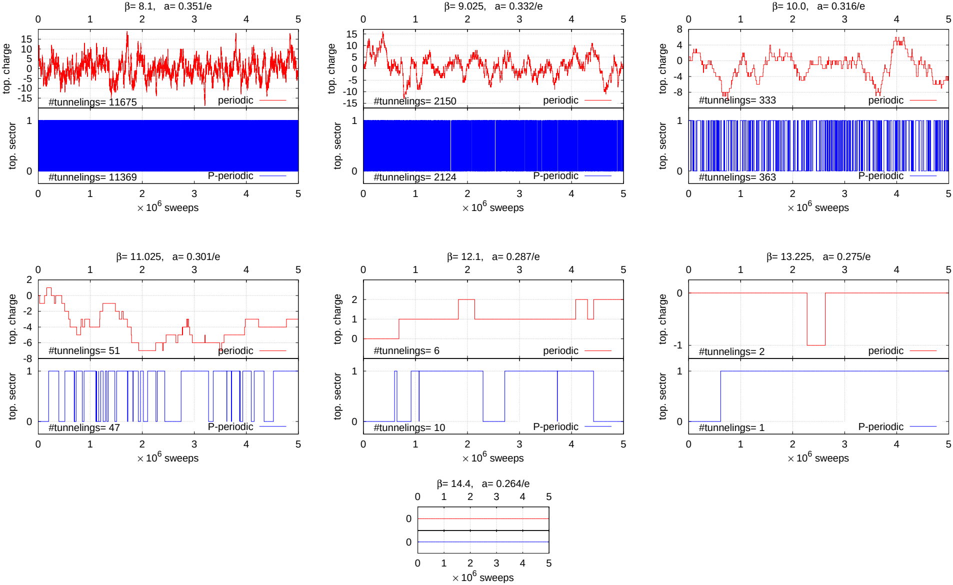
- Links $U_\mu(x,t) = \exp(iA_\mu(x,t))$
- Plaquette
  $U_P(x,t) = \exp\left[iA_1(x,t) + iA_0(x+1,t) - iA_1(x,t+1) - iA_0(x,t)\right]$
- Topological charge has a bulk term and a boundary term

$$Q = \frac{1}{2\pi}\left[\sum_{x,t} \arg\left[U_P(x,t)\right] - 2\sum_{x} A_1(x,t=0)\right] \pmod 2$$

with $-\pi \leq \arg\left[U_P(x,t)\right] < \pi$.

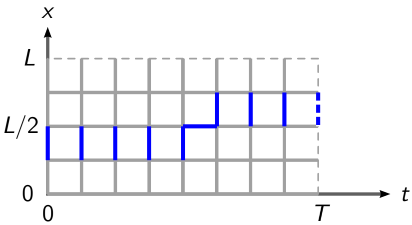


Tunneling between sectors



Equivalence of sectors

- Non-orientability $\longrightarrow$ $S = 0,\ Q = 1$ configuration: $U_Z(x,t)$



$U = 1, \quad A = 0$

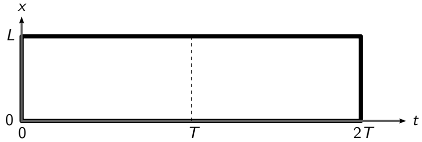
$U = -1, \quad A = \pi$

- The mapping $U \mapsto U \cdot U_Z$
  - bijection between the sectors $Q = 0$ and $Q = 1$,
  - leaves the action invariant,
  - leaves the integration measure invariant.

Outline

Construction using $2\mathcal{M}$

- Start with doubled manifold $2\mathcal{M}$



  - Periodic b.c. for both gauge fields and fermions
  - **D**: Dirac operator on $2\mathcal{M}$, satisfying $\gamma_5 \mathbf{D} \gamma_5 = \mathbf{D}^\dagger$
  - **P**: projection onto fermion fields with desired b.c.

