Leptonic decay-constant ratio f_K/f_π from clover-improved $N_f = 2+1$ QCD

Enno E. Scholz

Institute for Theoretical Physics, University of Regensburg

34th International Symposium on Lattice Field Theory

July, 24-30, 2016

Southampton, UK

- in collaboration with (Wuppertal U., JSC, CNRS Marseille, UR)

S. Dürr, Z. Fodor, C. Hoelbling, S. Krieg,
L. Lellouch, T. Lippert, T. Rae, A. Schäfer,
K.K. Szabo, L. Varnhorst

- preprint:

`arXiv:1601.05998[hep-lat]`

Leptonic decay-constant ratio f_K/f_π from lattice QCD using 2+1 clover-improved fermion flavors with 2-HEX smearing

Cabibbo-Kobayashi-Maskawa (CKM)-matrix elements

- (leptonic) decay constants $f_{\pi^\pm}$, $f_{K^\pm}$ $\to$ $\ell\,\nu_\ell$ of (charged) π-, K-meson $\longrightarrow$ V_{us} , V_{ud}
- leptonic decay ($\ell = e^\pm$, $\boldsymbol{\mu}^\pm$), experimentally measured: decay widhts

$$\frac{\Gamma(K^\pm \to \ell\nu_\ell)}{\Gamma(\pi^\pm \to \ell\nu_\ell)} = \frac{V_{\text{us}}^2}{V_{\text{ud}}^2} \frac{f_{K^\pm}^2}{f_{\pi^\pm}^2} \frac{M_{K^\pm}^2}{M_{\pi^\pm}^2} \frac{\left(1 - \frac{m_\ell^2}{M_{K^\pm}^2}\right)^2}{\left(1 - \frac{m_\ell^2}{M_{\pi^\pm}^2}\right)^2} \left(1 + \delta_{\text{em}}\right)$$

Marciano, 2004

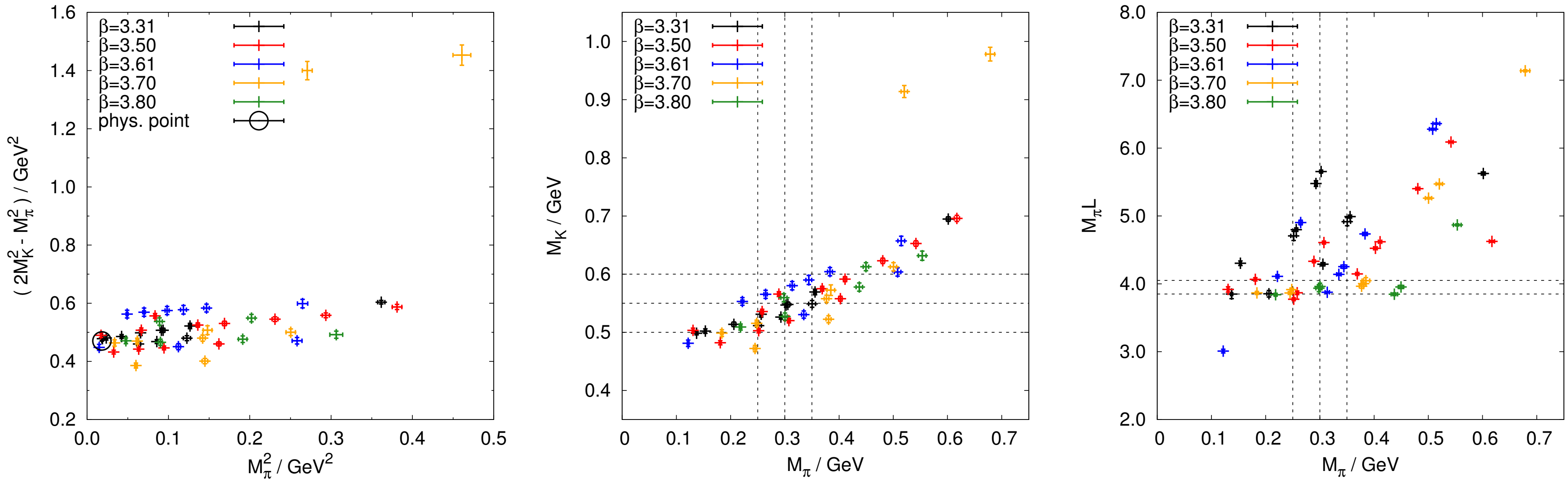
- measured on lattice (w/o isospin-splitting/EM)

$$\frac{f_K}{f_\pi}$$

- $V_{\text{us}}/V_{\text{ud}}$, with further input: first-row unitarity, V_{us}

ensembles

- generated by the Budapest-Marseille-Wuppertal (BMW)-Collaboration
- $N_f = 2 + 1$ tree-level clover-improved fermions, 2-HEX smearing
- tree-level Symanzik gluon action
- 47 ensembles in total
- five β-values: 3.31, 3.50, 3.61, 3.70, 3.80 ($1/a \simeq 1.7$–3.7 GeV, $a \simeq 0.12$–0.05 fm)
- M_π: 130–680 MeV, m_s physical (except two ensembles)



- have been used in other studies: quark masses, SU(2)-LECs and f_π, B_K, nucleon σ-term

extrapolation to the physical world

- interpolation to physical quark-masses $m_{\mathrm{ud}} \to m_{\mathrm{ud}}^{\mathrm{phys}}$, $m_{\mathrm{s}} \to m_{\mathrm{s}}^{\mathrm{phys}}$

- extrapolate to infinite volume $V \to \infty$

- extrapolate to the continuum limit $a \to 0$

- scale-setting $1/a$

variation over fits, ranges, . . . ("Wuppertal-method") for systematic uncertainty

- at least two different functional forms for each extrapolation

extrapolation to M_π^{phys}, M_K^{phys}

- NLO-SU(3): 2 fit-parameters

$$\frac{f_K}{f_\pi} = 1 + \frac{c_0}{2}\left\{\frac{5}{4}M_\pi^2 \log\left(\frac{M_\pi^2}{\mu^2}\right) - \frac{1}{2}M_K^2 \log\left(\frac{M_K^2}{\mu^2}\right) - \frac{3}{4}M_\eta^2 \log\left(\frac{M_\eta^2}{\mu^2}\right) + c_1[M_K^2 - M_\pi^2]\right\}$$

$$M_\eta^2 = \frac{1}{3}(4M_K^2 - M_\pi^2)\,, \quad c_0 = \frac{1}{(4\pi F_0)^2}\,, \quad c_1 = 128\pi^2 L_5(\mu)$$

- 3-,4-,6-parameter polynomial in M_π^2 (dominant) and $[M_K^2 - M_\pi^2]$ (flavor-symmetry constr.):

$$\frac{f_K}{f_\pi} = 1 + [M_K^2 - M_\pi^2]\left(c_0^{\text{3-par}} + c_1^{\text{3-par}}[M_K^2 - M_\pi^2] + c_2^{\text{3-par}} M_\pi^2\right)$$

$$\frac{f_K}{f_\pi} = 1 + [M_K^2 - M_\pi^2]\left(c_0^{\text{4-par}} + c_1^{\text{4-par}}[M_K^2 - M_\pi^2] + c_2^{\text{4-par}} M_\pi^2 + c_3^{\text{4-par}} M_\pi^4\right)$$

$$\frac{f_K}{f_\pi} = 1 + [M_K^2 - M_\pi^2]\left(c_0^{\text{6-par}} + c_1^{\text{6-par}}[M_K^2 - M_\pi^2] + c_2^{\text{6-par}} M_\pi^2 + c_3^{\text{6-par}} M_\pi^4 + c_4^{\text{6-par}} M_\pi^2[M_K^2 - M_\pi^2] + c_5^{\text{6-par}}[M_K^2 - M_\pi^2]^2\right)$$

- all fit-functions satisfy flavor-symmetry constraint: $\left.\frac{f_K}{f_\pi}\right|_{m_{\text{ud}}=m_{\text{s}}} = 1$

continuum limit extrapolation

- cut-off effects
  (clover-improved fermions, tree-level Symanzik gluons)

  $$\propto \alpha\, a$$

  with strong coupling $\alpha = g^2/(4\pi)$ at scale $1/a$
- α: logarithmic function of a
- two ansätze: a^2- and $a\alpha$-scaling

β	a^{-1}/GeV	$\alpha_{N_f=3}$	$\alpha_{N_f=4}$
3.31	1.670(07)	0.327	0.333
3.50	2.134(15)	0.286	0.295
3.61	2.576(28)	0.262	0.271
3.70	3.031(32)	0.244	0.254
3.80	3.657(37)	0.227	0.237

$$\frac{f_K}{f_\pi} = 1 + \left(\frac{f_K}{f_\pi}(M_\pi, M_K) - 1\right)\left(1 + c^{\text{disc}} a^2\right)$$

$$\frac{f_K}{f_\pi} = 1 + \left(\frac{f_K}{f_\pi}(M_\pi, M_K) - 1\right)\left(1 + c^{\text{disc}} \alpha a\right)$$

