# Results on the heavy-dense QCD phase diagram using complex Langevin

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SCIENCE WITHOUT BORDERS

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[Aarts, Attanasio, Jäger, Sexty, hep-lat/1606.05561]

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# CLE: Motivation

- *Sign problem*: Inclusion of a chemical potential to an Euclidean path integral makes the action complex
- Average values for observables then rely on precise cancellations of oscillating quantities
- In QCD this is manifested in the fermion determinant

$$[\det M(U,\mu)]^* = \det M(U,-\mu^*)$$

  which is complex for real chemical potential $\mu$
- Results from Hybrid Monte-Carlo simulations become unreliable when the sign problem is severe

# CLE: Stochastic quantization

## On the lattice

[Damgaard and Hüffel, Physics Reports]

- Evolve gauge links according to the Langevin equation

$$U_{x\mu}(\theta + \varepsilon) = \exp\left[X_{x\mu}\right] U_{x\mu}(\theta)\,,$$

where

$$X_{x\mu} = i\lambda^a (\varepsilon D^a_{x\mu} S\left[U(\theta)\right] + \sqrt{\varepsilon}\,\eta^a_{x\mu}(\theta))\,,$$

$\lambda^a$ are the Gell-Mann matrices, $\varepsilon$ is the stepsize, $\eta^a_{x\mu}$ are white noise fields satisfying

$$\langle \eta^a_{x\mu} \rangle = 0\,, \quad \langle \eta^a_{x\mu} \eta^b_{y\nu} \rangle = 2\delta^{ab}\delta_{xy}\delta_{\mu\nu}\,,$$

$S$ is the QCD action and $D^a_{x\mu}$ is defined as

$$D^a_{x\mu} f(U) = \frac{\partial}{\partial \alpha} f(e^{i\alpha\lambda^a} U_{x\mu})\Big|_{\alpha=0}$$

# CLE: Complexification I

## Complexification

[Aarts, Stamatescu, hep-lat/0807.1597]

- Allow gauge fields to be complex, i.e., $\mathbb{R} \ni A_\mu^a(x) \to A_\mu^a(x) \in \mathbb{C}$
- On the lattice this means SU$(3) \ni U_{x\mu} \to U_{x\mu} \in$ SL$(3, \mathbb{C})$
- Use $U_{x\mu}^{-1}$ instead of $U_{x\mu}^\dagger$ as
  - it keeps the action holomorphic;
  - they coincide on SU$(3)$ but on SL$(3, \mathbb{C})$ it is $U^{-1}$ that represents the backwards-pointing link.
- That means the plaquette is now

$$U_{x,\mu\nu} = U_{x\mu} U_{x+\mu,\nu} U_{x+\nu,\mu}^{-1} U_{x,\nu}^{-1} ,$$

and the Wilson action reads

$$S[U] = -\frac{\beta}{3} \sum_x \sum_{\mu<\nu} \mathbf{Tr} \left[ \frac{1}{2} \left( U_{x,\mu\nu} + U_{x,\mu\nu}^{-1} \right) - \mathbb{1} \right]$$

# CLE: Complexification II – Gauge cooling

- The $\mathrm{SL}(3,\mathbb{C})$ group is a non-compact manifold, which means the gauge links can get arbitrarily far from $\mathrm{SU}(3)$
- During simulations monitor the distance from the unitary manifold with

$$d = \frac{1}{N_s^3 N_\tau} \sum_{x,\mu} \mathbf{Tr} \left[ U_{x\mu} U_{x\mu}^\dagger - \mathbb{1} \right]^2 \geq 0$$

- Use gauge transformations to keep the system as close as possible to $\mathrm{SU}(3)$, i.e., minimise the imaginary part of $A_\mu^a(x)$

$$U_{x\mu} \to \Lambda_x U_{x\mu} \Lambda_{x+\mu}^{-1}\,, \quad \Lambda_x = \exp\left[-\varepsilon \alpha \lambda^a f_x^a\right]$$

where

$$f_x^a = 2\mathbf{Tr}\left[\lambda^a \left( U_{x\mu} U_{x\mu}^\dagger - U_{x-\mu,\mu}^\dagger U_{x-\mu,\mu} \right)\right]$$

- The parameter $\alpha$ and the number of cooling steps are chosen adaptively based on the distance $d$.

