Baryon interactions in lattice QCD: the direct method vs. the HAL QCD potential method

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1. Baryon interactions from lattice QCD

2. Direct measurement vs HAL QCD method
   - Formalisms
   - Direct Measurement
   - HAL QCD Measurement

3. Origin of Fake Signal in Direct Method

4. Summary
Baryon interactions from lattice QCD

Direct measurement vs HAL QCD method
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Origin of Fake Signal in Direct Method

Summary
2 Methods for Hadron Interaction from Lattice QCD

QCD ▶ Hadron Interaction ▶ Nuclear Physics

1. **Lüscher’s finite volume method** — Lüscher ’86, ’91
   energy shift of two-particle in “box” ▶ phase shift
   \[
   \Delta E_L = 2\sqrt{k^2 + m^2} - 2m \quad \Rightarrow \quad k \cot \delta(k) = \frac{1}{\pi L} \sum_{n \in \mathbb{Z}^3} \frac{1}{|n|^2 - (kL/2\pi)^2}
   \]

2. **HAL QCD method** — Ishii-Aoki-Hatsuda ’07
   NBS wave function ▶ potential ▶ phase shift

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**Graphs and Data**
- NN wave function \(\psi(r)\) for different targets.
- Effective potential \(V_{\text{eff}}(r)\) in MeV.
- Experiment vs lattice data comparison.

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NN Interactions from Lattice QCD

<table>
<thead>
<tr>
<th></th>
<th>Lüscher</th>
<th>HAL QCD</th>
<th>phys. point</th>
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<tbody>
<tr>
<td>dineutron ((1S_0))</td>
<td>bound</td>
<td>unbound</td>
<td>unbound</td>
</tr>
<tr>
<td>deuteron ((3S_1))</td>
<td>bound</td>
<td>unbound</td>
<td>bound</td>
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</table>

↑ inconsistencies between two methods, which is correct?

▶ Today we will clarify the origin of this puzzle
Baryon interactions from lattice QCD

Direct measurement vs HAL QCD method
- Formalisms
- Direct Measurement
- HAL QCD Measurement

Origin of Fake Signal in Direct Method

Summary
Lüscher’s Finite Volume Method

- “energy shift” in finite box $L^3$

\[
\Delta E_L = E_{BB} - 2m_B = 2\sqrt{k^2 + m_B^2} - 2m_B
\]

\[\Rightarrow \text{phase shift} \quad \delta(k)\]

\[
k \cot \delta(k) = \frac{1}{\pi L} \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{1}{|\mathbf{n}|^2 - (kL/2\pi)^2}
\]

↑ THEORY

↓ PRACTICE — “Direct Method”

- measure: plateau in effective mass

\[
\Delta E_{\text{eff}}(t) = \log \frac{R(t)}{R(t+1)} \rightarrow \Delta E_L
\]

\[
R(t) = \frac{G_{BB}(t)}{\{G_B(t)\}^2} \rightarrow \exp \left[ -(E_{BB} - 2m_B) t \right]
\]

with $G_{BB}(t)(G_B(t))$: BB(B) correlators
Time-dependent HAL QCD Method

■ Nambu-Bethe-Salpeter wave function

\[ R(\vec{r}, t) = \sum_n A_n \psi_n(\vec{r}) e^{-\left(\frac{E_n - 2m_B}{2}\right)t} + \mathcal{O}(e^{-\left(\frac{E_{\text{th}} - 2m_B}{2}\right)t}) \]

■ with elastic saturation \( R(r, t) \) satisfies

\[
\left[ \frac{1}{4m_B} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0 \right] R(\vec{r}, t) = \int d\vec{r}' U(\vec{r}, \vec{r}') R(\vec{r}', t)
\]

▶ “potential” using velocity expansion \( U(r, r') \simeq V(r) \delta(r - r') \)

\[
V(\vec{r}) = \frac{1}{4m_B} \frac{(\partial/\partial t)^2 R(\vec{r}, t)}{R(\vec{r}, t)} - \frac{(\partial/\partial t) R(\vec{r}, t)}{R(\vec{r}, t)} - \frac{H_0 R(\vec{r}, t)}{R(\vec{r}, t)}
\]

▶ This method does not require the ground state saturation.
Difficulties in Multi-Baryons

Lüscher’s method requires ground state saturation

\[ G_{NN}(t) = c_0 \exp(-E_0^{(NN)} t) + c_1 \exp(-E_1^{(NN)} t) + \cdots \approx c_0 \exp(-E_0^{(NN)} t) \]

- **S/N problem:** \([\text{mass number } A] \times [\text{light quark}] \times [t \to \infty]\)

\[ S/N \sim \exp [-A \times (m_N - (3/2)m_\pi) \times t] \]

- **smaller gap of scattering state:** \(\Delta E \sim \vec{p}^2/m \sim O(1/L^2)\)

\[ L = L_0 \quad \rightarrow \quad L = 2 \times L_0 \quad \rightarrow \quad L = \infty \]

