The leading order hadronic contribution of the anomalous magnetic moment of the muon with $O(a)$-improved Wilson fermions with Pade approximants from fits and time moments

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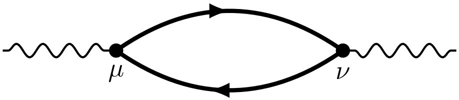
Outline

Determination of a_μ^{HLO}

The vacuum polarization tensor can be computed by

$$\Pi_{\mu\nu}(Q) = \int d^4x e^{iQx} \left\langle J_\mu^{(c)}(x) J_\nu^{(l)}(0) \right\rangle .$$

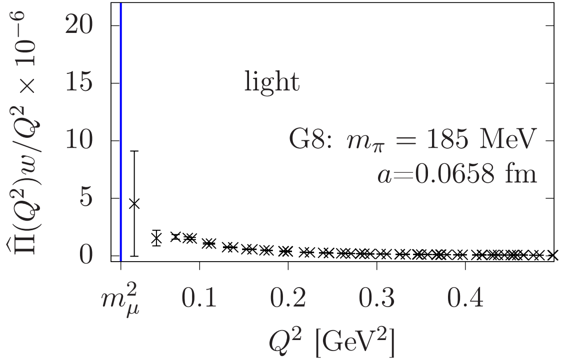


From Euclidean invariance and current conservation one finds

$$\Pi_{\mu\nu}(Q) = \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right) \Pi(Q^2).$$

The renormalized vacuum polarization function is given by

$$\widehat{\Pi}(Q^2) = 4\pi^2 \left(\Pi(Q^2) - \Pi(Q^2 = 0)\right) .$$



The anomalous magnetic moment of the muon is then given by the convolution integral:

$$a_\mu^{\mathrm{HLO}} = Z_V \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 \, \widehat{\Pi}(Q^2) \, \frac{w(Q^2/m_\mu^2)}{Q^2} ,$$

$$w(r) = \frac{16}{r^2 \left(1 + \sqrt{1 + 4/r}\right)^4 \sqrt{1 + 4/r}} .$$

Determination of a_μ^{HLO}: Hybrid approach

The vacuum polarization tensor can be computed by

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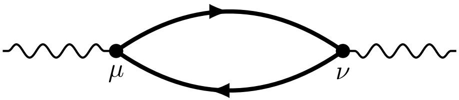

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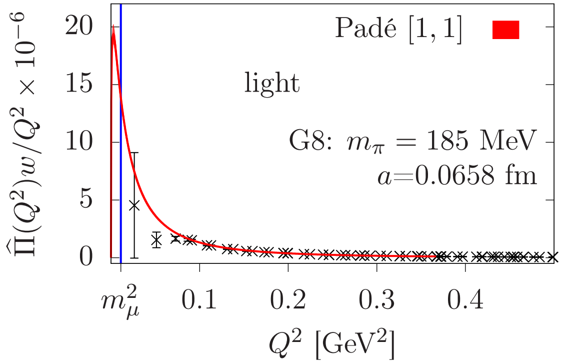


We split the convolution integral [Golterman et al., 2014]

$$\text{Continuous description: } a_{\mu,<}^{\text{HLO}} = Z_V \left(\frac{\alpha}{\pi}\right)^2 \int_0^{Q^2_{\text{cut}}} dQ^2\, \widehat{\Pi}_<(Q^2) \frac{w(Q^2/m_\mu^2)}{Q^2}, \qquad (1)$$

$$\text{Numerical integration: } a_{\mu,>}^{\text{HLO}} = Z_V \left(\frac{\alpha}{\pi}\right)^2 \int_{Q^2_{\text{cut}}}^{\infty} dQ^2\, \widehat{\Pi}_>(Q^2) \frac{w(Q^2/m_\mu^2)}{Q^2}. \qquad (2)$$

CLS-Ensembles with $N_f = 2$, update and changes to the setup

In our study we use $O(a)-$improved Wilson-fermions with $N_f = 2$ with partially twisted boundary conditions. The strange and charm quarks are partially quenched. In our analysis we use the extended frequentist method to estimate systematic errors [W-M Yao et al 2006 J. Phys. G: Nucl. Part. Phys. 33 1; S. Durr et al., Science **322** (2008) 1224]

Label	V	β	$m_\pi L$	a [fm]$^{(*)}$	m_π [MeV]	$N_{\text{meas}}[ud/s, c]$
A3	64×32^3	5.20	6.1	0.0755(9)(7)	495	1004
A4	64×32^3	5.20	4.7	0.0755(9)(7)	381	1600
A5	64×32^3	5.20	4.0	0.0755(9)(7)	331	1004
B6	96×48^3	5.20	5.2	0.0755(9)(7)	280	1224
E5	64×32^3	5.30	4.7	0.0658(7)(7)	437	4000
F6	96×48^3	5.30	5.0	0.0658(7)(7)	311	1200
F7	96×48^3	5.30	4.2	0.0658(7)(7)	265	1000
G8	128×64^3	5.30	3.9	0.0658(7)(7)	185	4652/820
N5	96×48^3	5.50	5.2	0.0486(4)(5)	441	1388
N6	96×48^3	5.50	4.0	0.0486(4)(5)	340	2236
O7	128×64^3	5.50	4.2	0.0486(4)(5)	268	2384/596

$^{(*)}$ We use the scale determined via f_K in [P. Fritzsch et al., Nucl. Phys. B **865** (2012) 397]