1 fit-parameter, flavor-symmetry constraint

scale-setting: mass-independent via M_Ω for each β **or** per-ensemble

infinite volume extrapolation

- ChPT: finite volume $M_{\pi,K}(L)$, $f_{\pi,K}(L)$ $\leftrightarrow$ infinite volume $M_{\pi,K}$, $f_{\pi,K}$

$$
\begin{aligned}
\frac{f_K(L)}{f_\pi(L)} &= \frac{f_K}{f_\pi}\Big(1 + c^{\mathsf{FV}}\Big[\frac{5}{8}\tilde{g}_1(M_\pi L) - \frac{1}{4}\tilde{g}_1(M_K L) - \frac{3}{8}\tilde{g}_1(M_\eta L)\Big]\Big) \\
\tilde{g}_1(z) &= \frac{24}{z}K_1(z) + \frac{48}{\sqrt{2}z}K_1(\sqrt{2}z) + \frac{32}{\sqrt{3}z}K_1(\sqrt{3}z) + \frac{24}{2z}K_1(2z) + \ldots \\
K_1(z) &= \sqrt{\frac{\pi}{2z}}\,e^{-z}\Big\{1 + \frac{3}{8z} - \frac{3\cdot 5}{2(8z)^2} + \frac{3\cdot 5\cdot 21}{6(8z)^3} - \frac{3\cdot 5\cdot 21\cdot 45}{24(8z)^4} + \ldots\Big\}
\end{aligned}
$$

- either full $\tilde{g}_1(z)$ or first term of expansion in z
- 1 fit-parameter
- flavor-symmetry constraint satisfied

functional forms for extrapolation — summary

- **4** functional forms for physical mass (NLO-SU(3) ChPT, 3-, 4-, 6-par. polynomial)
  2, 3, 4, or 6 fit-parameters
- **2** functional forms for continuum limit (a^2, $a\alpha$)
  1 fit-parameter
- **2** functional forms for infinite volume limit (full $\tilde{g}_1$, leading terms)
  1 fit-parameter
- **2** method for scale-setting (mass-independent, per-ensemble)

$$\mathbf{4 \times 2 \times 2 \times 2 = 32} \text{ combinations}$$

4, 5, 6 or 8 fit-parameters

fit ranges

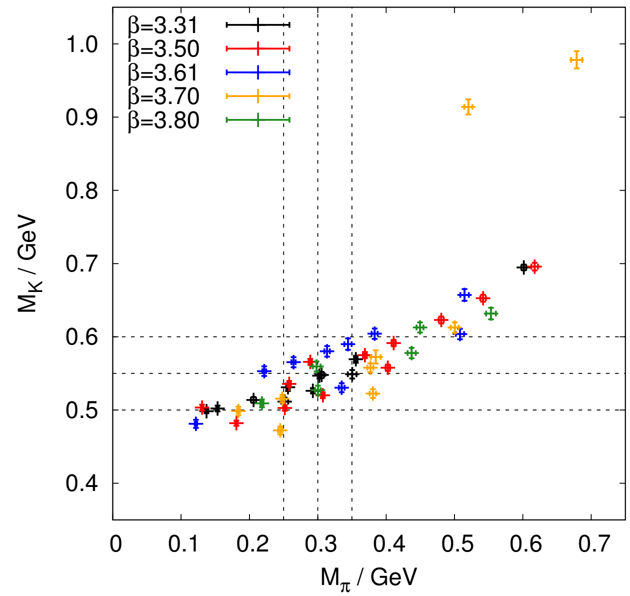
- 47 ensembles: $2^{47} \simeq \mathcal{O}(10^{14})$ combinations
  neither feasible nor sensible

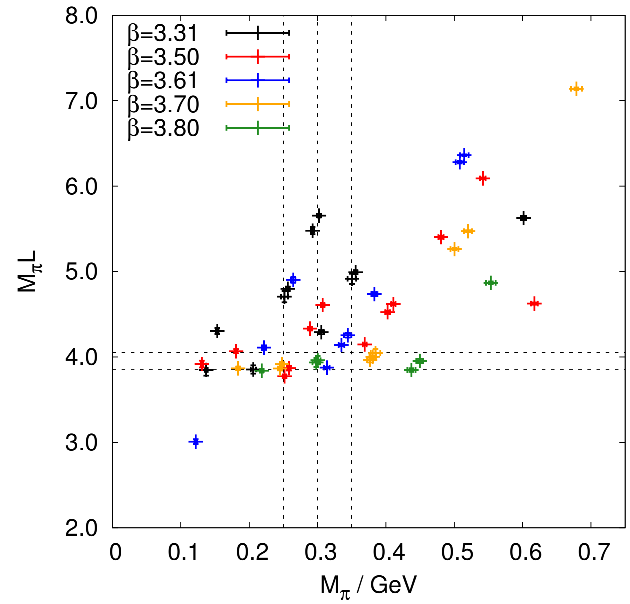
- M_π^{max}: none, 350 MeV, 300 MeV, 250 MeV
- M_K^{max}: none, 600 MeV, 550 MeV, 500 MeV
- $(M_\pi L)^{\text{min}}$: none, 3.85, 4.05
- β^{min}: 3.31, 3.50, 3.61

- ≥ 5 data-points: **63 combinations**
- only "true" fits

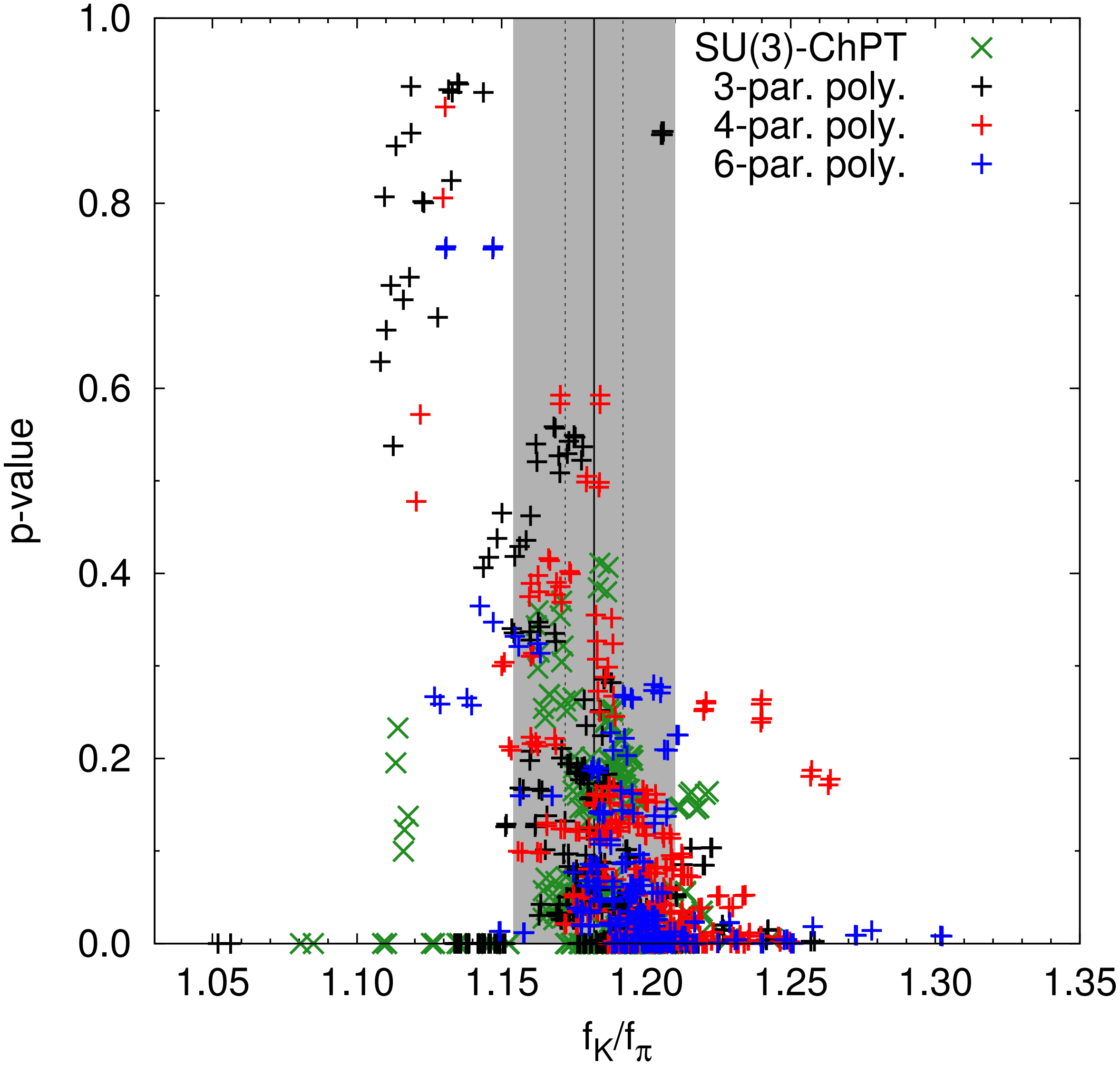
- theor. $\mathbf{32 \times 63 = 2016}$ fits

- effective: **1368 fits** $(n_{\text{d.o.f.}} \geq 1)$





results from 1368 fits



unweighted average:

$$\left.\frac{f_K}{f_\pi}\right|_{\text{flat}} = 1.191(08)_{\text{stat}}(24)_{\text{syst}}$$

