The multi-flavor Schwinger model with chemical potential

-

Overcoming the sign problem with MPS

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MPQ

in collaboration with

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Outline

1. Matrix Product States
2. Motivation
3. Multi-flavor Schwinger model on the lattice
4. Preliminary results
5. Conclusion & Outlook

Matrix Product States (MPS)

MPS ansatz

- Coming form quantum information theory
- MPS ansatz with open boundary conditions (OBC) for system with N sites

$$|\psi\rangle = \sum_{i_1, i_2, \ldots, i_N} A^{i_1} A^{i_2} \ldots A^{i_N} |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle$$

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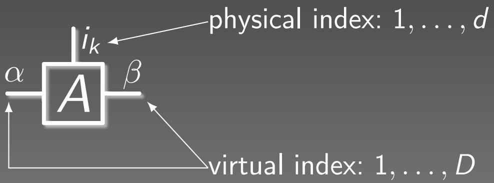
- Tensors $A^{i_k} \in \mathbb{C}^{D \times D}$



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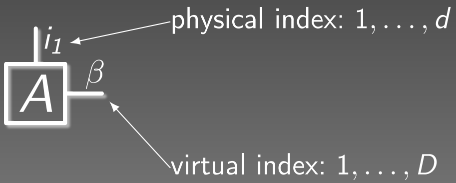
- Tensor $A^{i_1} \in \mathbb{C}^{1\times D}$



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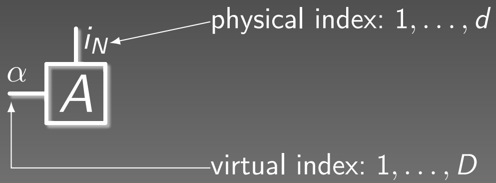
- Tensor $A^{i_N} \in \mathbb{C}^{D \times 1}$



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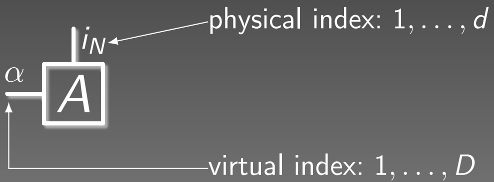

- D: **Bond dimension** of the MPS

Matrix Product States (MPS)

Scaling

- Number of parameters in the MPS with OBC

$|\psi\rangle =$ 

$$(N-2)D^2 d + 2Dd = \mathcal{O}(ND^2 d)$$

M. B. Hastings, Journal of Statistical Mechanics 2007 (2007)

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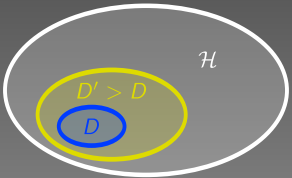
- Quantum information: Physically relevant states $D \ll d^{\lfloor \frac{N}{2} \rfloor}$



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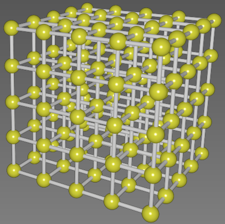
Matrix Product States (MPS)

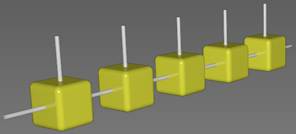
Monte Carlo

Expectation values	✓
Correlation functions	✓
Time evolution	✗
Sign problem free	✗
$3+1$ dimensions	✓

Matrix Product States

Ground states	✓
Low lying excitations	✓
Time evolution	✓
Sign problem free	✓
$3+1$ dimensions	✗





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Multi-flavor Schwinger model

Lattice Hamiltonian formulation

- Lattice formulation with Kogut-Susskind staggered fermions

$$H = -\frac{i}{2a}\sum_{n=1}^{N-1}\sum_{f=1}^{F}\left(\phi_{n,f}^{\dagger}\, e^{i\theta(n)}\, \phi_{n+1,f} - \text{h.c}\right) + \sum_{n=1}^{N}\sum_{f=1}^{F}\left((-1)^n m_f + \kappa_f\right)\phi_{n,f}^{\dagger}\phi_{n,f} + \frac{ag^2}{2}\sum_{n=1}^{N-1} L_n^2$$