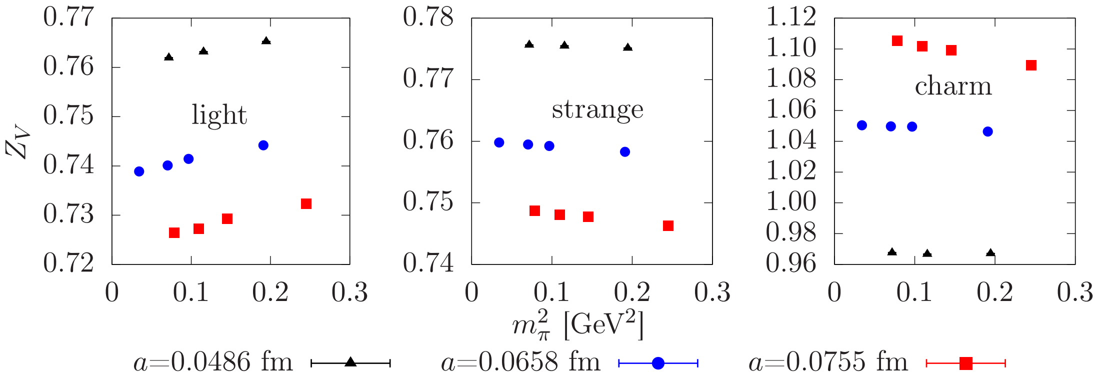
Non perturbative determination of Z_V

To take the mass dependence of Z_V into account we define the two- and three-point functions:

$$C_2(T/2) = \sum_{\vec{x}} \left\langle O(\vec{x}, T/2) O^\dagger(\vec{0}, 0) \right\rangle, \quad C_3(\tau, T/2) = \sum_{\vec{x},\vec{y}} \left\langle O(\vec{x}, T/2) J_0(\vec{y}, \tau) O^\dagger(\vec{0}, 0) \right\rangle,$$

where $O = \bar{\psi}_2 \gamma_5 \psi_1$, and $J_0 = \bar{\psi}_1 \gamma_0 \psi_1$. We take the ratio $R(\tau, T/2) = \frac{C_3(\tau, T/2)}{C_2(T/2)}$ and the difference $d(t) = R(t, T/2) - R(t + T/2, T/2) = Q_V \Rightarrow Z_V Q_V = 1$.

Light and strange quark contributions have similar renormalization coefficients. For the charm quark contribution the renormalization coefficients are larger and coarser lattice spacings receive stronger corrections.



Extraction of a_μ^{HLO} via fits

We use low order Padé approximants and polynomials to perform correlated fits with a singular value decomposition to the VPF in a small Q^2 interval:

$$\text{Padé [1,1]:}\ \Pi^{\mathrm{fit}}_{[1,1]}(Q^2) = \frac{A_1 Q^2}{B_1 + Q^2}, \tag{3}$$

$$\text{Padé [2,1]:}\ \Pi^{\mathrm{fit}}_{[2,1]}(Q^2) = Q^2 \left(A_0 + \frac{A_1}{B_1 + Q^2} \right), \tag{4}$$

$$\text{Polynomial:}\ \ \Pi^{\mathrm{fit}}(Q^2) = a + bQ^2. \tag{5}$$

We obtain the renormalized VPF by

$$Q^2 \leq Q^2_{\mathrm{cut}} : \quad \widehat{\Pi}_<(Q^2) = 4\pi^2 \left(\Pi^{\mathrm{fit}}_{[n,m]}(Q^2) - \Pi^{\mathrm{fit}}_{[n,m]}(Q^2 = 0) \right), \tag{6}$$

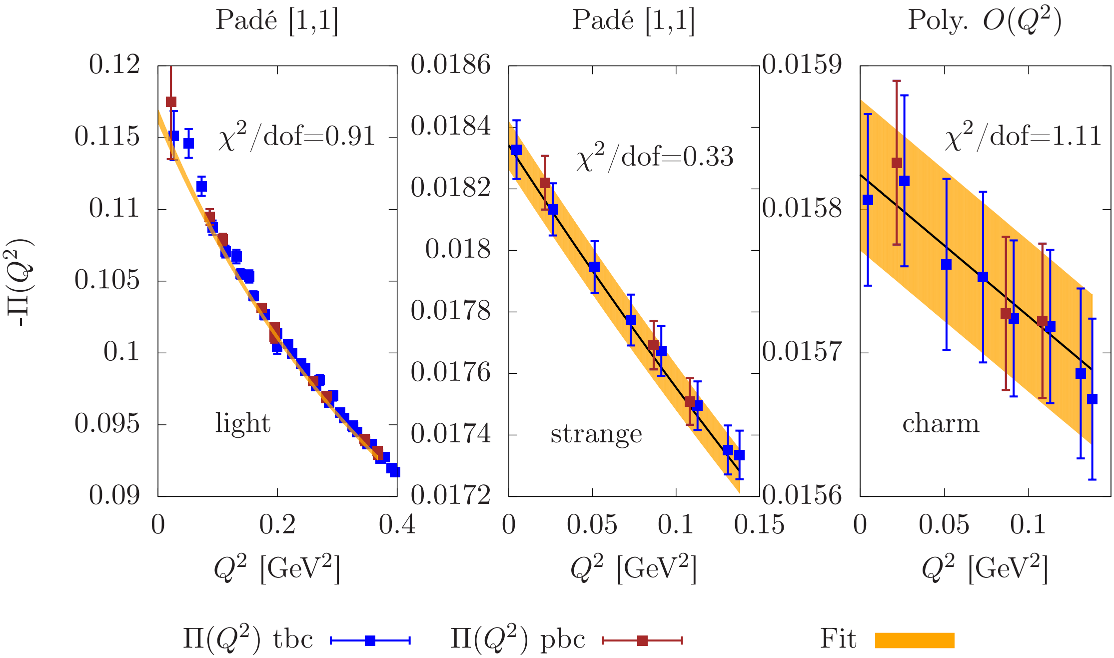
$$Q^2 > Q^2_{\mathrm{cut}} : \quad \widehat{\Pi}_>(Q^2) = 4\pi^2 \left(\Pi^{\mathrm{data}}(Q^2) - \Pi^{\mathrm{fit}}_{[n,m]}(Q^2 = 0) \right). \tag{7}$$