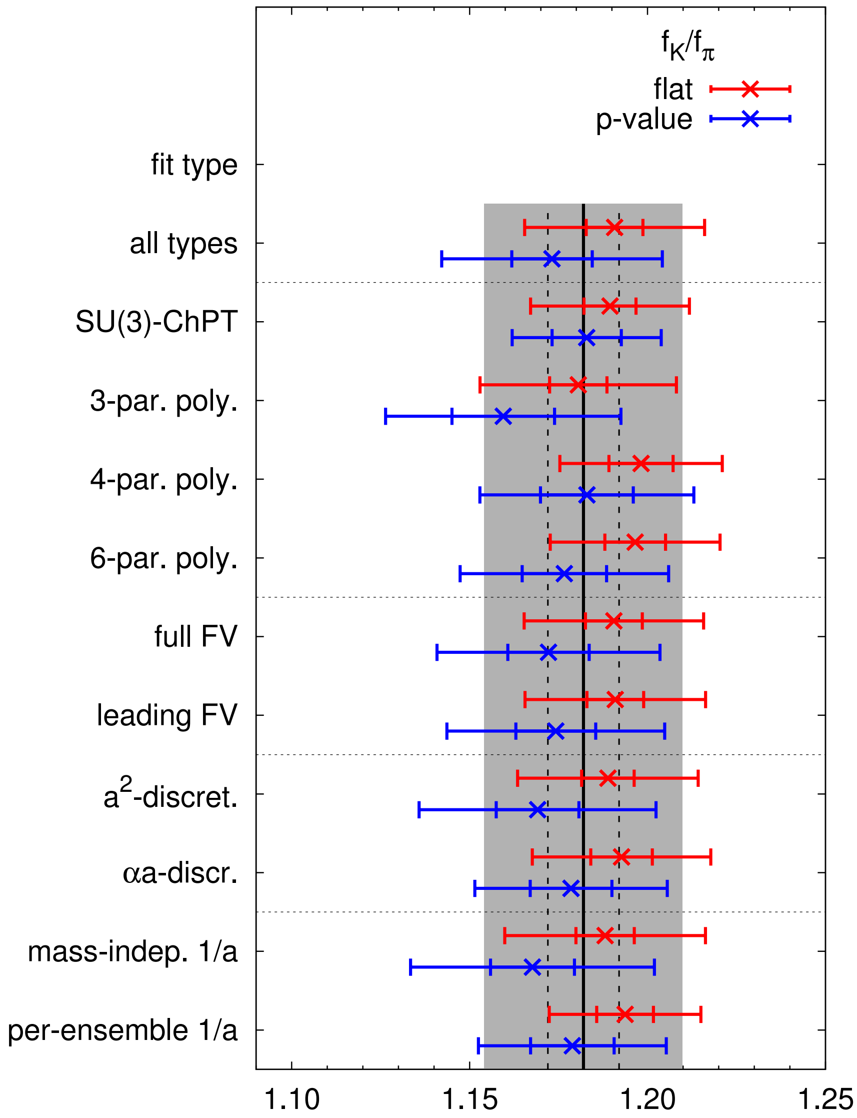
p-value weighted:

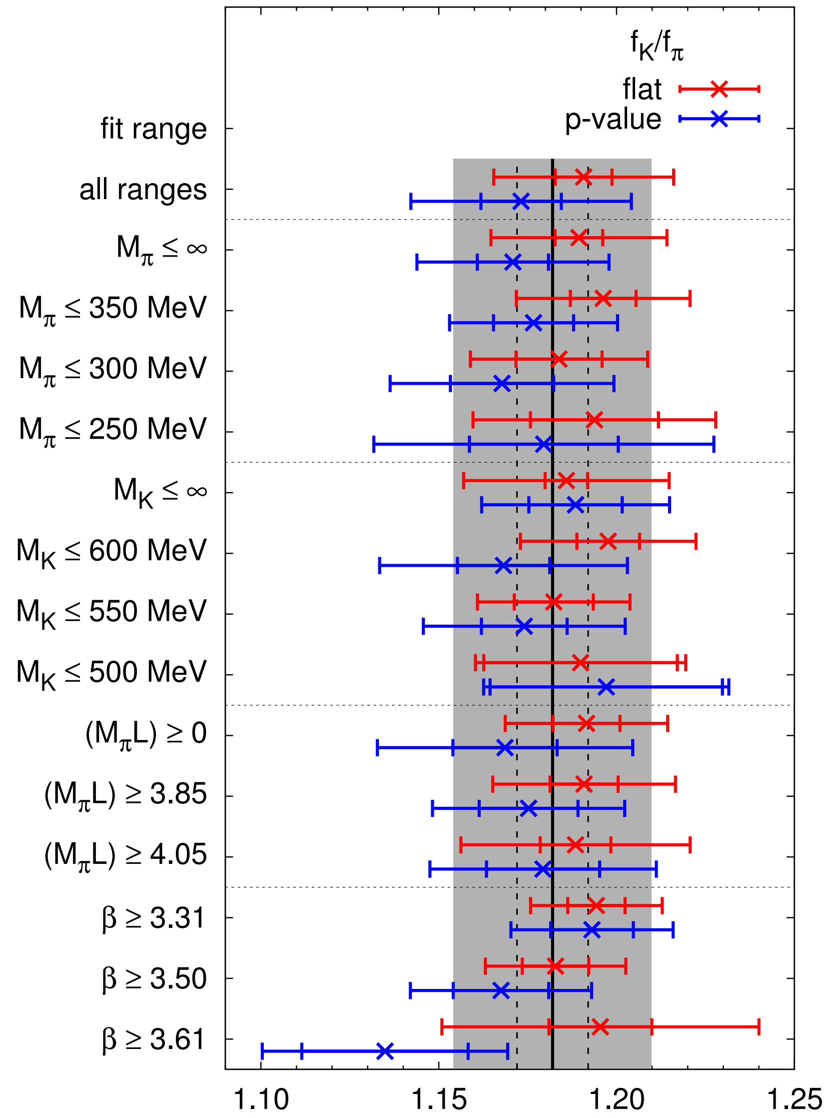
$$\left.\frac{f_K}{f_\pi}\right|_{p\text{-v.}} = 1.173(11)_{\text{stat}}(29)_{\text{syst}}$$

average:

$$\boldsymbol{\frac{f_K}{f_\pi} = 1.182\,\underbrace{(10)_{\text{stat}}(26)_{\text{syst}}}_{=(28)_{\text{comb}}}}$$

dependence on fit types, fit ranges





$f_{K^\pm}/f_{\pi^\pm}$, V_{us}, V_{ud}

- $m_{\mathrm{up}} \neq m_{\mathrm{down}}$, EM-effects: correction from ChPT [Gasser, Leutwyler (1985); Cirigliano, Neufeld (2011)]

$$\delta_{\mathrm{SU(2)}} = -0.0061(61) \rightarrow \frac{f_{K^\pm}}{f_{\pi^\pm}} = \frac{f_K}{f_\pi}\sqrt{1 + \delta_{\mathrm{SU(2)}}} = 1.178(10)_{\mathrm{stat}}(26)_{\mathrm{syst}}$$

$\delta_{\mathrm{SU(2)}}$: -0.0043(12) [Cirigliano, Neufeld (2011)] and FLAG-II, -0.0078(7) [Divitiis et al. (2012)]

- $V_{\mathrm{us}}/V_{\mathrm{ud}}$ from exp. measured decay widths [Moulson (2014), Rosner et al. (2015)]

$$\frac{V_{\mathrm{us}}}{V_{\mathrm{ud}}}\frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.27599(38)_{\mathrm{exp}} \rightarrow \frac{V_{\mathrm{us}}}{V_{\mathrm{ud}}} = 0.2343(20)_{\mathrm{stat}}(52)_{\mathrm{syst}}(03)_{\mathrm{exp}}$$

- using super-allowed nuclear β-decay $V_{\mathrm{ud}} = 0.97417(21)_{\mathrm{nuc}}$ [Hardy, Towner (2015)]

$$V_{\mathrm{us}} = 0.2282(19)_{\mathrm{stat}}(51)_{\mathrm{syst}}(03)_{\mathrm{exp\&nuc}}$$

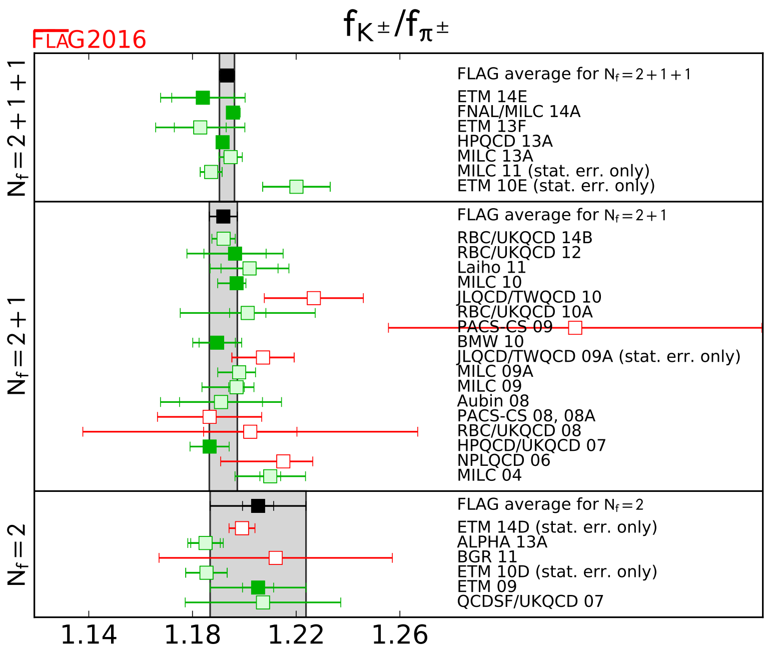
- first-row unitarity with $|V_{\mathrm{ub}}| = 4.12(37)(06) \cdot 10^{-3}$ [Rosner et al. (2015)]