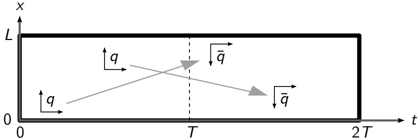
- On manifold $\mathcal{M}$

Dirac operator is given as

$$D_{x,y} = 2\left(\mathbf{PDP}\right)_{x,y} \qquad x, y \in \mathcal{M}$$

Construction of CP-boundaries

- τ: transformation on $2\mathcal{M}$
  shift in direction t, reflection in direction x



$$\tau \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} L - x \pmod{L} \\ y \\ z \\ t + T \pmod{2T} \end{pmatrix}$$

- Avoid complex determinant problem: add charge conjugation

$$\psi \to C\overline{\psi}^T \qquad \overline{\psi} \to -\psi^T C \qquad U_\mu \to U_\mu^*$$

  with $C = i\gamma_y\gamma_t = C^\dagger = C^{-1} = -C^T$

- Gauge fields: $t \to t + T, \quad x \to -x,$ charge conjugation

$$\begin{aligned} U_x(x, y, z, t+T) &= U_x^T(L - x - 1, y, z, t), \\ U_i(x, y, z, t+T) &= U_i^*(L - x, y, z, t), \qquad i = y, z, t \end{aligned}$$

- **T**: transformation of fermion fields:
  $t \to t + T, \quad x \to -x,$ charge conjugation

Construction of CP-boundaries

- Charge conjugation swaps ψ with $\overline{\psi}$
- Usual form $S = \overline{\psi}\mathbf{D}\psi$ not applicable
- Rewrite using vectors containing both ψ and $\overline{\psi}$

$$\begin{pmatrix} \psi^T & \overline{\psi} \end{pmatrix} \begin{pmatrix} & -\frac{1}{2}\mathbf{D}^T \\ \frac{1}{2}\mathbf{D} & \end{pmatrix} \begin{pmatrix} \psi \\ \overline{\psi}^T \end{pmatrix} = \frac{1}{2}\overline{\psi}\mathbf{D}\psi - \frac{1}{2}\psi^T\mathbf{D}^T\overline{\psi}^T = \overline{\psi}\mathbf{D}\psi$$

- Introduce

$$\xi = \begin{pmatrix} \psi \\ \overline{\psi}^T \end{pmatrix}, \qquad \tilde{\mathbf{D}} = \begin{pmatrix} & \mathbf{D}^T \\ -\mathbf{D} & \end{pmatrix} = -\tilde{\mathbf{D}}^T$$

- Then

$$S = \overline{\psi}\mathbf{D}\psi = -\frac{1}{2}\xi^T\tilde{\mathbf{D}}\xi$$

Convenient basis

- Change basis: $\psi, \overline{\psi} \quad \longrightarrow \quad$ eigenbasis of charge conjugation
  Lucini *et.al.* JHEP **1602** (2016) 076

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi + C\overline{\psi}^T \\ -i\psi + iC\overline{\psi}^T \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & C \\ -i & iC \end{pmatrix} \begin{pmatrix} \psi \\ \overline{\psi}^T \end{pmatrix} = V\xi$$

- Charge conjugation: $\eta \to \begin{pmatrix} \eta_1 \\ -\eta_2 \end{pmatrix} = \rho_3\, \eta, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $S = -\frac{1}{2}\xi^T \tilde{\mathbf{D}} \xi = -\frac{1}{2}\eta^T (V^{-1})^T \tilde{\mathbf{D}} (V^{-1}) \eta = -\frac{1}{2}\eta^T C \hat{\mathbf{D}} \eta$
- Using $C\mathbf{D}[U]^T C = \mathbf{D}[U^*]$

$$\hat{\mathbf{D}} = \frac{1}{2} \begin{pmatrix} \mathbf{D}[U] + C\mathbf{D}[U]^T C & i\mathbf{D}[U] - iC\mathbf{D}[U]^T C \\ -i\mathbf{D}[U] + iC\mathbf{D}[U]^T C & \mathbf{D}[U] + C\mathbf{D}[U]^T C \end{pmatrix} =$$

$$= \mathbf{D} \begin{bmatrix} \mathrm{Re}(U) & -\mathrm{Im}(U) \\ \mathrm{Im}(U) & \mathrm{Re}(U) \end{bmatrix} = \mathbf{D} \left[\mathrm{Re}(U) \cdot 1_{2\times 2} - i\, \mathrm{Im}(U) \cdot \rho_2 \right]$$