For the choice of Q^2_{cut} we consider:

Light quarks:	Strange quark:	Charm quark:
$N \leq 20$ points in the fit window	$N \ll 20$ points in the fit window	$N \ll 20$ points in the fit window
$Q^2_{\mathrm{cut}} \approx 0.5$ GeV^2	$Q^2_{\mathrm{cut}} \approx 0.5$ GeV^2	vary fit ansatz

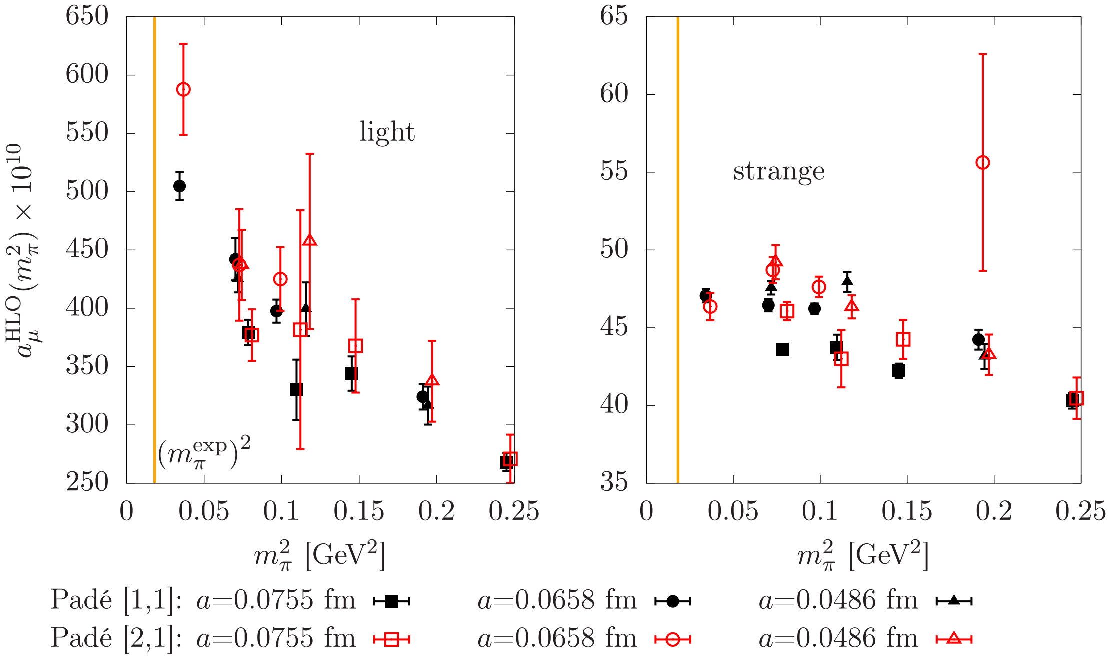
Extraction of a_μ^{HLO} via fits

G8, $m_\pi \approx 185$ MeV, $a = 0.0658$ fm, $\Pi(Q^2)$ with twisted boundary conditions in **blue** and periodic bondary conditions in **red**.



Extraction of a_μ^{HLO} via fits

Padé [2,1] is considerably less stable for small Q^2 intervals. We show results for the light quark contribution on the left, and for the strange quark contribution on the right.



Extraction of a_μ^{HLO} via time moments

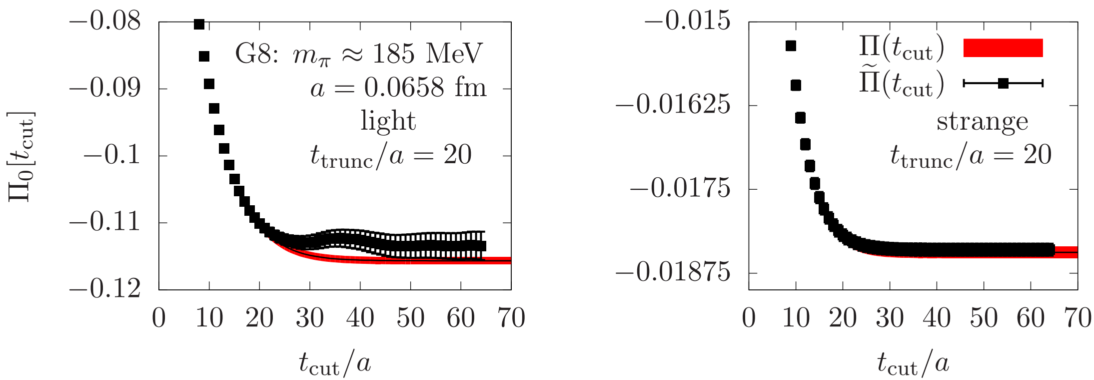
We truncate the sum to determine the time moments and use the amplitude A and mass m_V of the correlation function [Chakraborty et al., 2014, 2016]:

$$G_{2n}(t_{\mathrm{cut}}) = \frac{a^4}{3}\sum_{i=1}^{3}\sum_{t=0}^{t_{\mathrm{trunc}}}\sum_{\vec{x}} t^{2n}\left(\left\langle V_i(\vec{x},t)V_i(\vec{0},0)\right\rangle + \left\langle V_i(\vec{x},T-t)V_i(\vec{0},0)\right\rangle\right) + a^4\int_{t_{\mathrm{trunc}}}^{t_{\mathrm{cut}}} 2t^{2n}Ae^{-m_V t}dt, \quad (8)$$

$$\Pi_j(t_{\mathrm{cut}}) = (-1)^{j+1}\frac{G_{2j+2}(t_{\mathrm{cut}})}{(2j+2)!}. \quad (9)$$

$$\widetilde{G}_{2n}(t_{\mathrm{cut}}) = \frac{a^4}{3}\sum_{i=1}^{3}\sum_{t=0}^{t_{\mathrm{cut}}}\sum_{\vec{x}} t^{2n}\left(\left\langle V_i(\vec{x},t)V_i(\vec{0},0)\right\rangle + \left\langle V_i(\vec{x},T-t)V_i(\vec{0},0)\right\rangle\right), \quad (10)$$

$$\widetilde{\Pi}_j(t_{\mathrm{cut}}) = (-1)^{j+1}\frac{\widetilde{G}_{2j+2}(t_{\mathrm{cut}})}{(2j+2)!}. \quad (11)$$