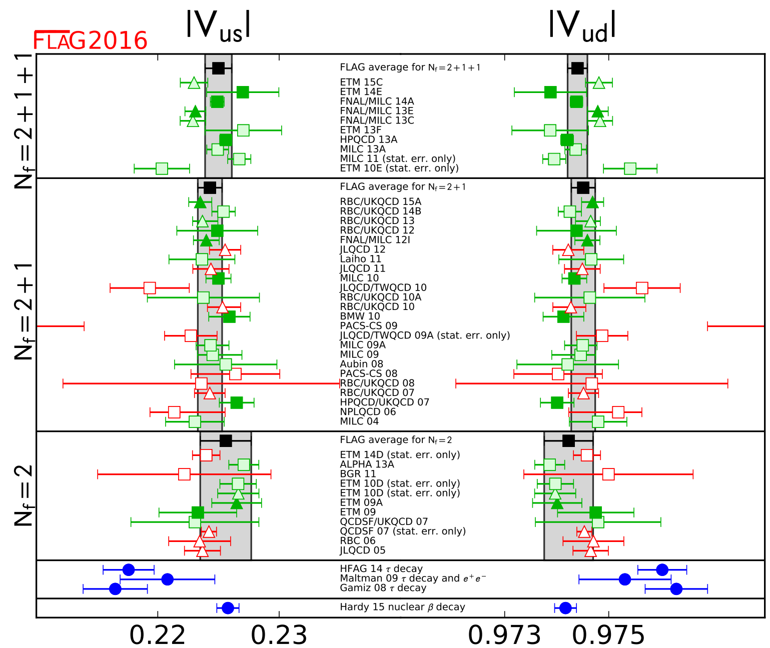
$$V_{\mathrm{ud}}^2 + V_{\mathrm{us}}^2 + |V_{\mathrm{ub}}|^2 - 1 = 0.0011(09)_{\mathrm{stat}}(23)_{\mathrm{syst}}(05)_{\mathrm{exp\&nuc}} = 0.0011(25)_{\mathrm{comb}}$$

- alternatively: w/o V_{ud} from nuclear β-decay using $|V_{\mathrm{ub}}|$ and first-row unitarity

$$V_{\mathrm{ud}} = 0.9736(04)_{\mathrm{stat}}(11)_{\mathrm{syst}}(01)_{\mathrm{exp}} \quad V_{\mathrm{us}} = 0.2281(18)_{\mathrm{stat}}(48)_{\mathrm{syst}}(03)_{\mathrm{exp}}$$

comparison with FLAG-2016





FLAG-2016 $N_f = 2+1$ avr.:

$$f_{K^\pm}/f_{\pi^\pm} = 1.192(5)$$

our result:

$$f_{K^\pm}/f_{\pi^\pm} = 1.178(28)$$

FLAG-2016 $N_f = 2+1$ avr.:

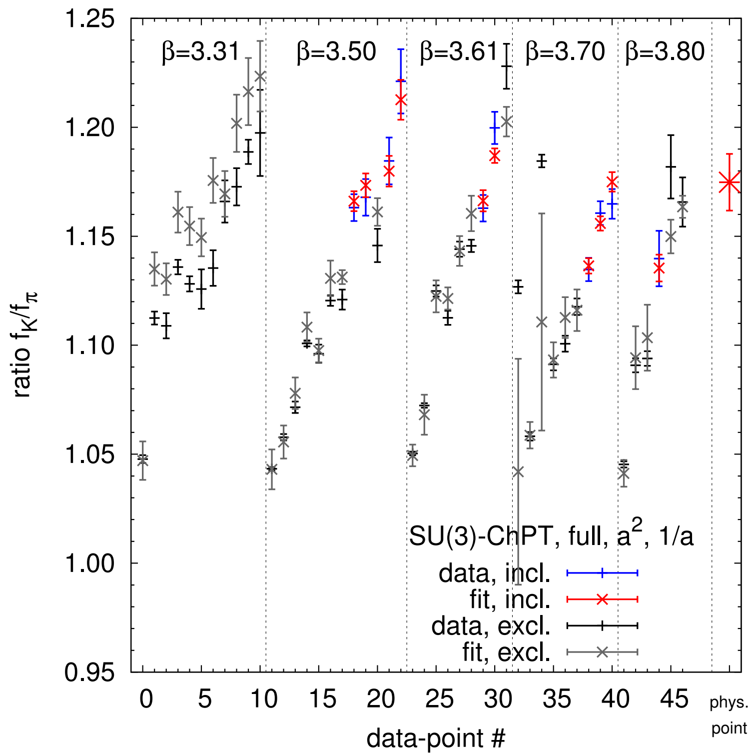
$$V_{\text{us}} = 0.2243(10),\ V_{\text{ud}} = 0.97451(23)$$

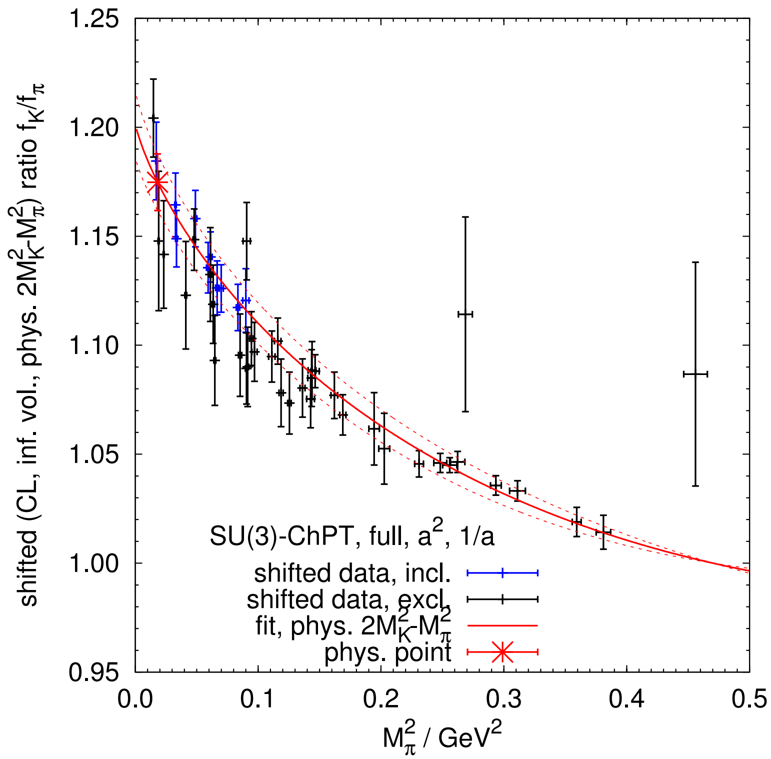
our results: (first row unit.+V_{ub})