- $\hat{\mathbf{D}}$ is a usual Dirac operator, but with 6×6 real component links

Dirac-operator with CP-boundary condition

- $\mathbf{T}$: transformation of fermion fields

$$t \to t + T, \quad x \to -x, \quad \text{charge conjugation}$$

$$(\mathbf{T}\eta)(x) = -\gamma_5 \gamma_x \rho_2 \rho_3 \, \eta(\tau x), \qquad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

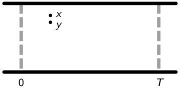
$$\mathbf{T}^\dagger = \mathbf{T}, \qquad \mathbf{T}^2 = \mathbf{1}, \qquad \left[\hat{\mathbf{D}}, \mathbf{T}\right] = 0$$

$\longrightarrow$ Define $\quad \mathbf{P}_\pm = \dfrac{\mathbf{1} \pm \mathbf{T}}{2}, \qquad \hat{\mathbf{D}}_\pm = \mathbf{P}_\pm \hat{\mathbf{D}} \mathbf{P}_\pm$

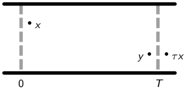
- $\hat{D}_\pm$: Dirac-operator on $\mathcal{M}$ $\qquad \hat{D}_\pm \sim \mathbf{P}_\pm \hat{\mathbf{D}} \mathbf{P}_\pm \big|_{\mathrm{Ran}(\mathbf{P}_\pm)}$

$$\left(\hat{D}_\pm\right)_{x,y} = \left(\hat{\mathbf{D}} \pm \mathbf{T}\hat{\mathbf{D}}\right)_{x,y} = \hat{\mathbf{D}}_{x,y} \mp \gamma_5 \gamma \rho_2 \rho_3 \, \hat{\mathbf{D}}_{\tau x,y} \qquad x, y \in \mathcal{M}$$

Bulk term: $\quad \hat{\mathbf{D}}_{x,y}$



Boundary term: $\quad \mp\gamma_5 \gamma \rho_2 \rho_3 \, \hat{\mathbf{D}}_{\tau x,y}$



Dirac-operator with CP-boundary condition

- $\gamma_5\rho_2$-Hermiticity

$$\left[\hat{\mathbf{D}}, \rho_2\right] = 0, \quad \gamma_5 \hat{\mathbf{D}} \gamma_5 = \hat{\mathbf{D}}^\dagger \quad \longrightarrow \quad \gamma_5\rho_2 \hat{\mathbf{D}} \gamma_5 \rho_2 = \hat{\mathbf{D}}^\dagger$$

M. A. Metlitski arXiv:1510.05663

- $\mathbf{T}$: $\quad t \to t + T, \quad x \to -x, \quad$ charge conjugation

$$\left(\mathbf{T}\eta\right)(x) = -\gamma_5\gamma_x\rho_2\rho_3\,\eta(\tau x), \qquad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

- $\left[\mathbf{T}, \gamma_5\rho_2\right] = 0 \quad \longrightarrow \quad \mathbf{P}_\pm = \dfrac{\mathbf{1} \pm \mathbf{T}}{2}$ preserves $\gamma_5\rho_2$-Hermiticity

  $\longrightarrow \quad \det\left(\hat{\mathbf{D}}_\pm\big|_{\mathrm{Ran}(\mathbf{P}_\pm)}\right) \in \mathbb{R}$, where $\hat{\mathbf{D}}_\pm = \mathbf{P}_\pm \hat{\mathbf{D}} \mathbf{P}_\pm$

- $\rho_2\mathbf{T} = -\mathbf{T}\rho_2 \quad \longrightarrow \quad \rho_2\mathbf{P}_\pm = \mathbf{P}_\mp\rho_2 \quad \longrightarrow \quad \rho_2\hat{\mathbf{D}}_\pm\rho_2 = \hat{\mathbf{D}}_\mp$