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Kinetic part + Coupling to gauge field — Mass term — Chemical potential — Electric energy

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- Gauss Law

$$L_n - L_{n-1} = Q_n = \sum_{f=1}^{F}\left[\phi^{\dagger}_{n,f}\phi_{n,f} - \frac{1}{2}\left(1-(-1)^n\right)\right]$$

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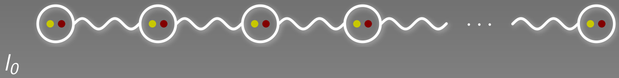

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$l_0 + Q_1 = l_1$

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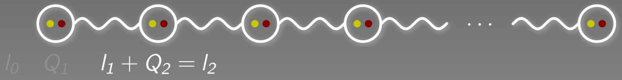

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$l_0 + Q_1 \quad + \quad Q_2 = l_2$

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$$l_0 + Q_1 \quad + \quad Q_2 \quad + \quad Q_3 = l_3$$

Multi-flavor Schwinger model

Lattice Hamiltonian formulation

- Dimensionless formulation on gauge invariant subspace

$$W = -x \sum_{n=1}^{N-1} \sum_{f=1}^{F} \left(\phi_{n,f}^{\dagger} \phi_{n+1,f} - \text{h.c} \right)$$
$$+ \sum_{n=1}^{N} \sum_{f=1}^{F} \left((-1)^n \mu_f + \nu_f \right) \phi_{n,f}^{\dagger} \phi_{n,f} + \sum_{n=1}^{N-1} \left(l_0 + \sum_{k=1}^{n} Q_n \right)^2$$

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x: $1/(ag)^2$

μ_f: $2m_f/ag^2$

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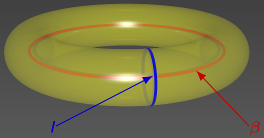
x: $1/(ag)^2$, μ_f: $2m_f/ag^2$, ν_f: $2\kappa_f/ag^2$

- Hamiltonian commutes with $N_i = \sum_{n=1}^{N} \phi^\dagger_{n,i}\phi_{n,i}$

Multi-flavor Schwinger model

Previous analytic work

- Analysis of the phase structure for the massless case $m/g = 0$ on a torus

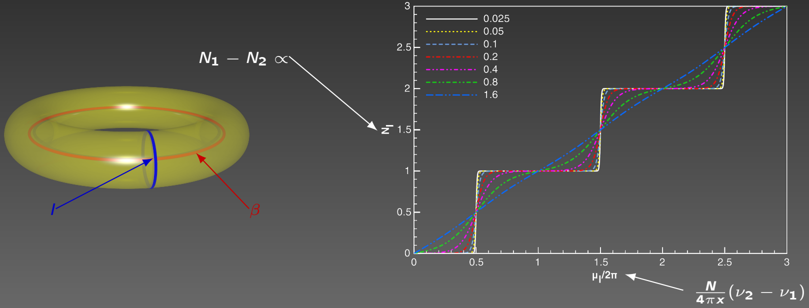


R. Narayanan, Phys. Rev. D 86, 125008 (2012)
R. Lohmayer, R. Narayanan, Phys. Rev. D 88, 105030 (2013)

Multi-flavor Schwinger model

Previous analytic work

- Analysis of the phase structure for the massless case $m/g = 0$ on a torus
- Result for the two flavor case



- Jumps in the particle number difference correspond to first oder phase transitions

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Preliminary results

MPS approach to the multi-flavor Schwinger model

- Focus on the two flavor case at zero temperature
- Fix $\nu_1 = 0$ and vary ν_2
- Use MPS with with OBC

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MPS approach to the multi-flavor Schwinger model

- Focus on the two flavor case at zero temperature
- Fix $\nu_1 = 0$ and vary ν_2
- Use MPS with with OBC
- Fixed physical volume