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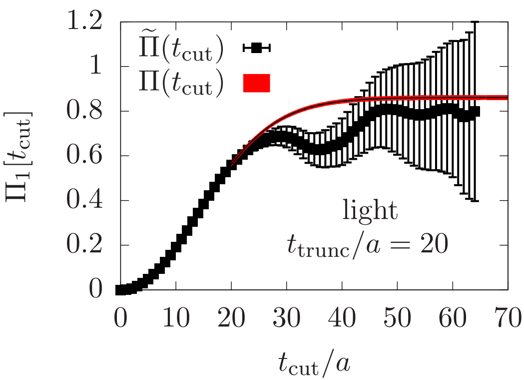
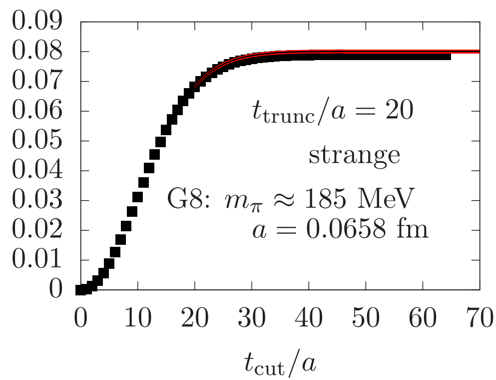




Extraction of a_μ^{HLO} via time moments

The Padé approximants can be written in terms of time moments as:

$$\text{Padé [1,1]: } \hat{\Pi}^{\mathrm{mom}}_{[1,1]}(Q^2) = \frac{\Pi_1^2 Q^2}{\Pi_1 - \Pi_2 Q^2}, \tag{12}$$

$$\text{Padé [2,1]: } \hat{\Pi}^{\mathrm{mom}}_{[2,1]}(Q^2) = \frac{\Pi_1 \Pi_2 Q^2 + (\Pi_2^2 - \Pi_1 \Pi_3) Q^4}{\Pi_2 - \Pi_3 Q^2}. \tag{13}$$

We obtain the renormalized VPF by

$$Q^2 \leq Q^2_{\mathrm{cut}} : \quad \widehat{\Pi}_<(Q^2) = 4\pi^2 \hat{\Pi}^{\mathrm{mom}}_{[n,m]}(Q^2), \tag{14}$$

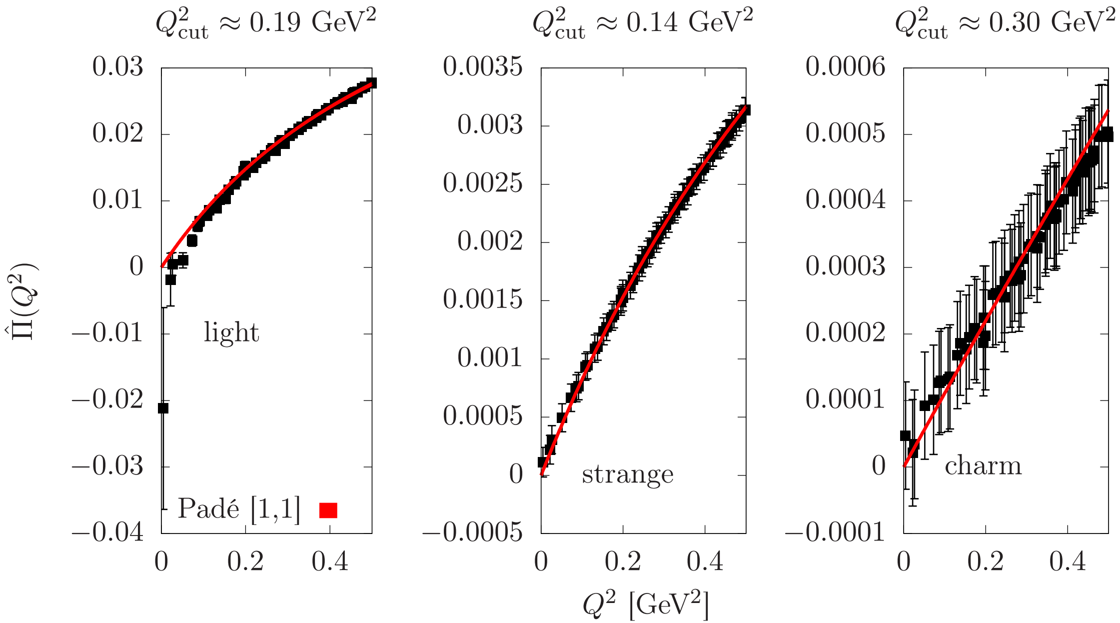
$$Q^2 > Q^2_{\mathrm{cut}} : \quad \widehat{\Pi}_>(Q^2) = 4\pi^2 \left(\Pi^{\mathrm{data}}(Q^2) - \Pi_0\right). \tag{15}$$