  $\longrightarrow \quad \hat{\mathbf{D}}_+ \sim \hat{\mathbf{D}}_- \quad \longrightarrow \left\{\mathrm{eig}(\hat{\mathbf{D}}_\pm)\right\} = \left\{\mathrm{eig}(\hat{\mathbf{D}})\right\}$ with multiplicity 2

- $\hat{\mathbf{D}}$ is an ordinary Dirac-operator, with $\mathrm{Re}\left\{\mathrm{eig}(\hat{\mathbf{D}})\right\} > 0$ close to the continuum

Conclusions

- Conclusions
  - Two topological classes of gauge fields
  - $U(1)$ in 2d: numerical tests
  - Fermions on non-orientable manifolds are possible: CP-boundaries
- Open issues and Outlook
  - Investigate effects of 2 sectors for QCD
  - Implement dynamical HMC

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Grassmann integral → Pfaffian

- $S = \overline{\psi}\mathbf{D}\psi = -\frac{1}{2}\xi^T\tilde{\mathbf{D}}\xi$
- $\int \mathrm{d}\xi \, \exp\left(-\frac{1}{2}\xi^T\tilde{\mathbf{D}}\xi\right) = \mathrm{pf}\left(\tilde{\mathbf{D}}\right)$
- $\mathrm{pf}(M)$: Pfaffian of $2n \times 2n$ matrix M
  - $\mathrm{pf}(M) = \frac{1}{2^n\, n!}\sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma)\prod_{i=1}^{n} M_{\sigma(2i-1),\sigma(2i)}$
  - $\mathrm{pf}(M) = \mathrm{pf}\left(\frac{M-M^T}{2}\right)$
  - $\mathrm{pf}\left(A^T M A\right) = \det(A)\,\mathrm{pf}(M)$
  - if $M = -M^T$ then $\mathrm{pf}(M)^2 = \det(M)$
- In our case

$$\mathrm{pf}\left(\tilde{\mathbf{D}}\right) = \mathrm{pf}\begin{pmatrix} & \mathbf{D}^T \\ -\mathbf{D} & \end{pmatrix} = \det(\mathbf{D}) = \int \mathrm{d}\psi\,\mathrm{d}\overline{\psi}\,\exp\left(\overline{\psi}\mathbf{D}\psi\right)$$

$\mathcal{O}(a)$-improvement and $\mathbf{D}_{\text{ov}}$ in C-eigenbasis

$$\overline{\psi}\mathbf{D}[U]\psi = -\frac{1}{2}\eta^T C\,\mathbf{D}\begin{bmatrix}\mathrm{Re}(U) & -\mathrm{Im}(U)\\ \mathrm{Im}(U) & \mathrm{Re}(U)\end{bmatrix}\eta \qquad (1)$$

- Eqn. (1) is valid for $\mathbf{D}[U]$ linear in links.

$U \mapsto \begin{bmatrix}\mathrm{Re}(U) & -\mathrm{Im}(U)\\ \mathrm{Im}(U) & \mathrm{Re}(U)\end{bmatrix}$ is a representation equivalent to $\mathbf{3} \oplus \mathbf{3}^*$

- Eqn. (1) is valid for $\mathbf{D}[U]$ linear in products of links
  $\longrightarrow$ valid for clover improved Wilson operator.
- Eqn. (1) is valid in some more general cases, e.g. for

$$\mathbf{D}_{\text{ov}}[U] = 1 + \gamma_5\,\mathrm{sgn}\big(\gamma_5\,(\mathbf{D}_{\text{W}}[U] - m_0)\,\big)$$

overlap operator:

$$\overline{\psi}\mathbf{D}_{\text{ov}}[U]\psi = -\frac{1}{2}\eta^T C\,\left(1 + \gamma_5\,\mathrm{sgn}\left\{\gamma_5\left(\mathbf{D}_{\text{W}}\begin{bmatrix}\mathrm{Re}(U) & -\mathrm{Im}(U)\\ \mathrm{Im}(U) & \mathrm{Re}(U)\end{bmatrix} - m_0\right)\right\}\right)\eta$$