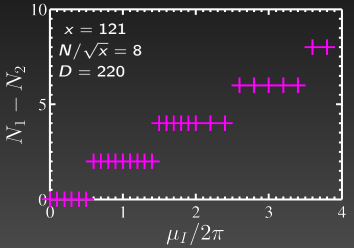
Bond dimension: $D \in [40, 220]$

→ Lattice spacing: $x \in [9, 121]$

→ Volume: $\frac{N}{\sqrt{x}} \in \{2, 6, 8\}$

Preliminary results

Massless case $m/g = 0$



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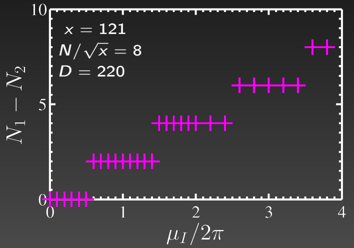

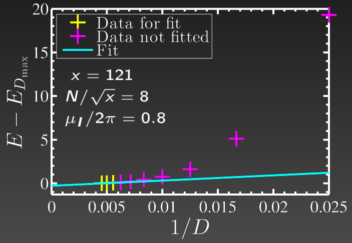

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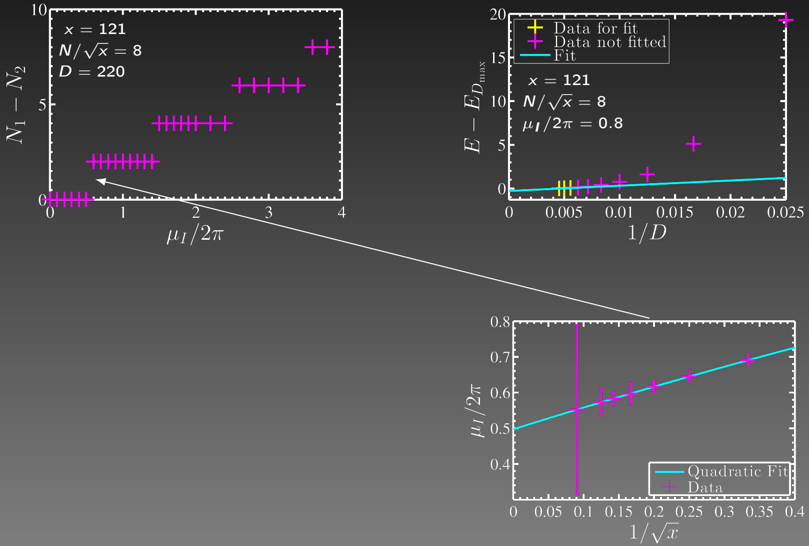

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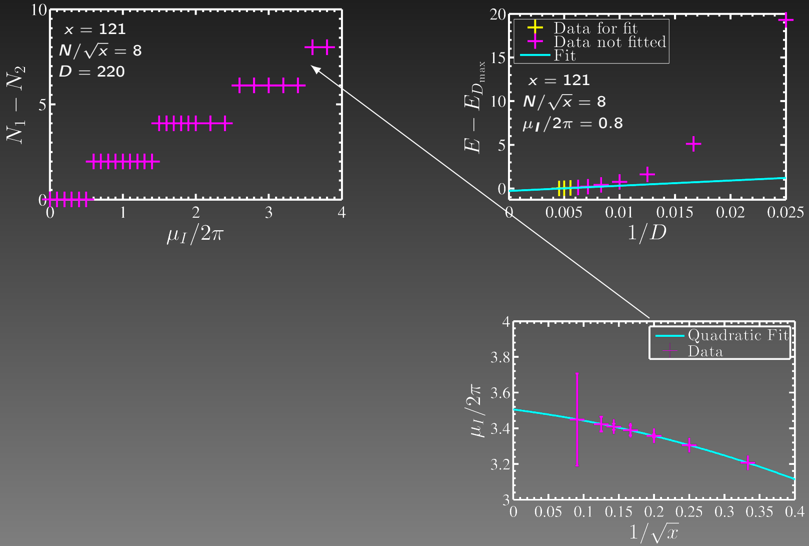