We want the switch from expansion with the time moments to the data to be smooth:

$$Q^2_{\mathrm{cut}} = \min_{0.1<Q^2<0.5\ \mathrm{GeV}^2} \left| \hat{\Pi}^{\mathrm{mom}}_{[n,m]}(Q^2) - \left(\Pi^{\mathrm{data}}(Q^2) - \Pi_0\right) \right|. \tag{16}$$

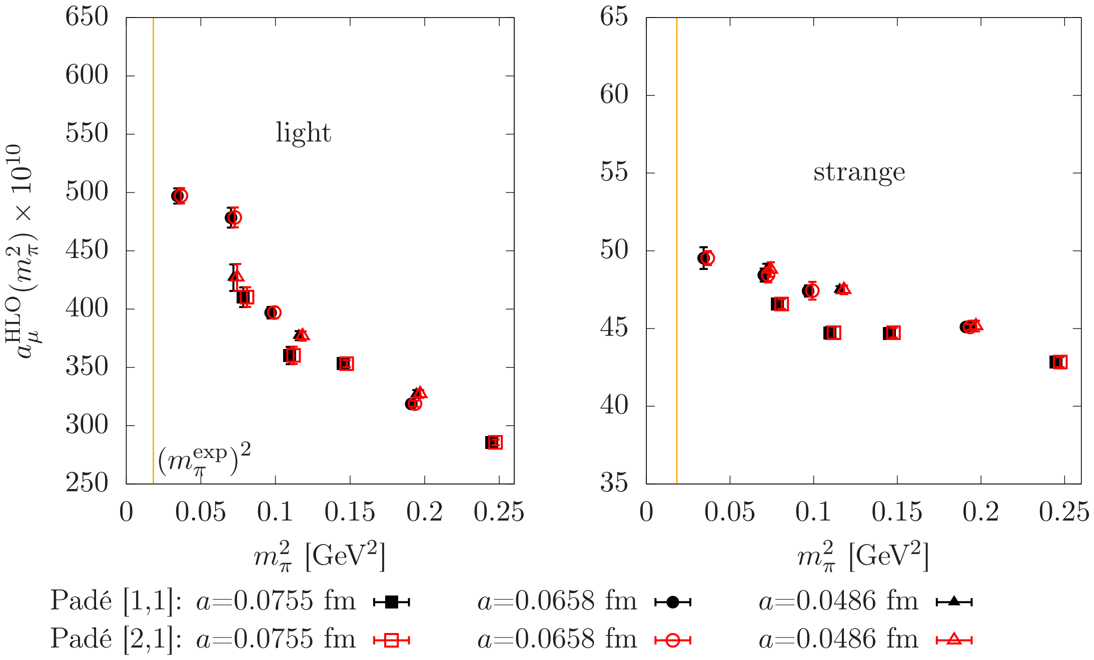
Extraction of a_μ^{HLO} via time moments

G8, $m_\pi \approx 185$ MeV, $a = 0.0658$ fm, we show Padé [1,1].



Extraction of a_μ^{HLO} via time moments

The different Padé ansätze yield well compatible results. We choose Padé [1,1], which only needs the moments $\Pi_{0,1,2}$.

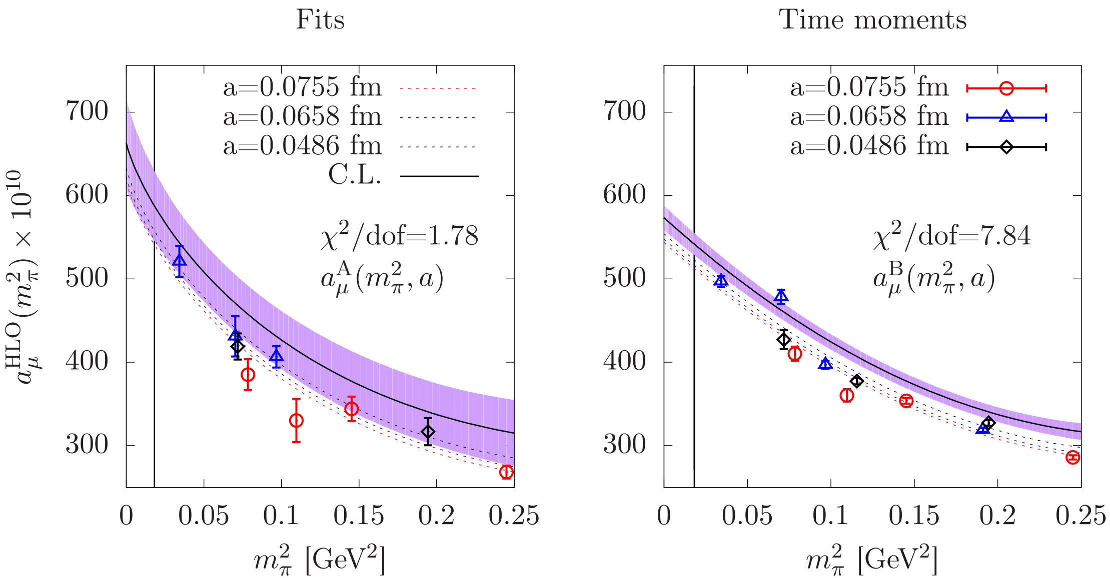


Extrapolation to physical a_μ^{HLO}: light quarks

We use uncorrelated fits with the ansätze:

$$a_\mu^{\mathrm{A}}(m_\pi^2, a) = b_0 + b_1 m_\pi^2 + b_2 m_\pi^2 \log(m_\pi^2) + b_3 a, \quad (17)$$