Pion propagator

- Interpolating operators in η basis

$$\begin{aligned} \mathcal{O}_{\pi^-} &= \overline{\psi}_u \gamma_5 \psi_d &= -\frac{1}{2}\eta_u^T \gamma_5 C\,(1-\rho_2)\eta_d \\ \overline{\mathcal{O}}_{\pi^-} &= -\overline{\psi}_d \gamma_5 \psi_u &= \frac{1}{2}\eta_d^T \gamma_5 C\,(1-\rho_2)\eta_u \end{aligned}$$

- Correlator between $x, y \in \mathcal{M}$

$$\left\langle \mathcal{O}_{\pi^-}(x)\,\overline{\mathcal{O}}_{\pi^-}(y)\right\rangle = -\frac{1}{4}\left\langle (\eta_u^T)_x \gamma_5 C\,(1-\rho_2)\left(\eta_d\right)_x \left(\eta_d^T\right)_y \gamma_5 C\,(1-\rho_2)\left(\eta_u\right)_y \right\rangle$$

- Grassmann integral for η

$$\frac{\int \mathrm{d}\eta\ \eta_{i_1}\,\eta_{i_2}\,\eta_{i_3}\,\eta_{i_4}\ \exp\left(-\frac{1}{2}\eta^T M \eta\right)}{\int \mathrm{d}\eta\ \exp\left(-\frac{1}{2}\eta^T M \eta\right)} = \frac{1}{8}\sum_{\sigma\in S_4} \mathrm{sgn}(\sigma)\left(M^{-1}\right)_{\sigma(i_1),\sigma(i_2)}\left(M^{-1}\right)_{\sigma(i_3),\sigma(i_4)}$$

- We choose $m_u = m_d$ (N.B.: $M = C\hat{D}$, $\left[\rho_2, \hat{\mathbf{D}}\right] = 0$ but $\left[\rho_2, \hat{D}\right] \neq 0$)

$$\left\langle \mathcal{O}_{\pi^-}(x)\,\overline{\mathcal{O}}_{\pi^-}(y)\right\rangle = \frac{1}{4}\mathrm{Tr}\left[\left(\hat{D}^{-1}\right)_{y,x}\rho_2\left(\hat{D}^{-1}\right)^{\dagger}_{x,y}\rho_2 + \left(\hat{D}^{-1}\right)_{y,x}\left(\hat{D}^{-1}\right)^{\dagger}_{x,y}\right]$$

2 remarks

1. Strategy to increase statistics
   - Usual: put sources at several different $t \geq 0$
   - Here: shift gauge configuration with $-t$, and put source at 0
2. Correlator can be expressed using $\hat{\mathbf{D}}^{-1}$

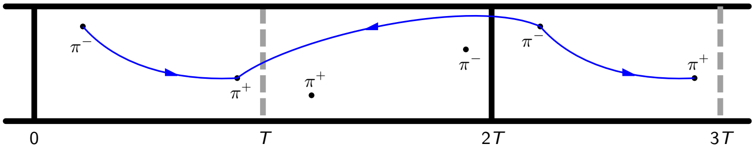
$$\left(\hat{D}^{-1}\right)_{x,y} = 2\left(\mathbf{P}_{\pm}\hat{\mathbf{D}}^{-1}\mathbf{P}_{\pm}\right)_{x,y} \qquad x, y \in \mathcal{M}$$

$$\left\langle \mathcal{O}_{\pi^-}(x)\,\overline{\mathcal{O}}_{\pi^-}(y)\right\rangle =$$

$$= \frac{1}{4}\mathrm{Tr}\left[\left(\hat{D}^{-1}\right)_{y,x}\rho_2\left(\hat{D}^{-1}\right)^{\dagger}_{x,y}\rho_2 + \left(\hat{D}^{-1}\right)_{y,x}\left(\hat{D}^{-1}\right)^{\dagger}_{x,y}\right] =$$

