Abelian color cycles: a new approach to strong coupling expansion and dual representations for non-abelian lattice gauge theory

Carla Marchis, Christof Gattringer

University of Graz

Lattice 2016, 29/07/2016

Der Wissenschaftsfonds.

Dual approach

Method developed in order to overcome the sign problem.

General strategy:

- The theory is exactly mapped to new d.o.f., the **dual variables**, by factorizing and expanding the Boltzmann factor;
- The conventional d.o.f. are integrated out, leading to **constraints for the dual variables**;
- These constraints allow for geometrical interpretation of the system:

$$\text{dual variables for matter} \longleftrightarrow \text{loops}$$
$$\text{dual variables for gauge fields} \longleftrightarrow \text{surfaces;}$$

- Dual partition sum has only real and positive contributions.

Dual approach

Successful in overcoming the complex action problem of various abelian theories:

- Silver Blaze phenomenon for a charged scalar ϕ^4 field;
  [Gattringer and Kloiber, Nucl.Phys. **B869** (2013)]
- Abelian Gauge-Higgs systems;
  [Mercado, Gattringer and Schmidt, Phys.Rev.Lett. **111** (2013)]
- O(N) and CP(N-1) models with chemical potential;
  [Bruckmann, Gattringer, Kloiber and Sulejmanpasic, Phys.Lett. **B749** (2015)]
- Massless fermions with chemical potential and U(1) gauge fields;
  [Gattringer and Sazonov, Phys.Rev. **D93** (2016)]
- 2-dimensional O(3) model at $\mu \neq 0$.
  [Bruckmann, Wed @ 12:10]

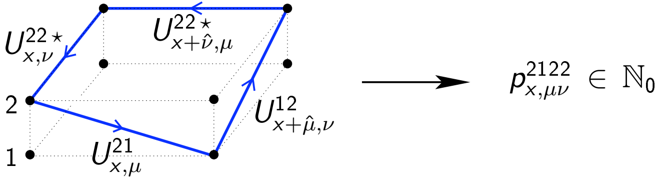
Abelian color cycles...

SU(2) lattice gauge action:

$$S_G[U] = -\frac{\beta}{2} \sum_{x,\mu<\nu} \mathrm{Tr}\, U_{x,\mu} U_{x+\hat{\mu},\nu} U^{\dagger}_{x+\hat{\nu},\mu} U^{\dagger}_{x,\nu}$$

$$= -\frac{\beta}{2} \sum_{x,\mu<\nu} \sum_{a,b,c,d=1}^{2} U^{ab}_{x,\mu} U^{bc}_{x+\hat{\mu},\nu} U^{dc\,\star}_{x+\hat{\nu},\mu} U^{ad\,\star}_{x,\nu}.$$

Abelian color cycles $U^{ab}_{x,\mu} U^{bc}_{x+\hat{\mu},\nu} U^{dc\,\star}_{x+\hat{\nu},\mu} U^{ad\,\star}_{x,\nu}$: paths in color space along plaquettes.



For SU(2) there are a total of 2^4 abelian color cycles (ACC).

...enable the reordering of link elements...

Expand the Boltzmann factor and reorder the link elements:

$$Z = \int D[U] \prod_{x,\mu<\nu} \prod_{a,b,c,d=1}^{2} e^{\frac{\beta}{2} U^{ab}_{x,\mu} U^{bc}_{x+\hat{\mu},\nu} U^{dc\,\star}_{x+\hat{\nu},\mu} U^{ad\,\star}_{x,\nu}}$$

$$= \int D[U] \prod_{x,\mu<\nu} \prod_{a,b,c,d=1}^{2} \sum_{p^{abcd}_{x,\mu\nu}=0}^{\infty} \frac{\left(\frac{\beta}{2}\right)^{p^{abcd}_{x,\mu\nu}}}{p^{abcd}_{x,\mu\nu}!} \left(U^{ab}_{x,\mu} U^{bc}_{x+\hat{\mu},\nu} U^{dc\,\star}_{x+\hat{\nu},\mu} U^{ad\,\star}_{x,\nu}\right)^{p^{abcd}_{x,\mu\nu}}$$

$$= \sum_{\{p\}} \prod_{x,\mu<\nu} \prod_{a,b,c,d} \frac{\left(\frac{\beta}{2}\right)^{p^{abcd}_{x,\mu\nu}}}{p^{abcd}_{x,\mu\nu}!} \prod_{x,\mu} \int dU_{x,\mu} \prod_{a,b} \left(U^{ab}_{x,\mu}\right)^{N^{ab}_{x,\mu}[p]} \left(U^{ab\,\star}_{x,\mu}\right)^{\bar{N}^{ab}_{x,\mu}[p]}$$