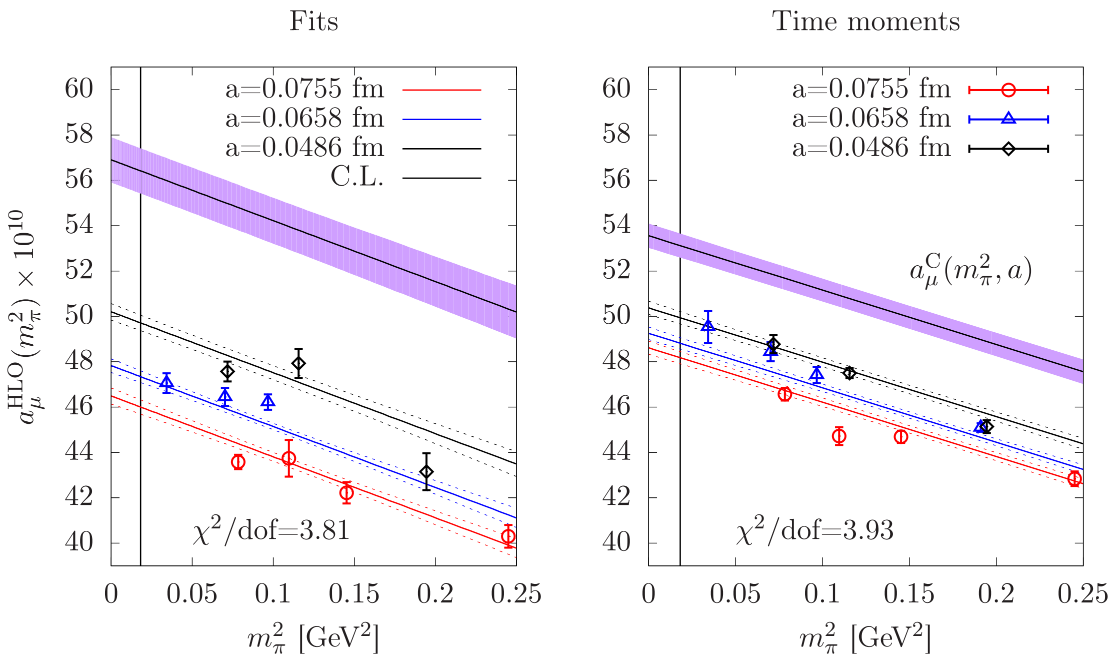
$$a_\mu^{\mathrm{B}}(m_\pi^2, a) = b_0 + b_1 m_\pi^2 + b_2 m_\pi^4 + b_3 a. \quad (18)$$



Extrapolation to physical a_μ^{HLO}: strange quark

We use an uncorrelated fit with the ansatz:

$$a_\mu^{\mathrm{C}}(m_\pi^2, a) = b_0 + b_1 m_\pi^2 + b_2 a \qquad (19)$$



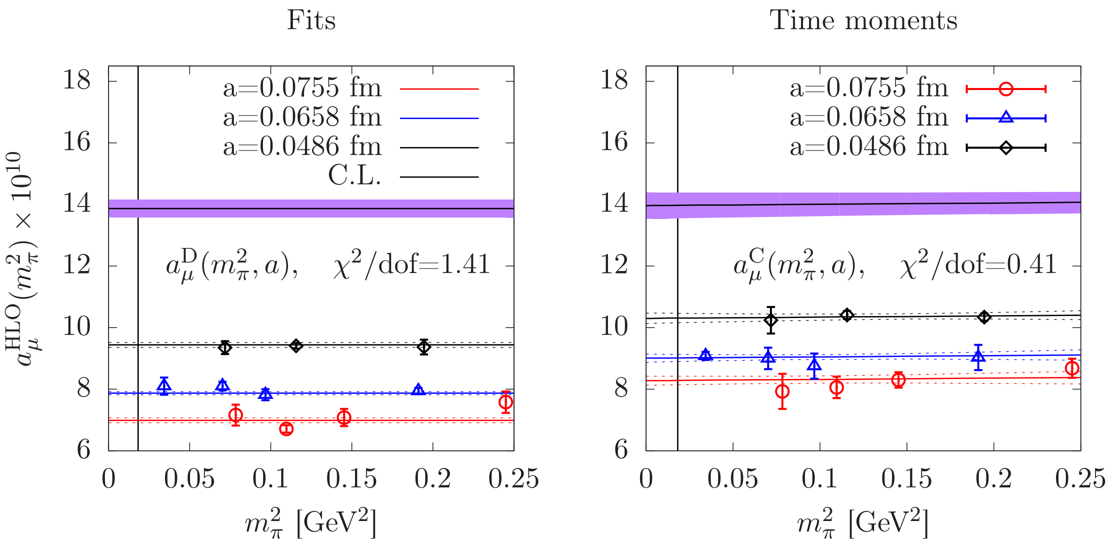
Extrapolation to physical a_μ^{HLO}: charm quark

We use uncorrelated fits with the ansätze:

$$a_\mu^{\mathrm{C}}(m_\pi^2, a) = b_0 + b_1 m_\pi^2 + b_2 a, \qquad (17)$$

$$a_\mu^{\mathrm{D}}(m_\pi^2, a) = b_0 + b_1 a \qquad (20)$$

Corrections due to lattice artifacts of $O(50\%)$.



Systematic effects and results

Systematic effects:

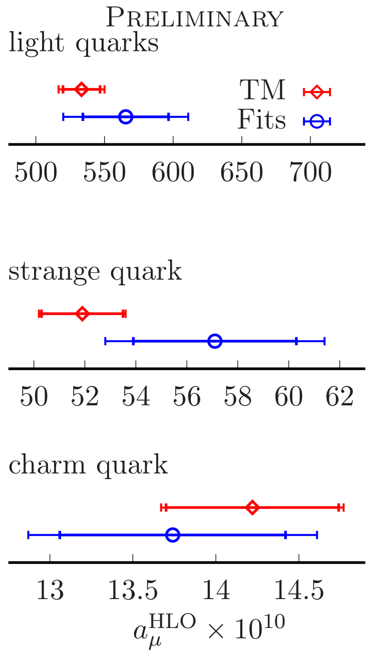
- cuts:
  - $m_\pi < 400$ MeV,
  - $a < 0.070$ fm,
  - and $m_\pi < 400$ MeV & $a < 0.070$ fm.
- Fit ansätze for the chiral and continuum extrapolation: $a_\mu^{\mathrm{A}}(m_\pi^2, a)$, $a_\mu^{\mathrm{B}}(m_\pi^2, a), \ldots$
- fits: Q^2-dependence,
- time moments: $t_{\mathrm{trunc}}-$dependence.