$$V_{\text{us}} = 0.2281(51),\ V_{\text{ud}} = 0.9736(12)$$

BACKUP

NLO SU(3)-ChPT fit



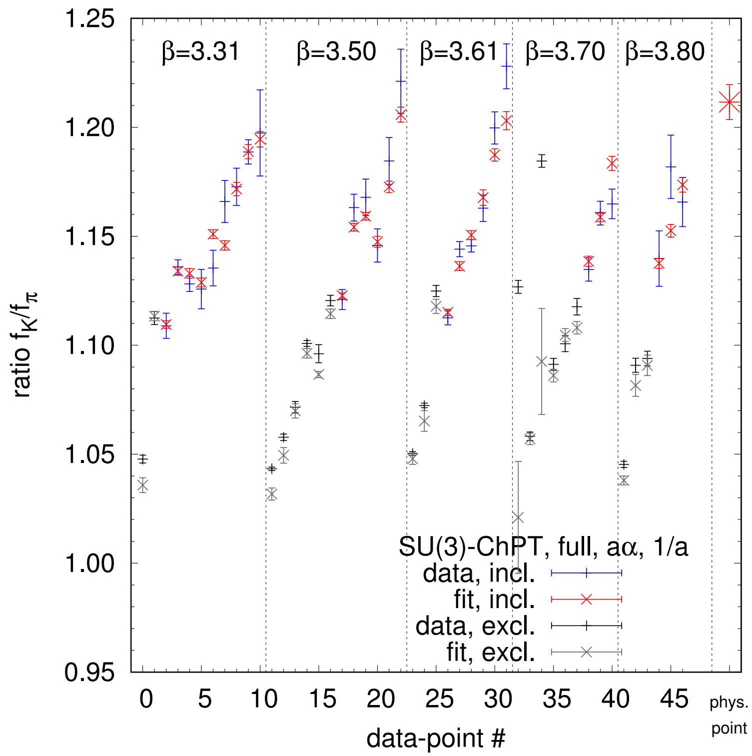


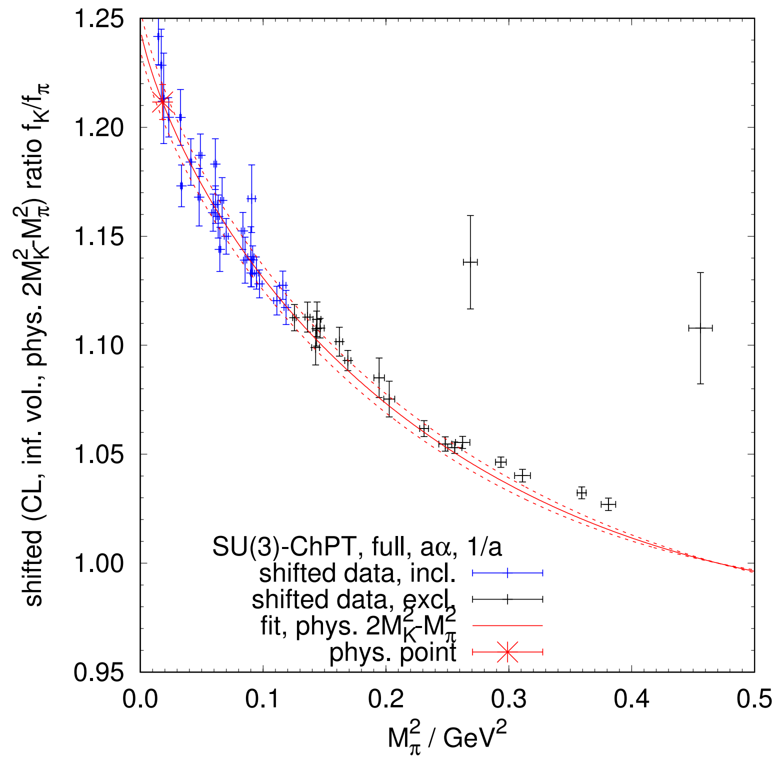
mass-independent scale, full FV, a^2-discretization

$M_\pi \leq 300\,\text{MeV}$, $M_K \leq 600\,\text{MeV}$, $(M_\pi L) \geq 3.85$, $\beta \geq 3.50$

$$\boldsymbol{\chi^2 = 7.6,\ n_{\text{d.o.f.}} = 6,\ p = 0.27}$$

NLO SU(3)-ChPT fit



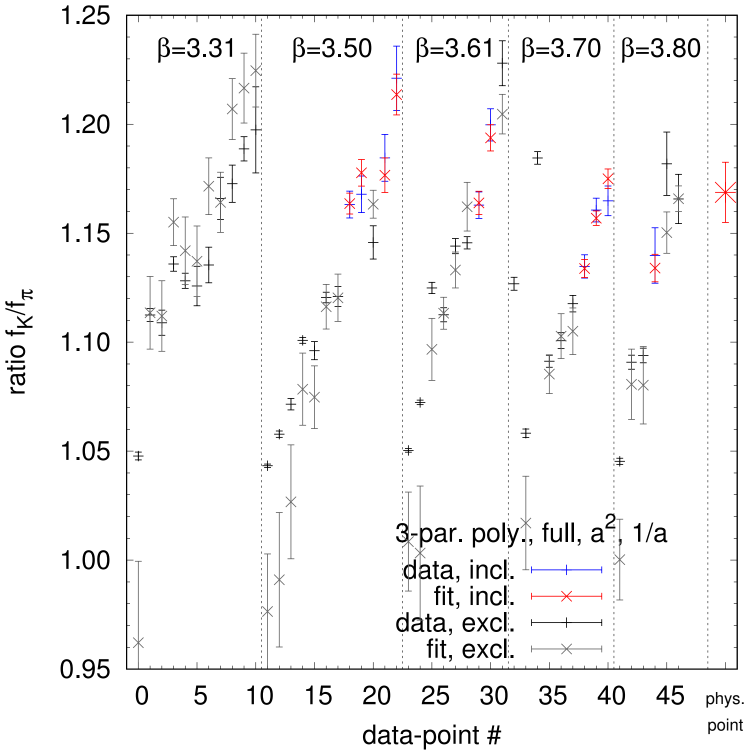


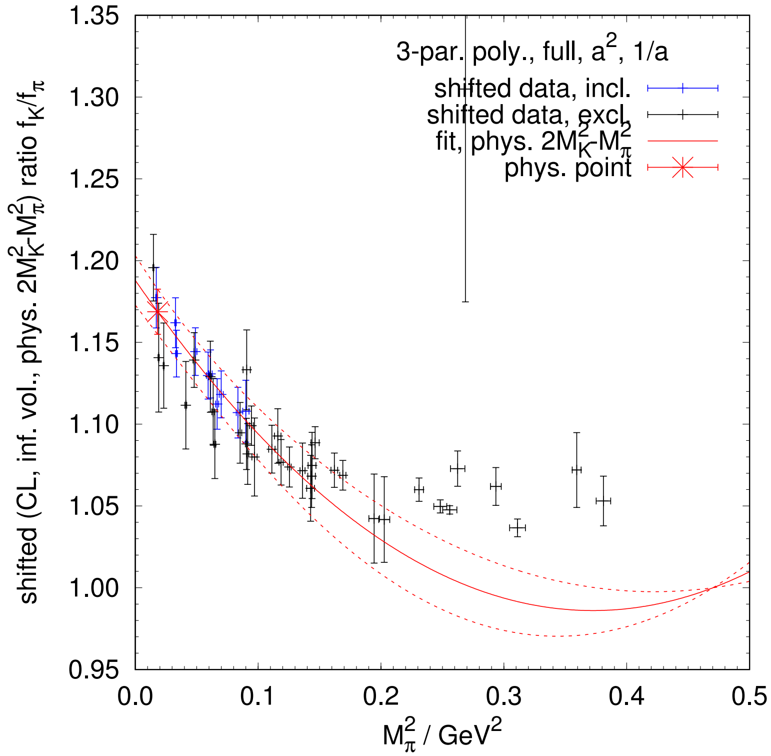
mass-independent scale, full FV, $a\alpha$-discretization

$M_\pi \leq 350\,\text{MeV}$, $M_K \leq 600\,\text{MeV}$

$\boldsymbol{\chi^2 = 47}$, $\boldsymbol{n_{\text{d.o.f.}} = 23}$, $\boldsymbol{p = 2 \cdot 10^{-3} \approx 0}$

3-parameter polynomial fit



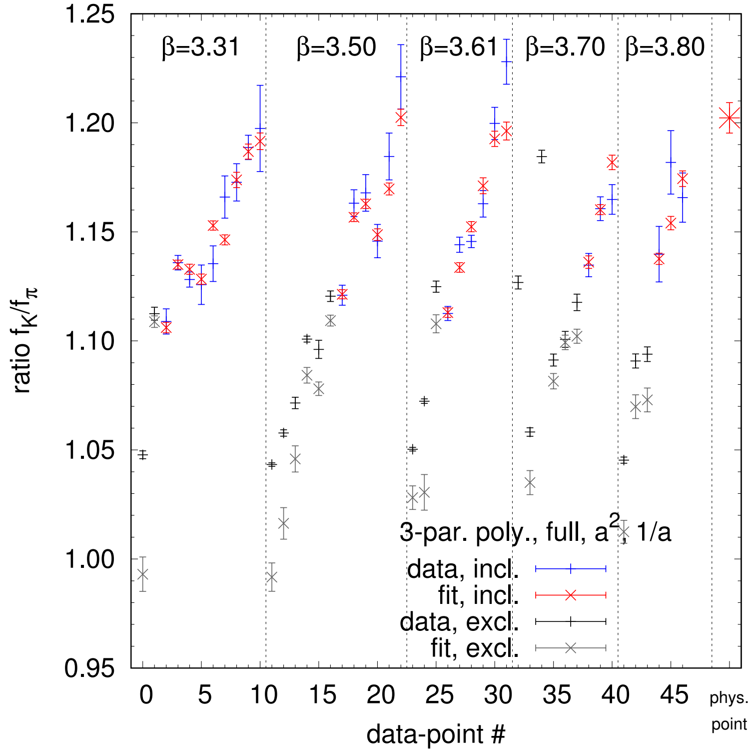


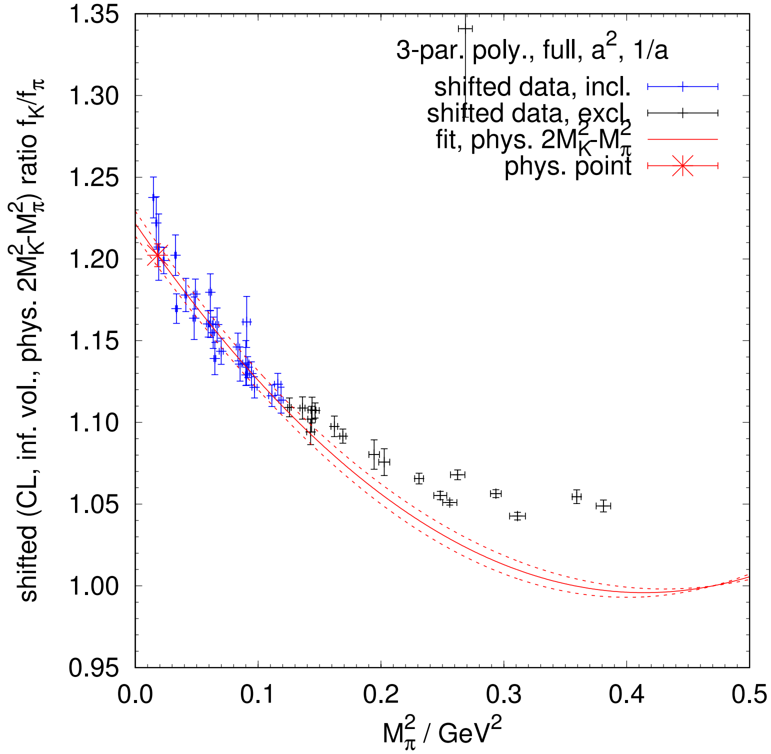
mass-independent scale, full FV, a^2-discretization