$p^{abcd}_{x,\mu\nu} \in \mathbb{N}_0$ **cycle occupation numbers**

..and the integration over the Haar measure

Explicit parametrization of SU(2):

$$U = \begin{pmatrix} \cos\theta\, e^{i\alpha} & \sin\theta\, e^{i\beta} \\ -\sin\theta\, e^{-i\beta} & \cos\theta\, e^{-i\alpha} \end{pmatrix}; \qquad \alpha,\beta \in [-\pi,\pi],\ \theta \in [0,\pi/2]$$

with normalized Haar measure

$$dU = 2d\theta \sin\theta \cos\theta \frac{d\alpha}{2\pi}\frac{d\beta}{2\pi}.$$

Partition sum

$$Z = \sum_{\{p\}} W_\beta[p](-1)^{\sum_{x,\mu} J^{21}_{x,\mu}} \prod_{x,\mu} \delta(J^{11}_{x,\mu} - J^{22}_{x,\mu})\delta(J^{12}_{x,\mu} - J^{21}_{x,\mu}).$$

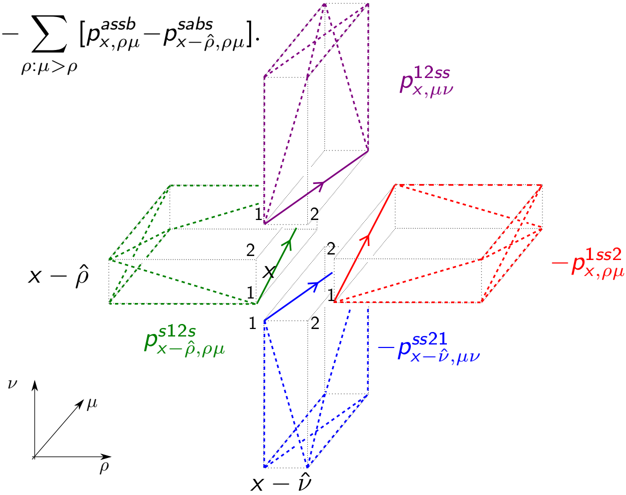
Geometrical interpretation

Fluxes

$$J^{ab}_{x,\mu} = \sum_{\nu:\mu<\nu} [p^{abss}_{x,\mu\nu} - p^{ssba}_{x-\hat{\nu},\mu\nu}] - \sum_{\rho:\mu>\rho} [p^{assb}_{x,\rho\mu} - p^{sabs}_{x-\hat{\rho},\rho\mu}].$$

Notation

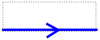
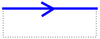
$$p^{abss}_{x,\mu\nu} = \sum_{c,d=1}^{2} p^{abcd}_{x,\mu\nu}$$



Constraints

$$Z = \sum_{\{p\}} W_\beta[p](-1)^{\sum_{x,\mu} J^{21}_{x,\mu}} \prod_{x,\mu} \delta(J^{11}_{x,\mu} - J^{22}_{x,\mu})\delta(J^{12}_{x,\mu} - J^{21}_{x,\mu}).$$

Constraints:

$\forall x, \mu$  $\stackrel{!}{=}$  and  $\stackrel{!}{=}$ 

Minus sign:

$$(-1)^{\#\text{ flux crossings}}$$

Staggered fermions

Fermionic partition sum

$$Z_F[U] = \int D[\bar{\psi}, \psi] e^{-S_F[\bar{\psi}, \psi, U]}; \qquad \int D[\bar{\psi}, \psi] = \int \prod_x \prod_{a=1}^{2} d\bar{\psi}_x^a d\psi_x^a$$

$$\psi_x = \begin{pmatrix} \psi_x^1 \\ \psi_x^2 \end{pmatrix}, \quad \bar{\psi}_x = \begin{pmatrix} \bar{\psi}_x^1 & , \bar{\psi}_x^2 \end{pmatrix};$$

$$\eta_{x,1} = 1, \; \eta_{x,2} = (-1)^{x_1} \; \eta_{x,3} = (-1)^{x_1+x_2} \; \eta_{x,4} = (-1)^{x_1+x_2+x_3}.$$