Number of variations:

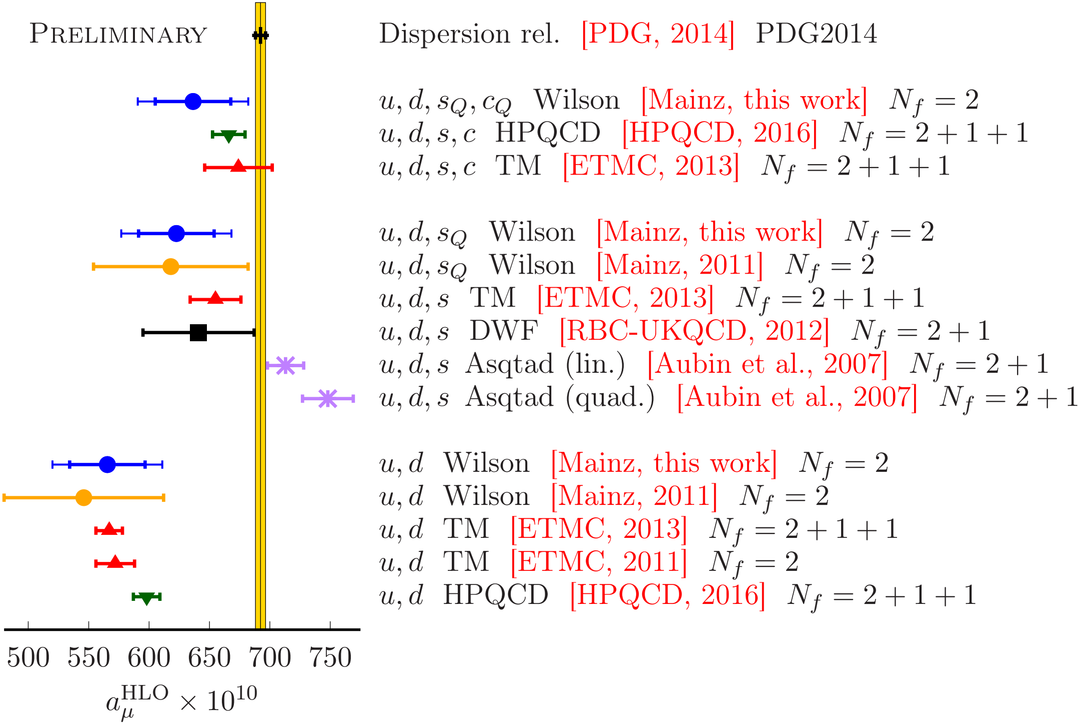
Flavor	Fits	Time moments
ud	16	24
s	8	12
c	16	24

Fractions of the systematic error (%):

Flav.	Fits			Time moments		
	LA	m_π	Q^2	LA	m_π	$\Delta t_{\mathrm{trunc}}$
ud	30.8	34.6	34.6	51.1	39.6	9.3
s	44.9	10.2	44.9	77.2	20.1	2.7
c	29.4	27.0	43.6	26.6	73.4	0.0



Results


PRELIMINARY
Dispersion rel. [PDG, 2014] PDG2014
u, d, s_Q, c_Q Wilson [Mainz, this work] $N_f = 2$
u, d, s, c HPQCD [HPQCD, 2016] $N_f = 2+1+1$
u, d, s, c TM [ETMC, 2013] $N_f = 2+1+1$
u, d, s_Q Wilson [Mainz, this work] $N_f = 2$
u, d, s_Q Wilson [Mainz, 2011] $N_f = 2$
u, d, s TM [ETMC, 2013] $N_f = 2+1+1$
u, d, s DWF [RBC-UKQCD, 2012] $N_f = 2+1$
u, d, s Asqtad (lin.) [Aubin et al., 2007] $N_f = 2+1$
u, d, s Asqtad (quad.) [Aubin et al., 2007] $N_f = 2+1$
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u, d TM [ETMC, 2013] $N_f = 2+1+1$
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u, d HPQCD [HPQCD, 2016] $N_f = 2+1+1$
500 550 600 650 700 750
$a_\mu^{\mathrm{HLO}} \times 10^{10}$

Conclusions and outlook

Conclusions

- We discussed two methods to extract $a_\mu^{\rm HLO}$ via a combination of a continuous discription in the low Q^2 regime and a numerical procedure for large Q^2.
- Results based on time moments are more precise but show fluctuations in the chiral behaviour $\rightarrow$ fits to the physical point are more difficult.
- We studied different sources of systematic effects for both methods.
- Our 4 flavor result shows a difference of $\sim 1.2\sigma$ when compared with the result from phenomenology. We did not yet include FSE, sea quark effects for the strange and charm, disconnected diagrams, and iso-spin breaking effects.

Outlook

- The next generation of CLS ensembles with $N_f = 2+1$ are ready to be processed.
- Development of the formalism and code for iso-spin symmetry breaking for Wilson fermions is underway.
- We are currently working on a more precise estimate for the disconnected contribution to $a_\mu^{\rm HLO}$.

Thank you for your attention!