# Heavy-dense QCD

## Heavy-dense approximation

[Aarts, Stamatescu, hep-lat/0807.1597]

- Heavy quarks $\rightarrow$ spatial part of fermion determinant does not contribute, but temporal part is exact ($\kappa \rightarrow 0$):

$$\det M(U,\mu) = \prod_{\vec{x}} \left\{ \det \left[ 1 + (2\kappa e^{\mu})^{N_\tau} \mathcal{P}_{\vec{x}} \right]^2 \times \det \left[ 1 + \left(2\kappa e^{-\mu}\right)^{N_\tau} \mathcal{P}_{\vec{x}}^{-1} \right]^2 \right\}$$

- Polyakov loop

$$\mathcal{P}_{\vec{x}} = \prod_{\tau} U_4(\vec{x},\tau)$$

- Exhibits the sign problem: $[\det M(U,\mu)]^* = \det M(U,-\mu^*)$
- $\mu = \mu_c^* \equiv -\ln(2\kappa)$ marks the transition to higher densities at zero temperature ($N_\tau \rightarrow \infty$)

# Simulation setup and observables

## Simulation setup

- Gauge coupling $\beta = 5.8$
  - Lattice spacing (approximate) $a \sim 0.15$ fm
- Hopping parameter $\kappa = 0.04$
  - Critical chemical potential $\mu_c^0 = 2.53$
- Lattice volumes $V = 6^3, 8^3, 10^3$
- Number of flavours $N_f = 2$
- Temporal extents/temperatures

| $N_\tau$ | 28 | 24 | 20 | 16 | 14 | 12 | 10 |
|---|---|---|---|---|---|---|---|
| $T$ [MeV] | 48 | 56 | 67 | 84 | 96 | 112 | 134 |
| $N_\tau$ | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
| $T$ [MeV] | 168 | 192 | 224 | 268 | 336 | 447 | 671 |

# Simulation setup and observables

## Observables

- "Real part" of the Polyakov loop

$$P^s = \frac{1}{2}\left(\langle P\rangle + \langle P^{-1}\rangle\right) ,$$

where

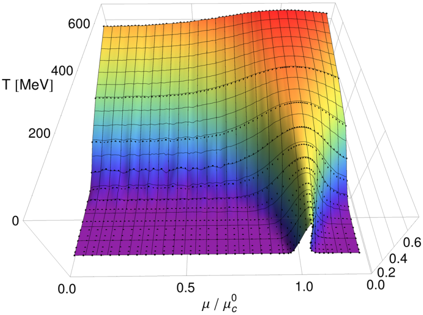
$$\langle P\rangle = \frac{1}{V}\sum_{\vec{x}}\langle P_{\vec{x}}\rangle , \qquad P_{\vec{x}} = \frac{1}{3}\mathrm{Tr}\,\mathcal{P}_{\vec{x}}$$

$$\langle P^{-1}\rangle = \frac{1}{V}\sum_{\vec{x}}\langle P_{\vec{x}}^{-1}\rangle , \qquad P_{\vec{x}}^{-1} = \frac{1}{3}\mathrm{Tr}\,\mathcal{P}_{\vec{x}}^{-1}$$

- Quark density

$$\langle n\rangle = \frac{T}{V}\frac{\partial \ln Z}{\partial \mu}$$

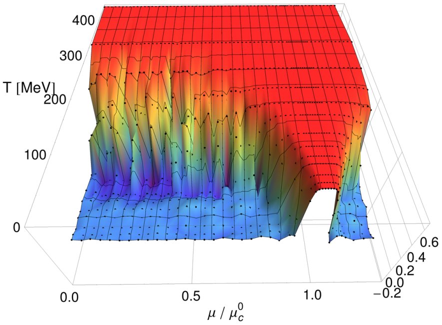
# HDQCD at $V = 10^3$, $\beta = 5.8$ and $\kappa = 0.04$