Decomposition into color bilinears of the fermionic action:

$$S_F[\bar{\psi}, \psi, U] = \sum_x \left(m\bar{\psi}_x \psi_x + \frac{1}{2} \sum_\mu \eta_{x,\mu} [\bar{\psi}_x U_{x,\mu} \psi_{x+\hat{\mu}} - \bar{\psi}_{x+\hat{\mu}} U_{x,\mu}^\dagger \psi_x] \right)$$

$$= \sum_x \left(m \sum_{a=1}^{2} \bar{\psi}_x^a \psi_x^a + \frac{1}{2} \sum_\mu \eta_{x,\mu} \sum_{a,b=1}^{2} [\bar{\psi}_x^a U_{x,\mu}^{ab} \psi_{x+\hat{\mu}}^b - \bar{\psi}_{x+\hat{\mu}}^b U_{x,\mu}^{ab\,\star} \psi_x^a] \right)$$

Staggered fermions

Expansion of the exponential and reordering of the terms:

$$
\begin{aligned}
Z_F[U] = & \int D[\bar{\psi},\psi] \prod_{x} \prod_{a=1}^{2} \sum_{s_x^a=0}^{1} \left(-m\bar{\psi}_x^a \psi_x^a\right)^{s_x^a} \\
& \times \prod_{x,\mu} \prod_{a,b=1}^{2} \sum_{k_{x,\mu}^{ab}=0}^{1} \left(-\frac{\eta_{x,\mu}}{2}\bar{\psi}_x^a U_{x,\mu}^{ab} \psi_{x+\hat{\mu}}^b\right)^{k_{x,\mu}^{ab}} \sum_{\bar{k}_{x,\mu}^{ab}=0}^{1} \left(\frac{\eta_{x,\mu}}{2}\bar{\psi}_{x+\hat{\mu}}^b U_{x,\mu}^{ab\,\star} \psi_x^a\right)^{\bar{k}_{x,\mu}^{ab}} \\
= & \frac{1}{2^{2V}} \sum_{\{s,k,\bar{k}\}} (2m)^{\sum_{x,a} s_x^a} \prod_{x,\mu} \prod_{a,b} (U_{x,\mu}^{ab})^{k_{x,\mu}^{ab}} (U_{x,\mu}^{ab\star})^{\bar{k}_{x,\mu}^{ab}} \\
& \times \int \prod_{x,a} d\bar{\psi}_x^a d\psi_x^a \prod_{x,a} \left(\psi_x^a \bar{\psi}_x^a\right)^{s_x^a} \prod_{x,\mu} \prod_{a,b} \left(-\eta_{x,\mu}\bar{\psi}_x^a \psi_{x+\hat{\mu}}^b\right)^{k_{x,\mu}^{ab}} \left(\eta_{x,\mu}\bar{\psi}_{x+\hat{\mu}}^b \psi_x^a\right)^{\bar{k}_{x,\mu}^{ab}}
\end{aligned}
$$

The Grassmann integral is saturated by monomers ($s_x^a = 1$), dimers ($k_{x,\mu}^{ab} = \bar{k}_{x,\mu}^{ab} = 1$) and loops (chain of $k_{x,\mu}^{ab} = 1$ and $\bar{k}_{x,\mu}^{ab} = 1$). Only loops introduce signs.

Interaction with the gauge fields

Dual partition sum

$$Z = \sum_{\{p,s,k,\bar{k}\}} C_{MDL}[s,k,\bar{k}] W_\beta[p,k,\bar{k}] W_m[s] \prod_{x,\mu} (-1)^{J^{21}_{x,\mu}+k^{21}_{x,\mu}+\bar{k}^{21}_{x,\mu}} \prod_L \mathrm{sign}(L)$$
$$\times \prod_{x,\mu} \delta\left(J^{11}_{x,\mu} + k^{11}_{x,\mu} - \bar{k}^{11}_{x,\mu} - [J^{22}_{x,\mu} + k^{22}_{x,\mu} - \bar{k}^{22}_{x,\mu}]\right)$$
$$\times \delta\left(J^{12}_{x,\mu} + k^{12}_{x,\mu} - \bar{k}^{12}_{x,\mu} - [J^{21}_{x,\mu} + k^{21}_{x,\mu} - \bar{k}^{21}_{x,\mu}]\right)$$

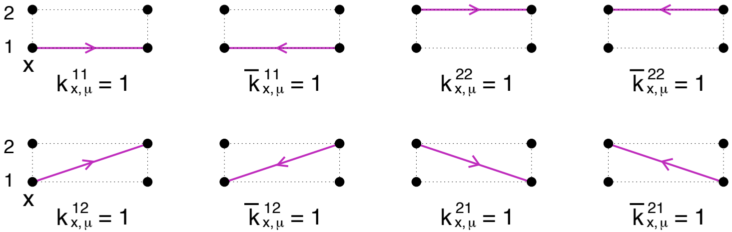
$$\mathrm{sign}(L) = -(-1)^{\#\mathrm{plaquettes}}(-1)^{\mathrm{length}/2}(-1)^{\mathrm{temp.winding}}$$