$M_\pi \leq 300\,\text{MeV},\ M_K \leq 600\,\text{MeV},\ (M_\pi L) \geq 3.85,\ \beta \geq 3.50$

$$\boldsymbol{\chi^2 = 5.8,\ n_{\text{d.o.f.}} = 5,\ p = 0.33}$$

3-parameter polynomial fit



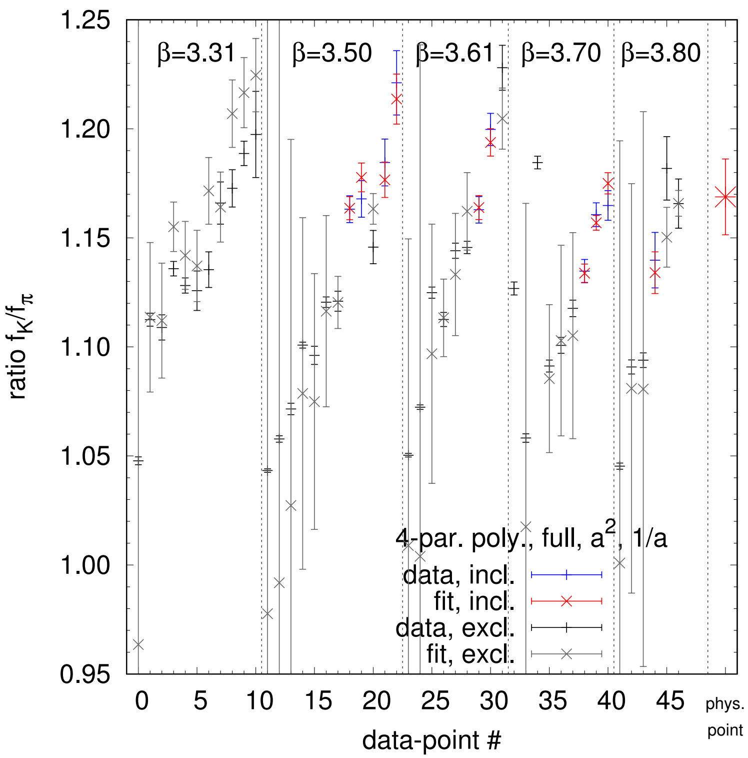


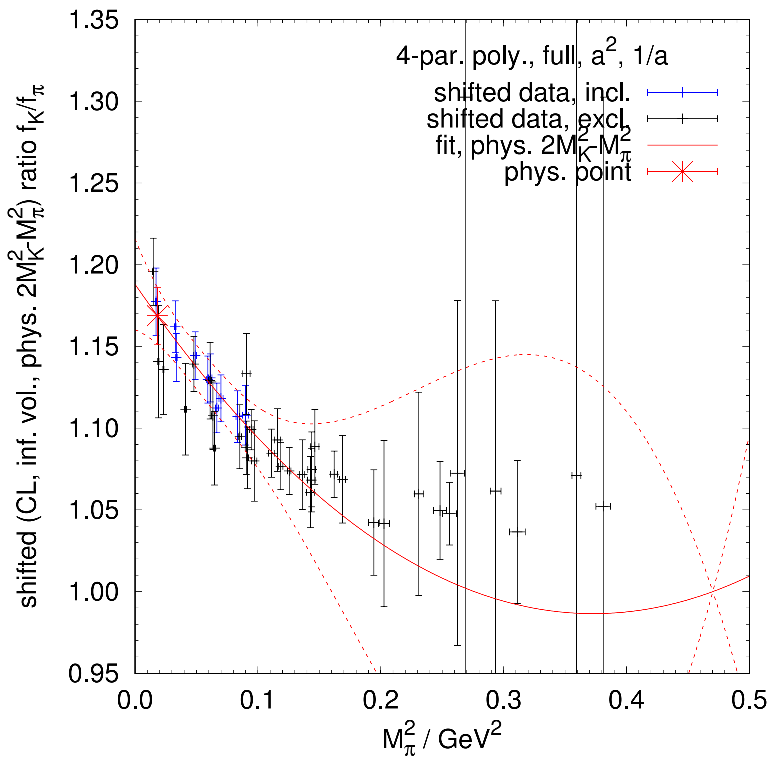
mass-independent scale, full FV, a^2-discretization

$M_\pi \leq 350\,\text{MeV},\ M_K \leq 600\,\text{MeV}$

$$\boldsymbol{\chi^2 = 54,\ n_{\text{d.o.f.}} = 22,\ p = 1.7 \cdot 10^{-4} \approx 0}$$

4-parameter polynomial fit



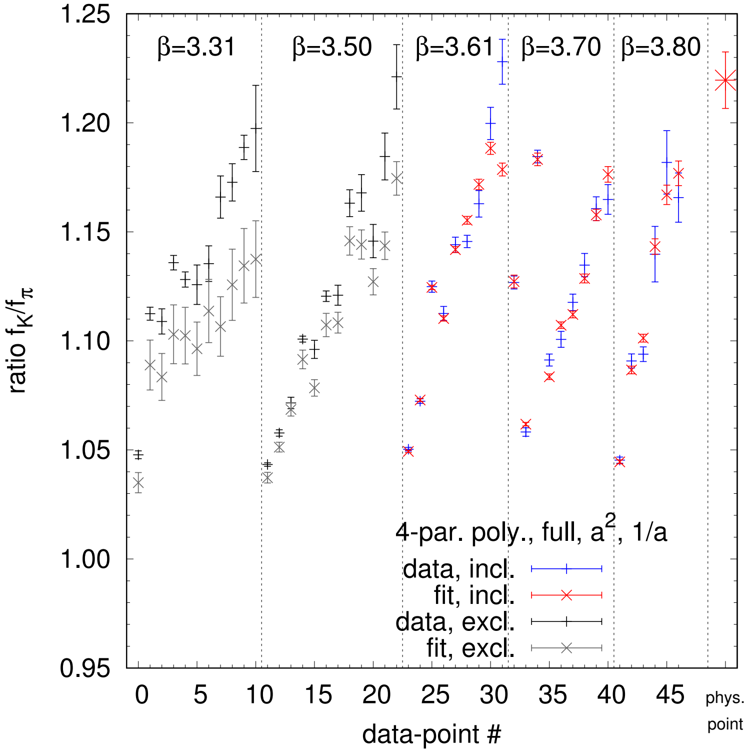


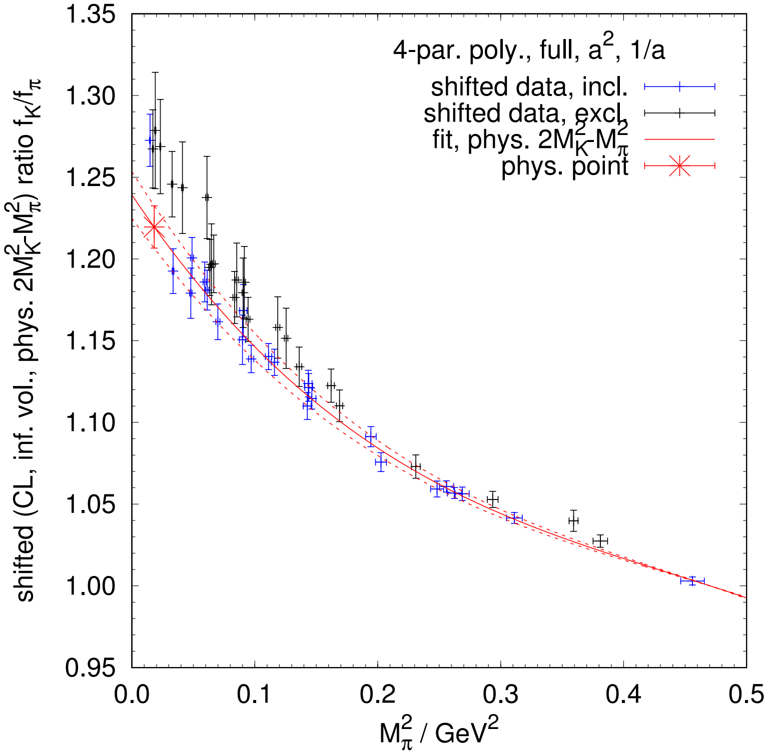
mass-independent scale, full FV, a^2-discretization