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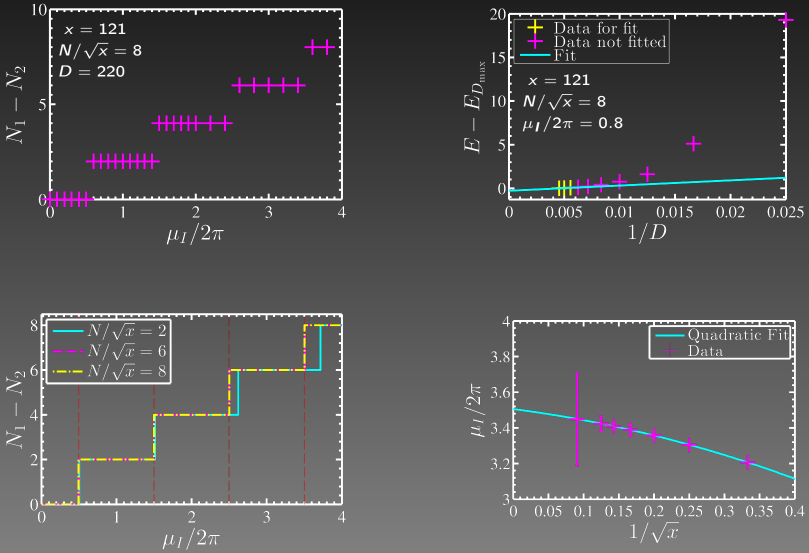
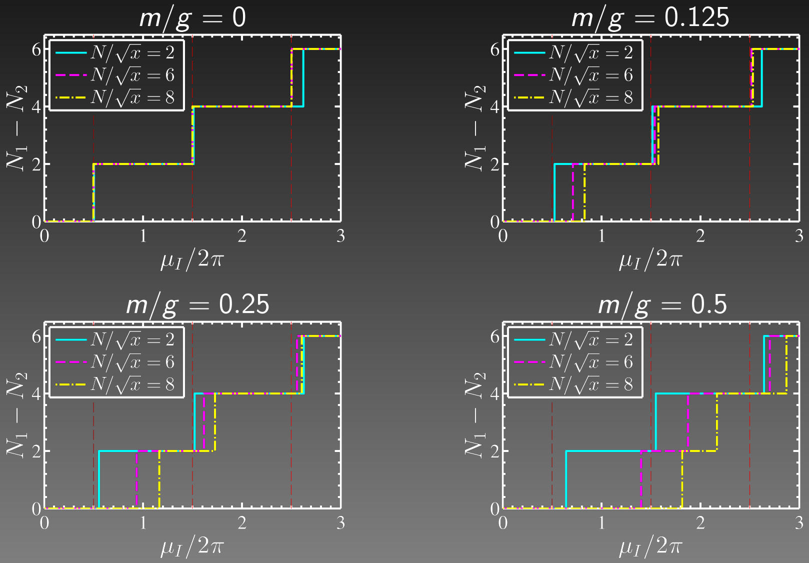


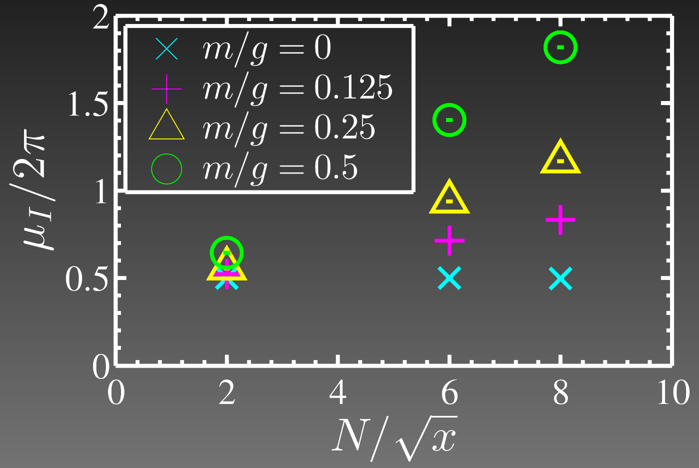
Preliminary results

Massive case



Preliminary results

Scaling with volume of the 1. jump



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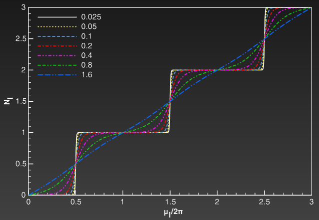
Conclusion & Outlook