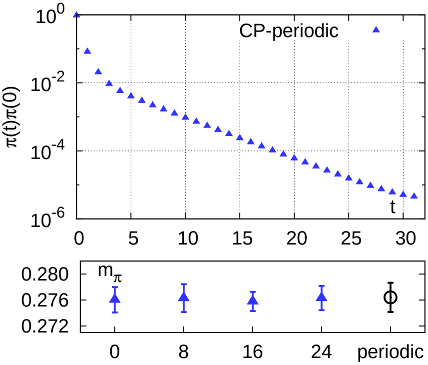
$$= \frac{1}{2}\mathrm{Tr}\left[\left(\hat{\mathbf{D}}^{-1}\right)_{y,x}\left(\hat{\mathbf{D}}^{-1}\right)^{\dagger}_{x,y}\right]$$

Terms containing $\mathbf{T}$ cancel



Numerical test

- $16^3 \times 32$ quenched with CP-boundaries, $\beta = 4.35$, $w_0 = 1.57$
- Wilson-Dirac operator, 4 steps of stout smearing with $\varrho = 0.125$, bare quark mass $m_0 = -0.16$



- Backward propagation suppressed
- Translational invariance

Suppressed backward propagation

$$\left\langle \mathcal{O}_{\pi^-}(t)\,\overline{\mathcal{O}}_{\pi^-}(\bar{t})\right\rangle =$$

$$= \mathrm{Tr}\left[CP\, e^{-(T-t)H}\,\mathcal{O}_{\pi^-}\; e^{-(t-\bar{t})H}\,\overline{\mathcal{O}}_{\pi^-}\; e^{-\bar{t}H}\right] =$$

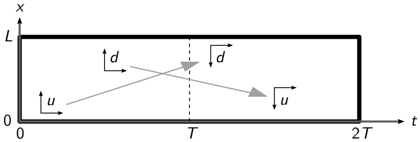
$$= \sum_{n,k} \langle n|\, CP\, e^{-(T-t)H}\,\mathcal{O}_{\pi^-}\,|k\rangle\,\langle k|\, e^{-(t-\bar{t})H}\,\overline{\mathcal{O}}_{\pi^-}\; e^{-\bar{t}H}\,|n\rangle =$$

$$= \sum_{n,k} \langle CP(n)|\,\mathcal{O}_{\pi^-}\,|k\rangle\,\langle k|\,\overline{\mathcal{O}}_{\pi^-}\,|n\rangle\; \exp\left[-TE_n - (t-\bar{t})(E_k - E_n)\right]$$

- Lowest term: $n = \text{vacuum}$, $k = \pi^-$ $\longrightarrow$ $\exp\left(-(t-\bar{t})M_\pi\right)$
- Missing: $n = \pi^+$, $k = \text{vacuum}$
- 2nd lowest: $n = \pi^- + \pi^+$, $k = \pi^-$
  $\longrightarrow$ $\exp\left[-TE_{\pi^-+\pi^+} + (t-\bar{t})(E_{\pi^-+\pi^+} - M_\pi)\right]$

P-boundaries for 2 degenerate flavors: $u \leftrightarrow d$

- τ: same as P-boundaries: $t \to t + T, \quad x \to -x$



$$\tau \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} L - x \pmod{L} \\ y \\ z \\ t + T \pmod{2T} \end{pmatrix}$$

- Gauge fields: same as P-boundaries
- $\mathbf{T}$: $t \to t + T, \quad x \to -x, \quad u \leftrightarrow d$

$$(\mathbf{T}\psi)(x) = i\gamma_5\gamma_x\tau_1\,\psi(\tau x), \qquad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ in flavor space}$$

- If $m_u = m_d$ then $[\mathbf{D}, \mathbf{T}] = 0$
- $\gamma_5\tau_3$ works: $[\gamma_5\tau_3, \mathbf{T}] = 0$

$$\longrightarrow \gamma_5\tau_3\mathbf{D}_\pm\gamma_5\tau_3 = \mathbf{D}_\pm^\dagger \quad \longrightarrow \quad \det(\mathbf{D}_\pm) \in \mathbb{R}$$