$M_\pi \leq 300\,\text{MeV}$, $M_K \leq 600\,\text{MeV}$, $(M_\pi L) \geq 3.85$, $\beta \geq 3.50$

$$\boldsymbol{\chi^2 = 5.8,\ n_{\text{d.o.f.}} = 4,\ p = 0.21}$$

4-parameter polynomial fit



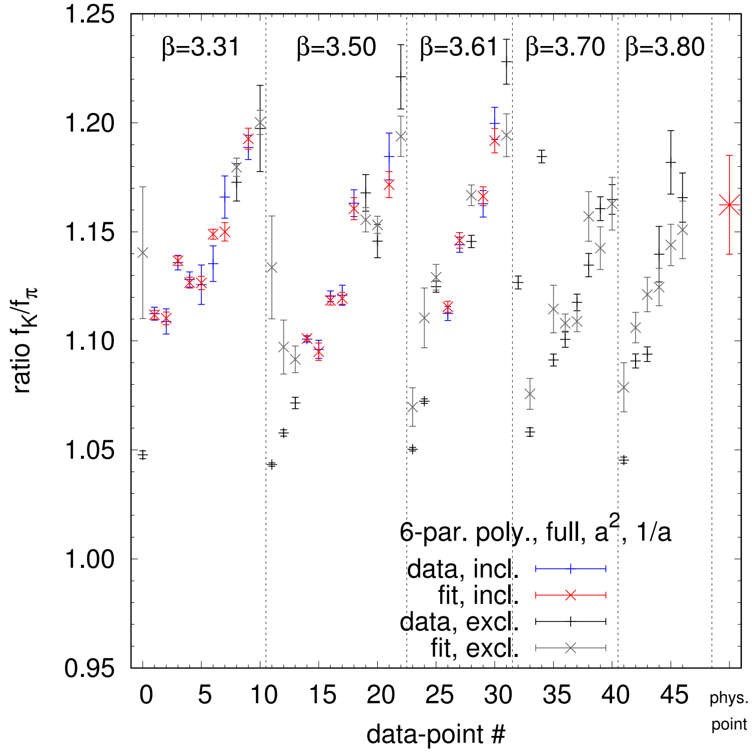


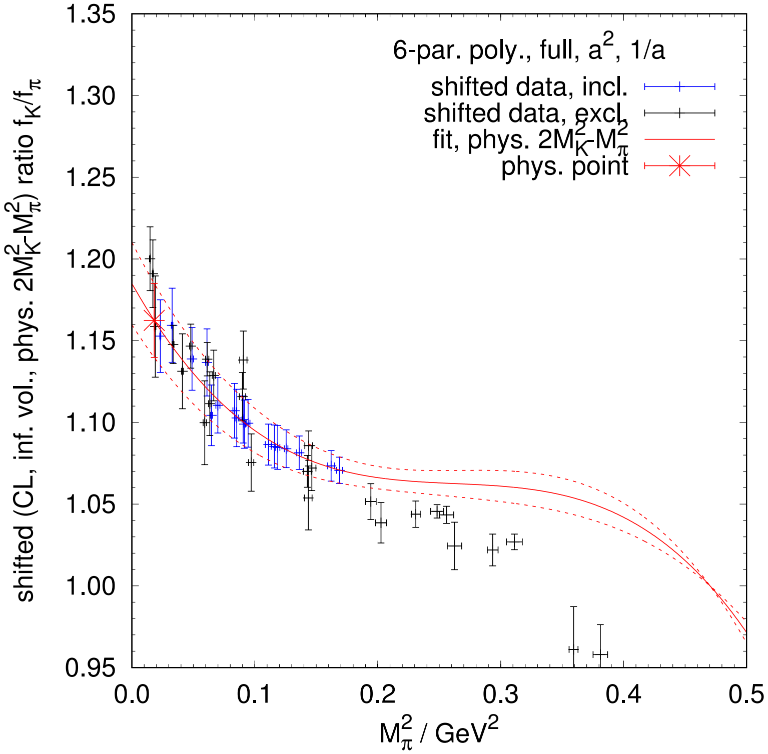
mass-independent scale, full FV, a^2-discretization

$\beta \geq 3.61$

$$\boldsymbol{\chi^2 = 72,\ n_{\text{d.o.f.}} = 18,\ p = 2 \cdot 10^{-8} \approx 0}$$

6-parameter polynomial fit



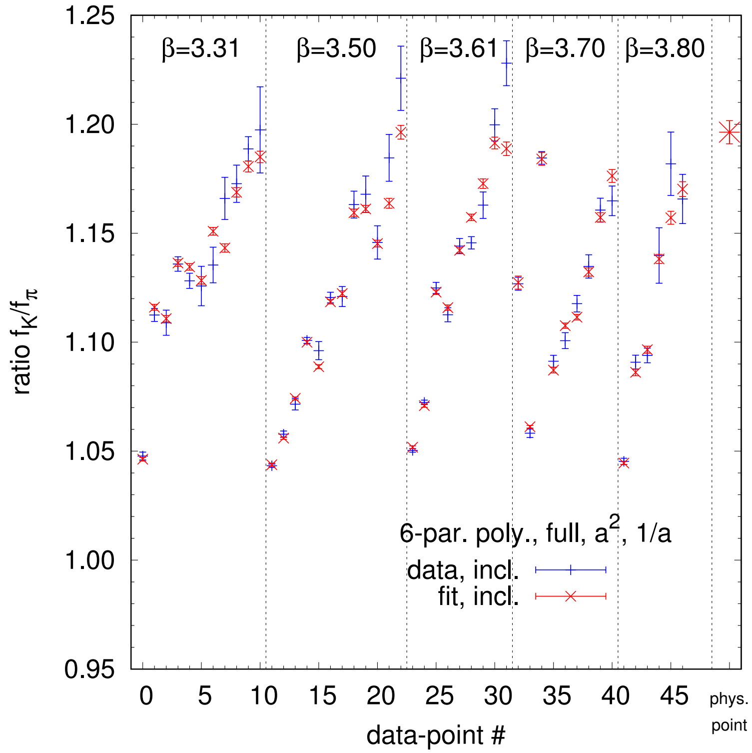


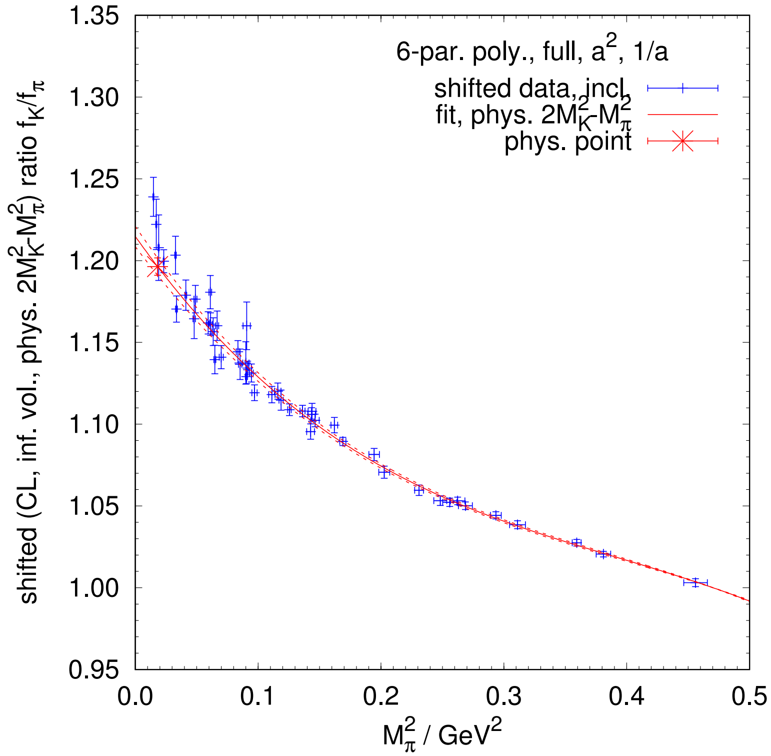
mass-independent scale, full FV, a^2-discretization

$m_K \leq 600\,\text{MeV}$, $(M_\pi L) \geq 4.05$

$\boldsymbol{\chi^2 = 11}$, $\boldsymbol{n_{\text{d.o.f.}} = 10}$, $\boldsymbol{p = 0.32}$

6-parameter polynomial fit





mass-independent scale, full FV, a^2-discretization

all points

$$\chi^2 = 95,\ n_{\text{d.o.f.}} = 39,\ p = 1.3 \cdot 10^{-6} \approx 0$$

flavor-symmetry constraint

$m_{\mathrm{ud}} = m_{\mathrm{s}}$ implies $M_\pi = M_K$, $f_\pi = f_K$

$$\left.\frac{f_K}{f_\pi}\right|_{m_{\mathrm{ud}}=m_{\mathrm{s}}} = 1$$

also at (any) finite volume and lattice spacing

only functional forms satisfying this constraint

extrapolation to physical masses

- we use M_π, M_K instead of (bare, renormalized) quark masses

$$\begin{aligned} M_\pi^2|_{\text{LO-ChPT}} &= 2B_0 m_{\text{ud}} \\ M_K^2|_{\text{LO-ChPT}} &= B_0(m_{\text{ud}} + m_{\text{s}}) \\ (2M_K^2 - M_\pi^2)|_{\text{LO-ChPT}} &= 2B_0 m_{\text{s}} \end{aligned}$$

- correction in isospin-limit (FLAG-recommendation)

$$M_\pi^{\text{phys}} = 134.8(0.3)\,\text{MeV} \quad M_K^{\text{phys}} = 494.2(0.4)\,\text{MeV}$$

