Tree level cut-off effects in gradient flow couplings with SF boundary conditions

Argia Rubeo, Stefan Sint
School of Mathematics, Trinity College, Dublin 2, Ireland
Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

Introduction

The gradient flow is a mapping $A_\mu(x) \to B_\mu(x,t)$ which defines a family of gauge fields parametrized by a continuous flow time t

$$\frac{\partial B_\mu}{\partial t} = D_\nu G_{\nu\mu} \quad B_\mu|_{t=0} = A_\mu$$

where $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$.
At leading order in momentum space the solution is

$$\tilde{B}_\mu = (e^{-tK(p,\alpha)})_{\mu\nu}\tilde{A}_\nu(p) \quad K(p,\alpha) = p^2\delta_{\mu\nu} + (\alpha-1)p_\mu p_\nu.$$

Our computation

Using the C++ library [4], we compute numerically (for $T = L$)

$$\hat{\mathcal{N}}(c, \frac{a}{L}) = \frac{c^4}{64}\sum_p \sin^2(p_0 x_0) \sum_{l>j=1}^{3} S_{lj}(p)$$

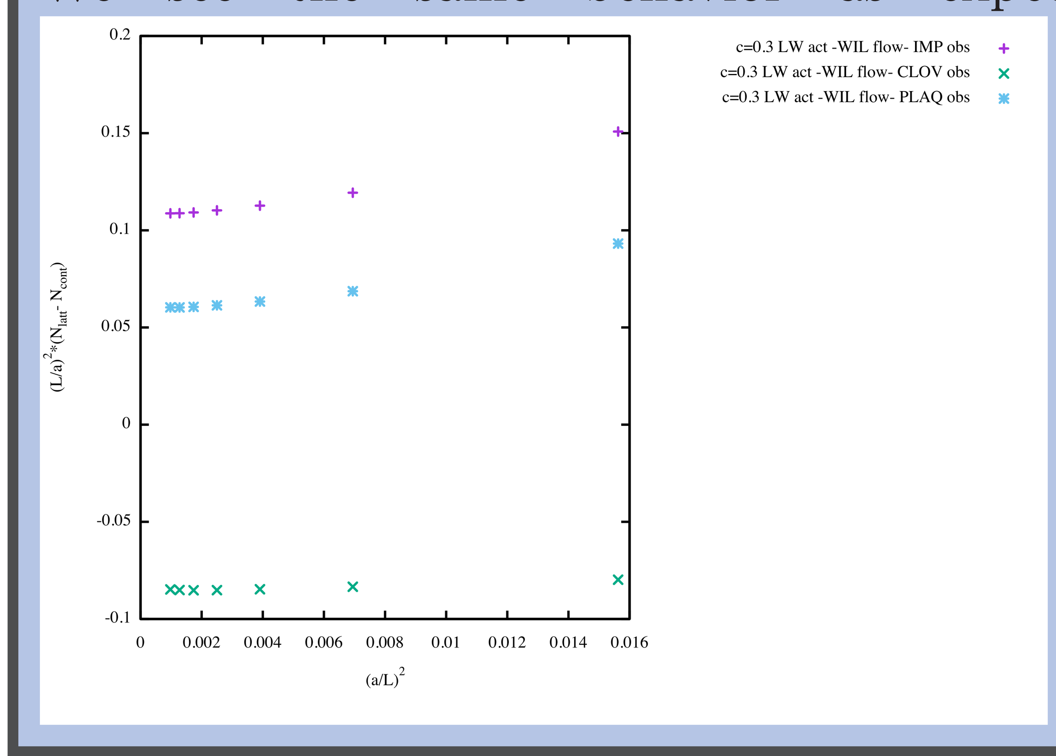
where $S_{\mu\nu}$ for the plaquette is given by

$$S_{\mu\nu}(p) = \hat{p}_\mu^2 \bar{D}_{\nu\nu}(p) + \hat{p}_\nu^2 \bar{D}_{\mu\mu}(p) - 2\hat{p}_\mu\hat{p}_\nu\bar{D}_{\mu\nu}(p)$$

for openSF boundary conditions the sine becomes cosine and p_0 takes different values [3].