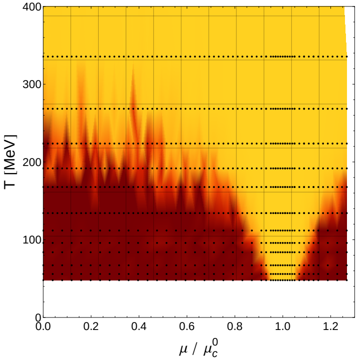



Average Polyakov loop (left) and quark density (right) as functions of temperature and chemical potential.
Regions of confinement ($\langle \mathcal{P} \rangle = 0$) and deconfinement are visible and also the smoothening of the transition to higher densities.

# Binder cumulant for the Polyakov loop at $V = 10^3$

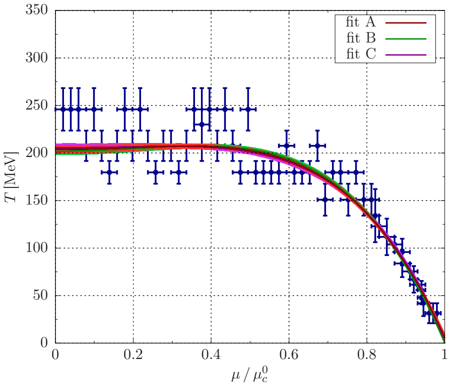
Binder cumulant: $B = 1 - \frac{\langle O^4 \rangle}{3\langle O^2 \rangle^2}$
Confinement is indicated by $B = 0$ while $B = 2/3$ means deconfinement




Left: Binder cumulant of the Polyakov loop as function of $\mu$ and $T$.
Right: Two dimensional projection of the Binder cumulant. Red colours indicated a value compatible with $0$, while yellow corresponds to $2/3$.

# Phase boundary at $V = 10^3$




Fit A: $$T_c = \sum_{k=0}^{n} a_k x^k$$

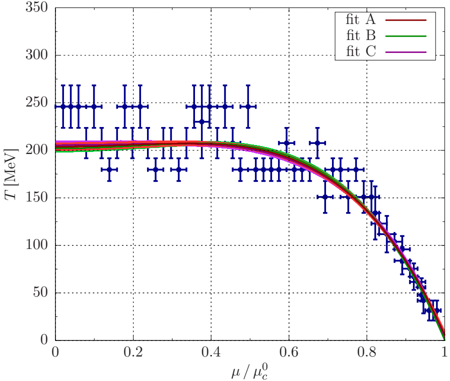
Fit B: $$T_c = \sum_{k=1}^{n} b_k (1-x)^k$$

Fit C: $$T_c = c_0 (1-x)^\alpha + \sum_{k=1}^{n} c_k (1-x)^k$$

With $x = (\mu/\mu_c^0)^2$.

- Fit B takes into account $T_c(\mu_c^0) = 0$
- Fit C has an additional term $0 < \alpha < 1$ to reproduce non-analytic behaviour at $x = 1$ from the Clausis-Clapeyron relation ($\partial T_c(\mu)/\partial\mu \to \infty$ at $\mu = \mu_c^0$)