- **Elastic**
- **Inelastic**
- **NN + π**

\( N N \quad \rightarrow \quad N N + \pi \)
Contamination of Scattering State and Fake Plateau

Example

\[ R(t) = b_0 e^{-\Delta E_{BB} t} + b_1 e^{-\delta E_{el} t} + c_0 e^{-\delta E_{inel} t} \]

with \( \delta E_{el} - \Delta E_{BB} = 50 \text{ MeV} \sim O(1/L^2) \), \( \delta E_{inel} - \Delta E_{BB} = 500 \text{ MeV} \sim O(\Lambda_{QCD}) \)

- g.s. saturation
  \( \Delta E_{BB}^{\text{eff}}(t) - \Delta E_{BB} \to 0 \)

- elastic saturation \( t \sim 1 \text{ fm} \)

- few % of contamination
  \( \Rightarrow \) “mirage” of plateau
  around \( t \sim 1 - 1.5 \text{ fm} \)
  much larger \( t \) for true g.s.

\( \Rightarrow \) a true ground state can be checked by quark source dependence

\( \Rightarrow \) HAL QCD — scattering state are not noises, but signals
Lattice Setup: Wall Source and Smeared Source

- **interaction** from both direct and HAL QCD methods

- **CHECK 2 quark sources** — mixture of excited states are different

- **wall source**
  standard of HAL QCD

- **smeared source**
  standard of direct method\(^\dagger\)

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**setup — 2 + 1 improved Wilson + Iwasaki gauge\(^\dagger\)**
- lattice spacing: \(a = 0.08995(40) \text{ fm}, a^{-1} = 2.194(10) \text{ GeV} \)
- lattice volume: \(32^3 \times 48, 40^3 \times 48, 48^3 \times 48, \text{ and } 64^3 \times 64 \)
  \[ m_\pi = 0.51 \text{ GeV}, m_N = 1.32 \text{ GeV}, m_K = 0.62 \text{ GeV}, m_\Xi = 1.46 \text{ GeV} \]

Energy Shift of $\Xi\Xi$: Smeared Src. vs. Wall Src.

$\Delta E_L^{\text{eff}}(t) \rightarrow \Delta E_L$ depends on quark source (smeared or wall)

$\Xi\Xi(^1S_0)$ at $48^3 \times 48$

$\Xi\Xi(^3S_1)$ at $48^3 \times 48$

- source dependence suggests these plateaux are "fake" signal

<table>
<thead>
<tr>
<th>$L \rightarrow \infty$</th>
<th>Smeared src.</th>
<th>Wall source</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta E_{\Xi\Xi}(^1S_0)$</td>
<td>$&lt; 0$ bound</td>
<td>$\simeq 0$ unbound</td>
</tr>
<tr>
<td>$\Delta E_{\Xi\Xi}(^3S_1)$</td>
<td>$&gt; 0$ unphysical</td>
<td>$\simeq 0$ unbound</td>
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cf. $\Delta E < 0 \Rightarrow$ binding or $\Delta E = 0 \Rightarrow$ scattering
Generalized Sink Operator

\[ C^{(g)}_{\Xi \Xi}(t) = \sum_{\vec{r}'} g(|\vec{r}'|) \sum_{\vec{R}} \langle \Xi(\vec{R} + \vec{r}', t)\Xi(\vec{R}, t) \bar{J}_{\Xi \Xi}(t = 0) \rangle \rightarrow \exp(-E_{\Xi \Xi} t) \]

\[ \Rightarrow \text{g.s. energy does not depend on } g(r) \]

- \( g(r) = 1 \): standard sink operator
- \( g(r) = 1 + A \exp(-Br) \): exp-type projection

**Smeared Src.**

one can make any “fake plateau”

**Wall Src.**

“stable”

\[
\begin{align*}
\Delta E_{\Xi \Xi}^{\text{eff}}(t) [\text{MeV}] & \quad \text{MeV} \\
\end{align*}
\]
HAL: Potential of $\Xi\Xi(1S_0)$ Smeared Src. vs Wall Src.

NBS wavefunction: $R^{\text{smear}}(r, t)$ or $R^{\text{wall}}(r, t)$

$$V_c(r) = \frac{1}{4m} \left( \frac{\partial^2}{\partial t^2} \right) R(r, t) - \frac{\partial}{\partial t} R(r, t) - \frac{H_0 R(r, t)}{R(r, t)}$$
HAL: Potential of $\Xi\Xi(1S_0)$ Smeared Src. vs Wall Src.