OpenSF results

We see the same behavior as expected.



Conclusion and outlook

At tree-level order we have studied the technically appealing framework of [3] for the implementation of SF and openSF boundary conditions. In both situations we confirm that $\mathrm{O}(a^2)$ improvement is achieved by the Zeuthen flow if combined with an improved action and observable. The boundary O(a) effect is eliminated by $c_t = 1$ which is implicit in our framework. Somewhat surprisingly, no higher order cutoff effects from the boundary seem to be present at this order, i.e. $\mathrm{O}(a^2)$ improvement holds as c is increased to 0.5 and beyond. In order to vary c_t we are currently looking at the staggered set-up of [5] to see if we can identify suitable improvement conditions for c_t which could then be tried out non-perturbatively.

References

[1] P. Fritzsch and A. Ramos.

[2] A. Ramos and S. Sint *Eur. Phys. J.* **C76** (2016), no. 1 15, [arXiv:1508.0555].

[3] M. Lüscher *JHEP* **06** (2014) 105, [arXiv:1404.5930].

[4] C. Sanderson and R. Curtin *Journal of Open Source Software* **Vol.1** (2016) 26.

[5] P. Perez-Rubio, S. Sint, and S. Takeda *JHEP* **07** (2011) 116, [arXiv:1105.0110].

Acknowledgements

We thank Alberto Ramos for helpful discussions and for providing an independent fortran code which we used to cross check our SF results.

Finite volume coupling

In the finite volume with SF boundary conditions, we use a non perturbative definition of the coupling through the magnetic energy density [1]

$$E_{mag}(t,x) = -\frac{1}{2}\mathrm{tr}\left(G_{ij}(t,x)G_{ij}(t,x)\right)$$

$$t^2 \langle E_{mag}(t,x)\rangle|_{\sqrt{8t}=cL, x_0=\frac{T}{2}} = \hat{\mathcal{N}}\left(c, \frac{a}{L}\right)\bar{g}^2_{GF}(L)$$

and we compute numerically the the normalization factor which in the continuum is

$$\mathcal{N}(c) = \lim_{\frac{a}{L}\to 0} \hat{\mathcal{N}}\left(c, \frac{a}{L}\right).$$

The gradient flow on the lattice can be defined through the equation

$$a^2\partial_t V_\mu(t,x) = -g_0^2 \partial_{x,\mu}(S_{lat}[V])V_\mu(t,x), \qquad V_\mu(0,x) = U_\mu(x)$$

in which $\partial_{x,\mu}$ stands for the differential operator with respect to the link variable $V_\mu(t,x) = \exp\{aB_\mu\}$.
We choose as lattice action $S_{lat} = S_W$ which define the *Wilson flow* and $S_{lat} = S_{LW}$ which define the *Zeuthen flow* through the modified equation [2]

$$a^2\partial_t V_\mu(t,x) = -g_0^2(1 + \frac{a}{12}D_\mu D_\mu^*)\partial_{x,\mu}(S_{LW}[V])V_\mu(t,x), \qquad V_\mu(0,x) = U_\mu(x)$$

Energy density discetizations: clover and plaquette

We consider the observable on the lattice defined by the *clover* and the *plaquette* discretization of $G_{\mu\nu}$:

$$G^{cl}_{\mu\nu} = \tilde{\partial}_\mu(1 - \frac{1}{2}a\,\partial_\nu^*)B_\nu - \tilde{\partial}_\nu(1 - \frac{1}{2}a\,\partial_\mu^*)B_\mu$$