Interaction with the gauge fields

Dual partition sum

$$Z = \sum_{\{p,s,k,\bar{k}\}} C_{MDL}[s,k,\bar{k}] W_\beta[p,k,\bar{k}] W_m[s] \prod_{x,\mu} (-1)^{J^{21}_{x,\mu} + k^{21}_{x,\mu} + \bar{k}^{21}_{x,\mu}} \prod_L \text{sign}(L)$$
$$\times \prod_{x,\mu} \delta\left(J^{11}_{x,\mu} + k^{11}_{x,\mu} - \bar{k}^{11}_{x,\mu} - [J^{22}_{x,\mu} + k^{22}_{x,\mu} - \bar{k}^{22}_{x,\mu}]\right)$$
$$\times \delta\left(J^{12}_{x,\mu} + k^{12}_{x,\mu} - \bar{k}^{12}_{x,\mu} - [J^{21}_{x,\mu} + k^{21}_{x,\mu} - \bar{k}^{21}_{x,\mu}]\right)$$

Dual variables for fermions:

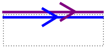


Interaction with the gauge fields

Dual partition sum

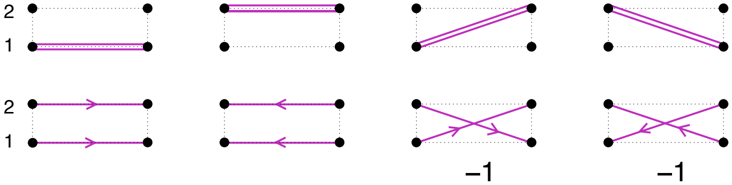
$$Z = \sum_{\{p,s,k,\bar{k}\}} C_{MDL}[s,k,\bar{k}] W_\beta[p,k,\bar{k}] W_m[s] \prod_{x,\mu} (-1)^{J^{21}_{x,\mu} + k^{21}_{x,\mu} + \bar{k}^{21}_{x,\mu}} \prod_L \text{sign}(L)$$
$$\times \prod_{x,\mu} \delta\left(J^{11}_{x,\mu} + k^{11}_{x,\mu} - \bar{k}^{11}_{x,\mu} - [J^{22}_{x,\mu} + k^{22}_{x,\mu} - \bar{k}^{22}_{x,\mu}]\right)$$
$$\times \delta\left(J^{12}_{x,\mu} + k^{12}_{x,\mu} - \bar{k}^{12}_{x,\mu} - [J^{21}_{x,\mu} + k^{21}_{x,\mu} - \bar{k}^{21}_{x,\mu}]\right)$$

Constraints:

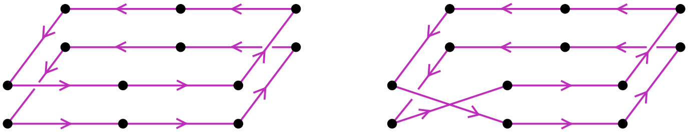
$\forall x, \mu$  $\overset{!}{=}$  and  $\overset{!}{=}$ 

Strong coupling ($\beta = 0$)

Building bricks for the construction of configurations compatible with the gauge constraints:

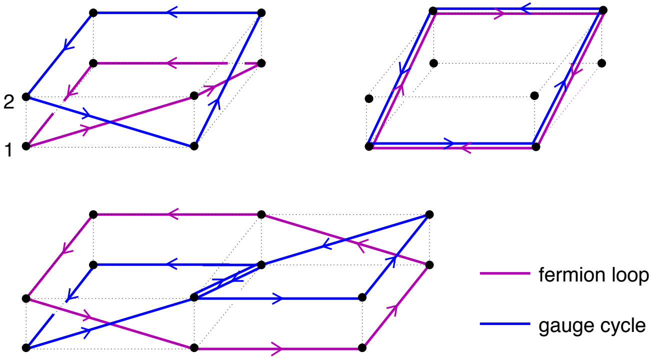


Examples of fermions loop



No minus sign

Strong coupling expansion



Conclusion and Outlook

- Development of a new approach, the ACC, that enable the dualization of non-abelian LGT.
- The weights for all terms of the strong coupling expansion are known in closed form.
- Up to $O(\beta^3)$ only positive terms.
- Configurations with negative sign are not excluded completely by the constraints. Possible resummations?
- The ACC concept can be easily generalized to other non-abelian gauge groups.

Thank you for your attention