# Phase boundary at $V = 10^3$

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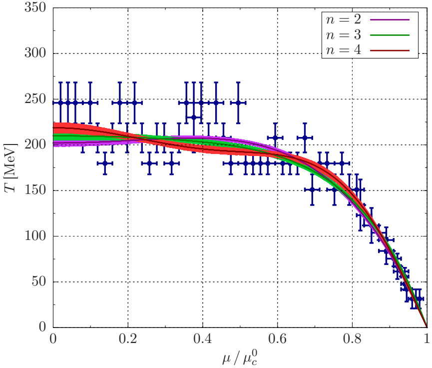
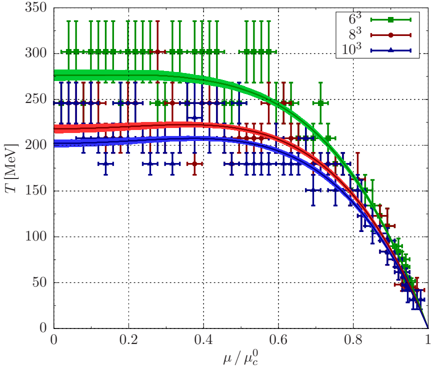
- The coefficients from fits A and B are compatible, showing that $T_c(\mu_c^0) = 0$ emerges naturally from our data
- Our lowest temperature is still away from $T = 0$ and the non-analytical behaviour cannot be captured by $\alpha$

# Phase boundary using fit B

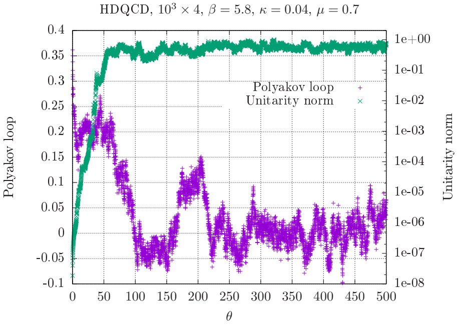


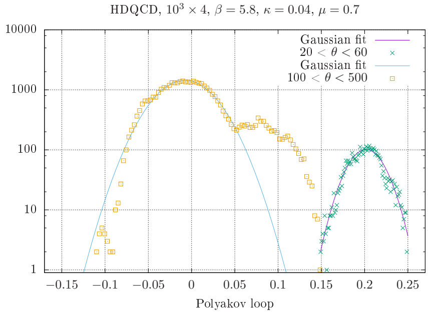




Left: Comparison of different orders for fit B at $V = 10^3$.
Right: Volume dependence of the phase boundary with $n = 2$.

# Instabilities I

A large unitarity norm leads to distributions that do not reflect the original theory.

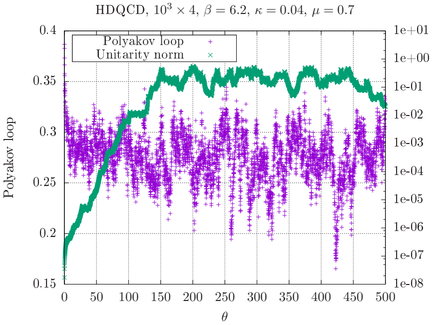


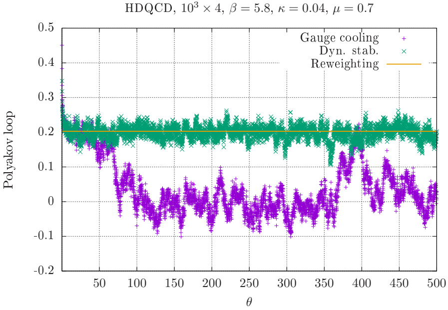




Left: Real part of the Polyakov loop and unitarity norms as functions of the Langevin time.
Right: Histogram of the Polyakov loop in the regions before and after the norm has increased.

# Instabilities II

## Exploring other possibilities




Left: Increasing $\beta$ from $5.8$ to $6.2$ pushed the instabilities further in Langevin time.
Right: The new technique of Dynamic Stabilisation (see Ben Jäger's talk).

# Summary and Outlook

## Summary

- Complex Langevin simulations allow the study of theories that exhibit the sign problem
- CLE + Gauge Cooling was successfully used to map the HDQCD phase diagram for real chemical potentials
- Instabilities can limit the amount of statistics available

## Outlook

- Map the vicinity of the phase boundary of QCD with fully dynamical quarks using the dynamic stabilisation technique (see Ben Jäger's talk)