$$G^{pl}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

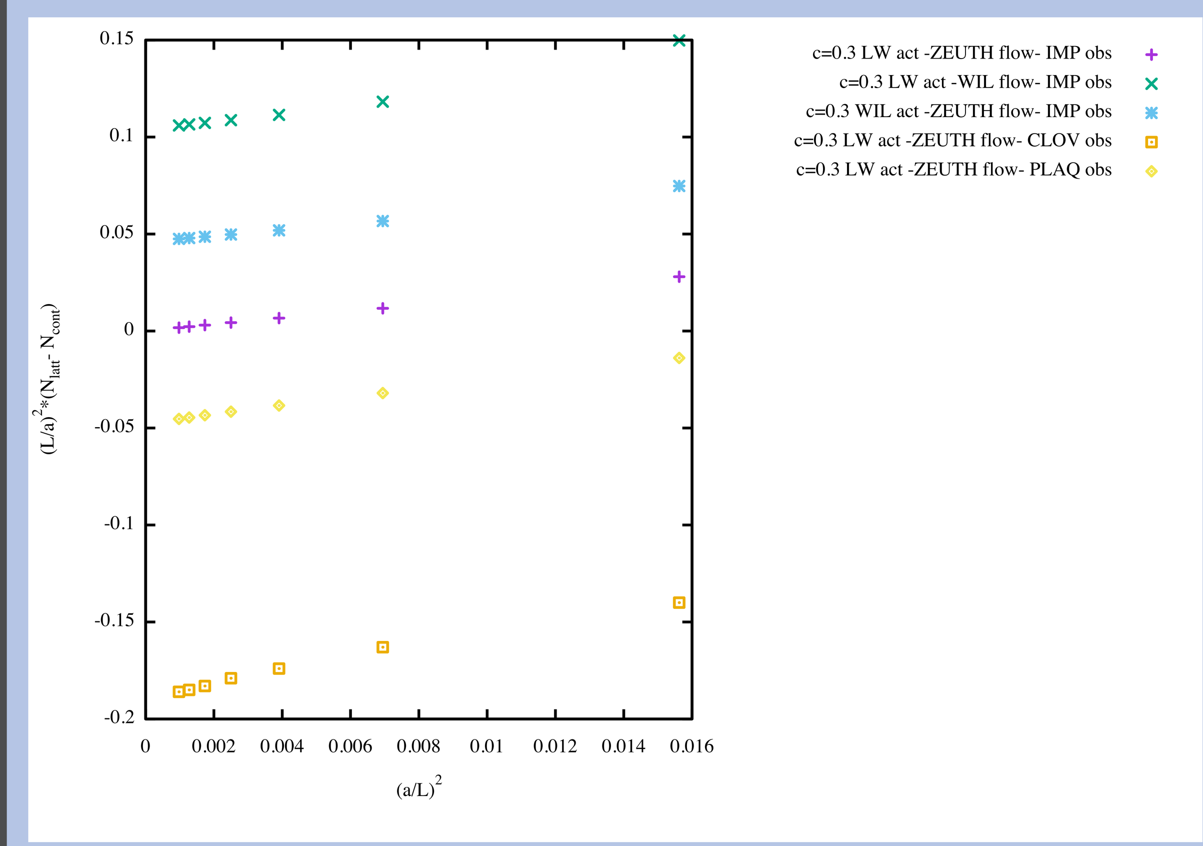
in terms of lattice derivatives as in [1].
We use SF/openSF boundary conditions as in [3], where the author implicitly applies an orbifolding principle. This gives the technical advantages of translation invariance which allows us to use the 4-momentum representation set-up:

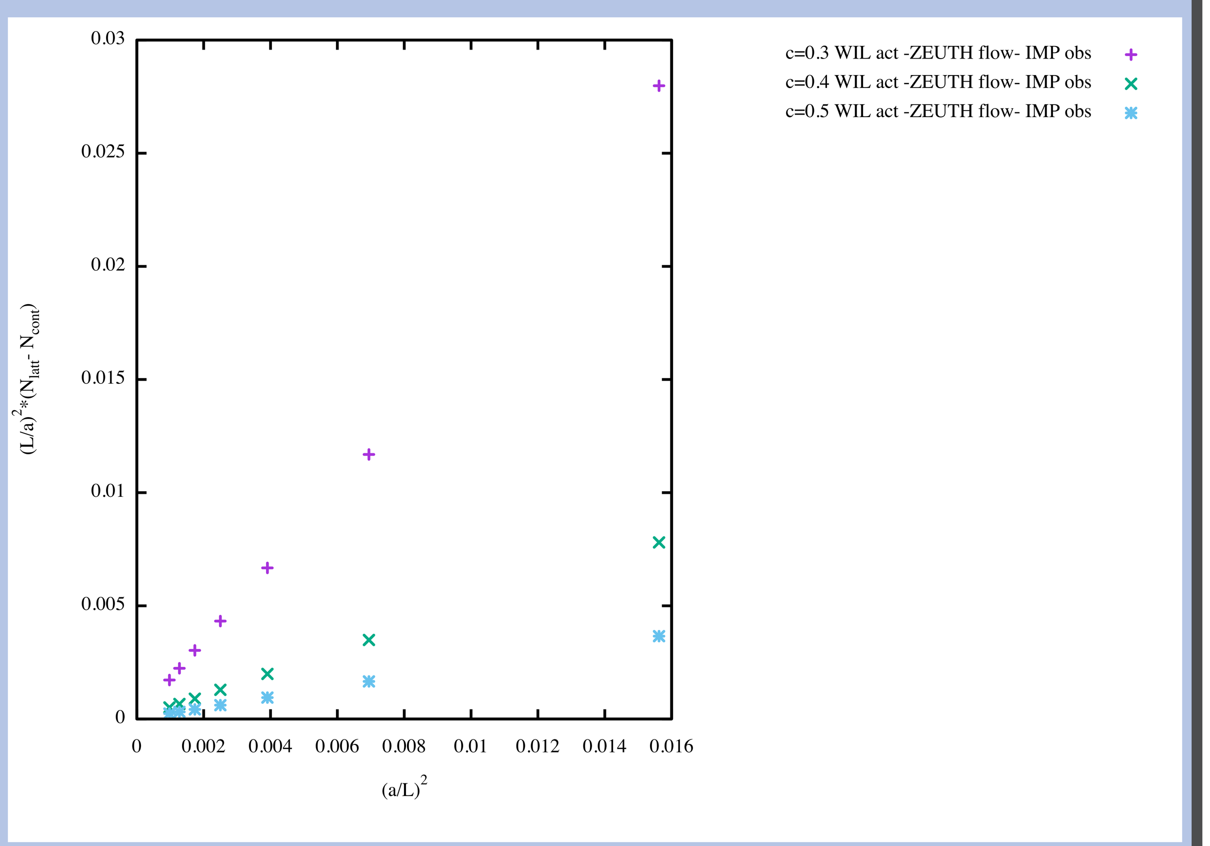
$$\langle \tilde{B}_\mu(p,t)\tilde{B}_\nu(q,s)\rangle = \delta(p+q)\bar{D}_{\mu\nu}(p,t,s;\lambda)$$

where the flow dependence at lowest order is given by the heat kernel

$$\bar{D}_{\mu\nu}(p,t,s;\lambda,\alpha) = \left(e^{-tK(p,\alpha)}\right)_{\mu\rho} D_{\rho\sigma}(p,\lambda)\left(e^{-sK(p,\alpha)}\right)_{\sigma\nu}.$$

SF results





On the left, $\mathrm{O}(a^2)$ improvement is seen only with Zeuthen flow, LW action and improved observable. We then "unimprove" one by one to distinguish the cutoff effects by Wilson flow, Wilson action or plaquette/clover observables, respectively.
On the right, increasing c, the boundary effects should become stronger while the bulk cutoff effects become smaller. Apparently, no $\mathrm{O}(a^2)$ or $\mathrm{O}(a^3)$ effects seem to be generated by the time boundaries.