NLO-SU(3)-ChPT extrapolation

$$\frac{f_K}{f_\pi} = 1 + \frac{c_0}{2}\left\{\frac{5}{4}M_\pi^2 \log\left(\frac{M_\pi^2}{\mu^2}\right) - \frac{1}{2}M_K^2 \log\left(\frac{M_K^2}{\mu^2}\right) - \frac{3}{4}M_\eta^2 \log\left(\frac{M_\eta^2}{\mu^2}\right) + c_1[M_K^2 - M_\pi^2]\right\}$$

$$M_\eta^2 \;=\; \frac{1}{3}(4M_K^2 \;-\; M_\pi^2)\,, \quad c_0 \;=\; \frac{1}{(4\pi F)^2}\,, \quad c_1 \;=\; 128\pi^2\, L_5(\mu)$$

2 fit-parameters, flavor-symmetry constraint satisfied

polynomial forms

expansion in

- M_π^2: dominant behaviour
- $[M_K^2 - M_\pi^2]$: flavor-symmetry constraint

3-, 4-, or 6-parameter fit:

$$\frac{f_K}{f_\pi} = 1 + [M_K^2 - M_\pi^2]\Big(c_0^{\text{3-par}} + c_1^{\text{3-par}}[M_K^2 - M_\pi^2] + c_2^{\text{3-par}} M_\pi^2\Big)$$

$$\frac{f_K}{f_\pi} = 1 + [M_K^2 - M_\pi^2]\Big(c_0^{\text{4-par}} + c_1^{\text{4-par}}[M_K^2 - M_\pi^2] + c_2^{\text{4-par}} M_\pi^2 + c_3^{\text{4-par}} M_\pi^4\Big)$$

$$\frac{f_K}{f_\pi} = 1 + [M_K^2 - M_\pi^2]\Big(c_0^{\text{6-par}} + c_1^{\text{6-par}}[M_K^2 - M_\pi^2] + c_2^{\text{6-par}} M_\pi^2 + c_3^{\text{6-par}} M_\pi^4 + c_4^{\text{6-par}} M_\pi^2[M_K^2 - M_\pi^2] + c_5^{\text{6-par}}[M_K^2 - M_\pi^2]^2\Big)$$

real world leptonic decay constant ratio

- measured in experiment: ratio of **charged** leptonic decay constants with 6 quark flavors (u,d,s,c,b,t)

$$\boldsymbol{\frac{f_{K^\pm}}{f_{\pi^\pm}}}$$

- suppressed in $N_f = 2+1$: dynamical charm, bottom, top, but $(m_{\text{charm}}/m_{\text{strange}})^2 \simeq 140 \ldots$
- (hard) isospin-breaking $m_{\text{up}} \neq m_{\text{down}}$, EM-effects (charges)

not taken into account in this lattice QCD simulation

- ChPT-analysis of these effects [Gasser, Leutwyler (1985); Cirigliano, Neufeld (2011)]

$$\frac{f_{K^\pm}}{f_{\pi^\pm}} = \frac{f_K}{f_\pi}\sqrt{1+\delta_{\text{SU(2)}}}$$

- $\delta_{\text{SU(2)}}$: -0.0043(12) [C.,N. (2011)] and FLAG-II, -0.0078(7) [Divitiis et al. (2012)]

$$\boldsymbol{\delta_{\text{SU(2)}} = -0.0061(61) \rightarrow \frac{f_{K^\pm}}{f_{\pi^\pm}} = 1.178(10)_{\text{stat}}(26)_{\text{syst}}}$$

CKM-matrix elements V_{ud}, V_{us}

$$\frac{V_{\text{us}}}{V_{\text{ud}}} \frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.27599(38)_{\text{exp}}$$

[Moulson (2014), Rosner et al. (2015)]

$$\frac{V_{\text{us}}}{V_{\text{ud}}} = 0.2343(20)_{\text{stat}}(52)_{\text{syst}}(03)_{\text{exp}}$$

- using super-allowed nuclear β-decay $V_{\text{ud}} = 0.97417(21)_{\text{nuc}}$ [Hardy, Towner (2015)]

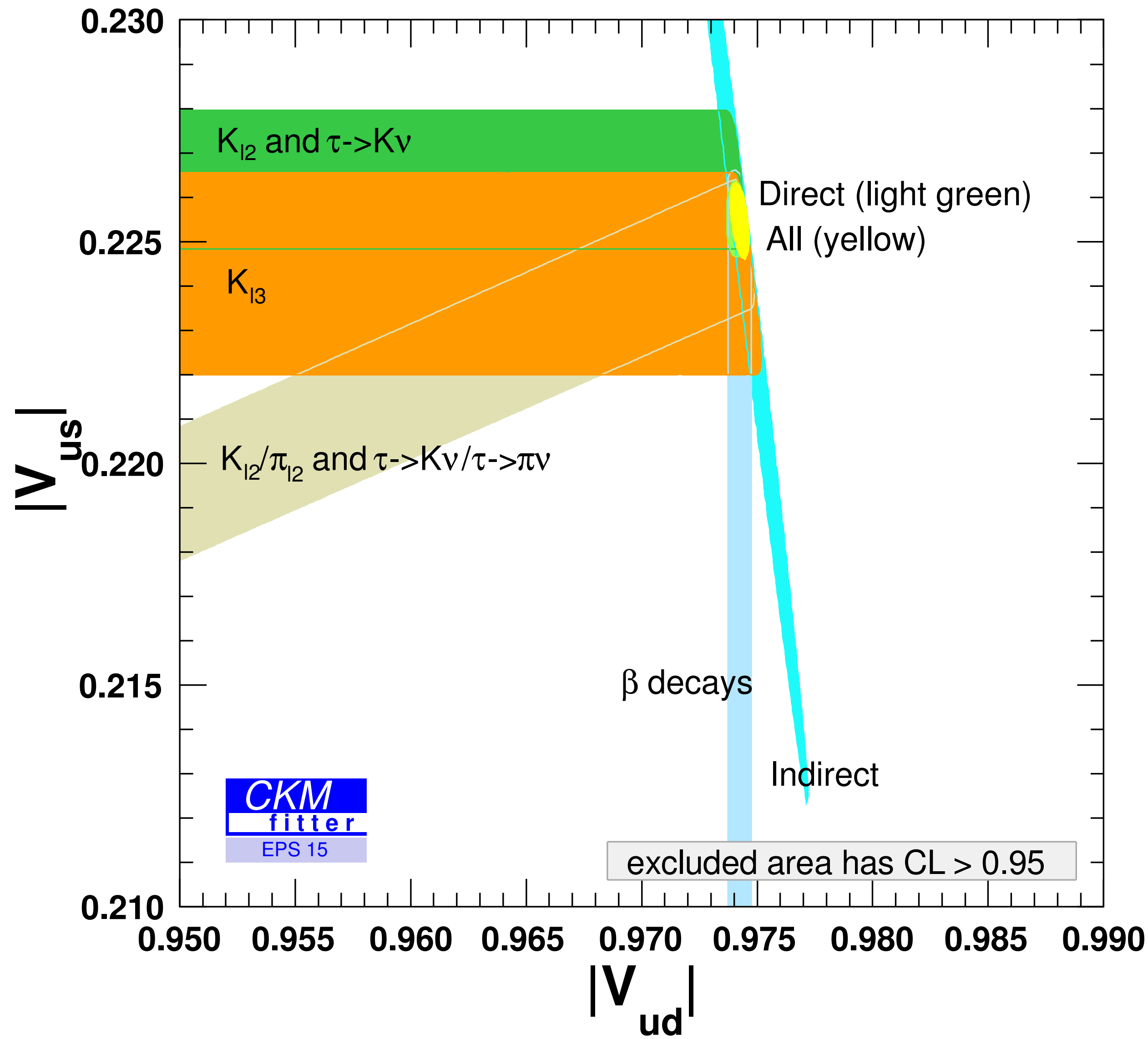
$$V_{\text{us}} = 0.2282(19)_{\text{stat}}(51)_{\text{syst}}(03)_{\text{exp\&nuc}}$$

- first-row unitarity ($|V_{\text{ub}}| = 4.12(37)(06) \cdot 10^{-3}$ [Rosner et al. (2015)])

$$V_{\text{ud}}^2 + V_{\text{us}}^2 + |V_{\text{ub}}|^2 - 1 = 0.0011(09)_{\text{stat}}(23)_{\text{syst}}(05)_{\text{exp\&nuc}} = 0.0011(25)_{\text{comb}}$$

- alternatively: w/o V_{ud} from nuclear β-decay **using $|V_{ub}|$ and first-row unitarity**

$$V_{\text{ud}} = 0.9736(04)_{\text{stat}}(11)_{\text{syst}}(01)_{\text{exp}} \quad V_{\text{us}} = 0.2281(18)_{\text{stat}}(48)_{\text{syst}}(03)_{\text{exp}}$$


0.230
0.225
0.220
0.215
0.210
$|V_{us}|$
0.950 0.955 0.960 0.965 0.970 0.975 0.980 0.985 0.990
$|V_{ud}|$
K_{l2} and τ->Kν
Direct (light green)
All (yellow)
K_{l3}
K_{l2}/π_{l2} and τ->Kν/τ->πν
β decays
Indirect
CKM fitter
EPS 15
excluded area has CL > 0.95