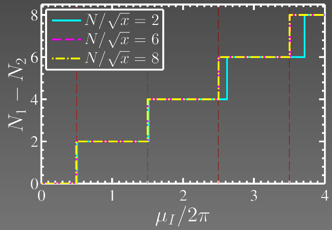
Conclusion

- **Good agreement** with analytic prediction
- **Sign problem** can be **overcome**
- Readily extends to massive case



Outlook

- MPS are very versatile
  - Finite temperature
  - Dynamical problems
  - Non-abelian gauge models
  - Generalizations to higher dimensions exist



Thank you for your attention!

More on Tensor Networks in Lattice Gauge Theory:

Plenary Talk by Dr. Shinji TAKEDA

A. Ground state calculation with MPS

Algorithm

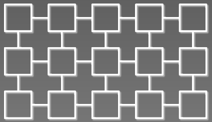
- State as MPS with OBC

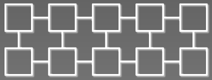
$|\psi\rangle =$ 

- Hamiltonian as Matrix Product Operator (MPO)

$H =$ 

- $E_0 = \min_{|\psi\rangle \in \mathcal{H}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \Leftrightarrow \min_{|\psi\rangle \in \mathcal{H}} \langle\psi|H|\psi\rangle - \lambda\langle\psi|\psi\rangle$

$\langle\psi|H|\psi\rangle =$ 

$\langle\psi|\psi\rangle =$ 

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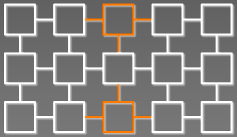
$$\Rightarrow \qquad H_{\text{eff}}\vec{A} = \lambda\vec{A}$$

$$H_{\text{eff}} =$$



B. Extracting the jumps

Eigenvalues inside a block

- Hamiltonian conserves particle numbers N_1 and N_2

$$H = \nu_1 N_1 + \nu_2 N_2 + H_{\text{aux}},$$

- Energy eigenvalues

$$E_{(N_1,N_2)}(\nu_1,\nu_2) = \nu_1 N_1 + \nu_2 N_2 + E_{\min}\left(H_{\text{aux}}|_{(N_1,N_2)}\right)$$

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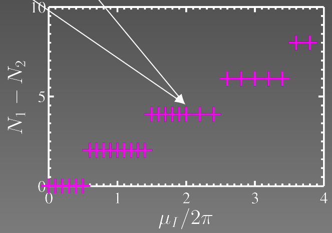

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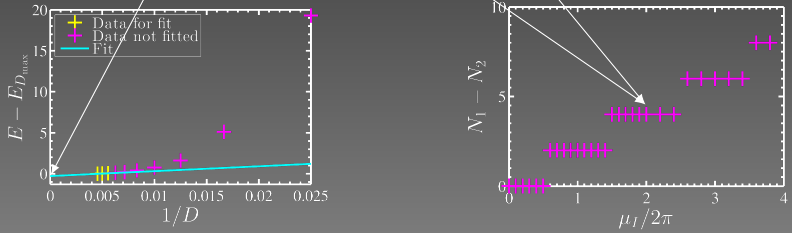

B. Extracting the jumps

Locating the jumps

- Transition from (N_1, N_2) to $(\bar{N}_1, \bar{N}_2) \Leftrightarrow E_{(N_1,N_2)} = E_{(\bar{N}_1,\bar{N}_2)}$

$$(\nu_2 - \nu_1)|_{\text{jump}} = \frac{E_{\text{min}}(H_{\text{aux}}|_{(\bar{N}_1,\bar{N}_2)}) - E_{\text{min}}(H_{\text{aux}}|_{(N_1,N_2)})}{\bar{N}_1 - N_1}$$