Backup

Extraction of a_μ^{HLO} via fits

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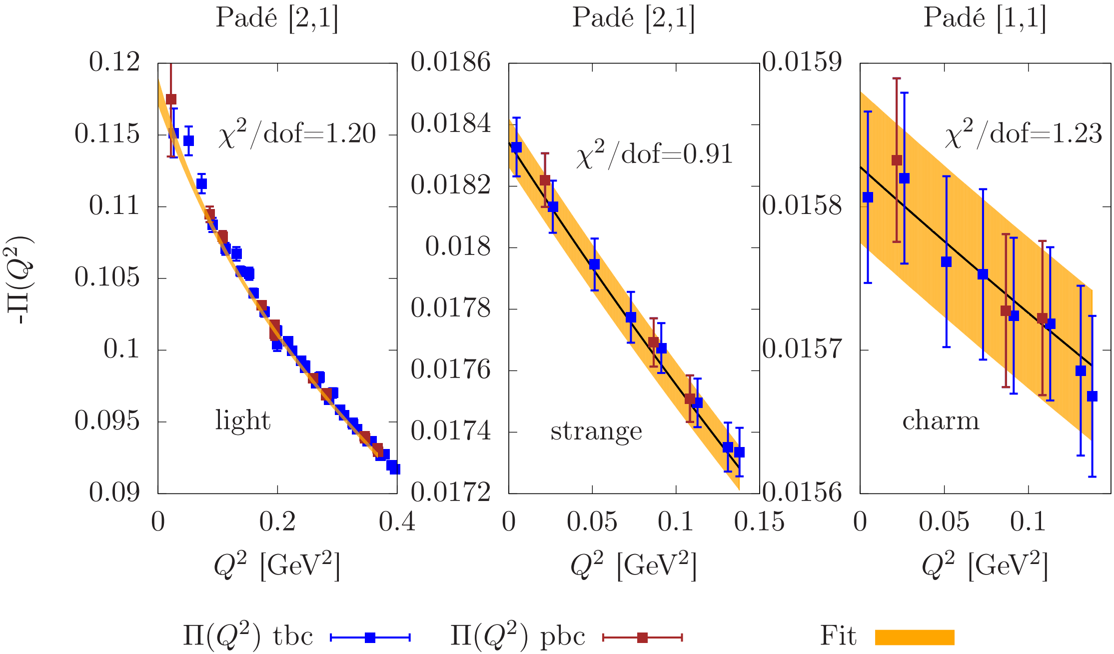

Extraction of a_μ^{HLO} via time moments

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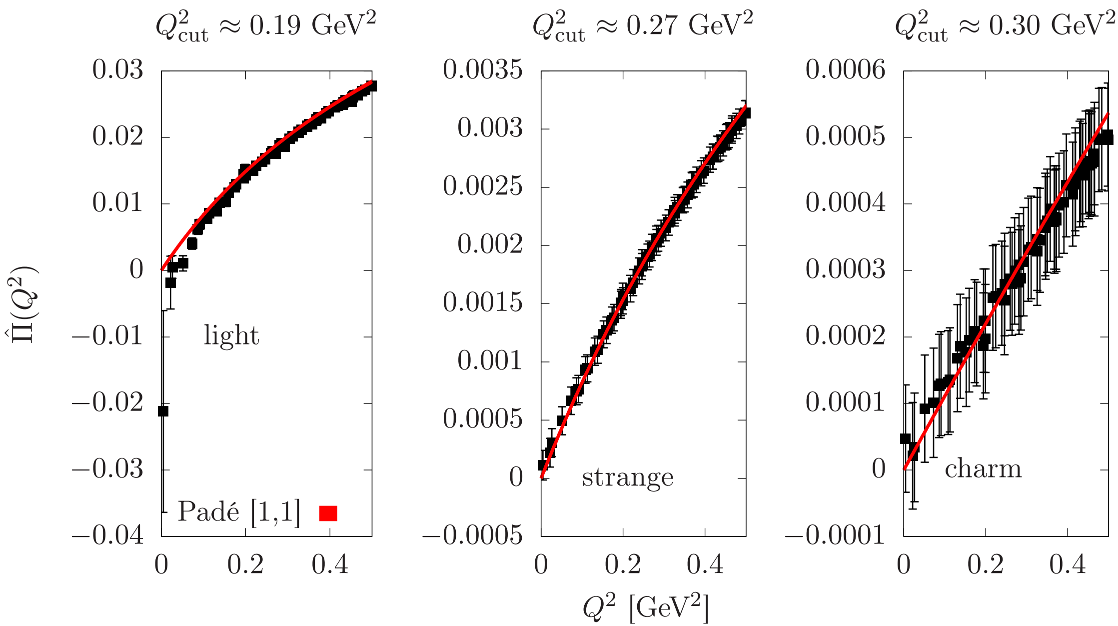

Extended frequentist method

Estimation of systematic errors

Systematic error: The central value is given by the median of the central values of all variations, such as

- picking subsets of data: $m_\pi < 400$ MeV, $a < 0.070$ fm
- different fit ansätze: $a_\mu^{\mathrm{A}}(m_\pi^2, a)$, $a_\mu^{\mathrm{B}}(m_\pi^2, a), \ldots$
- variations for Q^2_{cut} or t_{trunc}.

The central 68% give the systematic error.

Statistical error: Compute the median for each bootstrap sample. The statistical error is given by the central 68% of these medians.

The errors can be computed with weights (e.g. p-values of the fits).

Application to a_μ^{HLO}

Step 1: Determine $\widehat{\Pi}(Q^2)$ for each ensemble and all variations.

Step 2: Compute a_μ^{HLO} from $\widehat{\Pi}(Q^2)$.

Step 3: Perform the continuum and chiral extrapolation for all variations of the previous step and all variations of the extrapolation.

The effect of weights in the EFM