- **wall src.** — good convergence
- **smeared src.** — $t$-dep.
- **smeared src.** $\rightarrow$ **wall src.** for large $t$
Residual Diff. of Pot.: Next Leading Order Correction

Derivative expansion: \( U(r, r') = \{ V_0(r) + V_1(r) \nabla^2 \} \delta(r - r') \) (for \(^1S_0\))

\[
\left[ \frac{1}{4m} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0 \right] R(r, t) = \int d^3r' U(r, r') R(r', t)
\]

\[
\simeq V_0(r) R(r, t) + V_1(r) \nabla^2 R(r, t) + \cdots
\]

\( R^\text{smear} \) and \( R^\text{wall} \) \( \Rightarrow \) \( V_0(r) \) and \( V_1(r) \)

\[\blacktriangleright \text{HAL method works} \] — quark src. independent w/o g.s. saturation

□ Leading order approximation

□ Next leading order correction
HAL meets Lüscher: Energy Shift from Potential

- HAL QCD works well \textit{w/o g.s. saturation problem}
  
  use potential $\Rightarrow$ true "energy shift" in finite volume

\begin{itemize}
  \item Eigenequation in finite box $L^3$ with HAL QCD potential $V(r)$

  $[H_0 + V] \psi = \Delta E \psi$

  \end{itemize}

\begin{itemize}
  \item eigenvalue $\Delta E_0 \propto 1/L^3 \rightarrow 0 \Rightarrow$ scattering by Lüscher's formula

  \end{itemize}

- potential $V(r)$

\begin{itemize}
  \item $\Xi(1S_0)$ potential [MeV]

  \begin{itemize}
    \item $40^3, t = 12$
    \item $48^3, t = 12$
    \item $64^3, t = 12$
  \end{itemize}

  \end{itemize}

\begin{itemize}
  \item $\alpha/L^3$-fit eigenvalue

  \end{itemize}
Baryon interactions from lattice QCD

Direct measurement vs HAL QCD method
- Formalisms
- Direct Measurement
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Origin of Fake Signal in Direct Method

Summary
Wavefunction, Potential, Eigenvalues and Eigenfunctions

NBS wave function

- Wavefunction
- Potential
- Eigenvalues & Eigenfunctions

HAL method

- Solve $[H_0 + V] \psi = E \psi$
- Feed back
- Decomposition
- Projection
- (elastic scattering)

Ground state & excited states
Excited States in Wavefunction

▶ R-corr. decomposition by energy eigenmodes ▶ from HAL pot.

\[ R_{\text{wall/smear}}(\vec{r}, t) = \sum_n a_n^{\text{wall/smear}} \Psi_n(\vec{r}, t) \exp(-\Delta E_n t) \]

\[ R(\vec{p} = 0, t) = \sum_{\vec{r}} R(\vec{r}, t) = \sum_n b_n^{\text{wall/smear}} e^{-\Delta E_n t} \]

☐ ex. 1st excited state

- **wall source**
  \[ b_1/b_0 \ll 0.01 \]

- **smeared source**
  \[ b_1/b_0 \approx -0.1 \]

- with energy gap
  \[ E_1 - E_0 \approx 50 \text{ MeV} \]
  for \( L^3 = 48^3 \)

“contamination” of excited states \( b_n/b_0 \)

\[ |b_n/b_0| \\Xi(1S_0) \text{ at } t = 14 \]

\( \Delta E_n [\text{MeV}] \)
Origin of Fake Plateau — Contamination of Excited States

\[ \Delta E_{\text{eff}}(t) \equiv \log \frac{R(p = 0, t)}{R(p = 0, t + 1)} = \log \frac{\sum_n b_n \exp (-\Delta E_n t)}{\sum_n b_n \exp (-\Delta E_n (t + 1))} \]

“direct measurement” — reproduced by low-lying modes†

† eigenvalues \( \Delta E_n \), coefficients \( b_n^{\text{smear/wall}} \) for \( n = 0, 1, 2 \), at \( t = 14 \).
\[ \Delta E_{\text{eff}}(t) \equiv \log \frac{R(p = 0, t)}{R(p = 0, t + 1)} = \log \frac{\sum_n b_n \exp (-\Delta E_n t)}{\sum_n b_n \exp (-\Delta E_n (t + 1))} \]

- "direct measurement" — reproduced by low-lying modes\(^\dagger\)
- \textbf{g.s. saturation} of smeared source — \textbf{100 lattice units} \(\sim 10\ \text{fm} \) !!!

\(^\dagger\) eigenvalues \(\Delta E_n\), coefficients \(b_n^{\text{smea/wall}}\) for \(n = 0, 1, 2\), at \(t = 14\).
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4. Summary
Summary: Lüscher Direct vs HAL QCD

- “Direct method” — ground state saturation is extremely difficult
  - scattering states $\Rightarrow$ “fake plateau” ▶ Wrong Conclusion!
  - much smaller gap & larger noise @ phys. pt. $\Rightarrow$ almost impossible

- HAL QCD works well without g.s. saturation
  HAL QCD $\Rightarrow$ “correct” $\Delta E_L$ and input of Lüscher’s formula
- NBS corr. + “potential” $\Rightarrow$ excited states contamination
  and origin of fake plateau.

(even if you do not trust HAL QCD method)
fake plateau can be checked by Lüscher’s formula ▶ Aoki’s Talk
Demo: Contamination of Scattering State

Mock up data

\[ R(t) = b_0 e^{-\Delta E_{BB} t} + b_1 e^{-\delta E_{el} t} + c_0 e^{-\delta E_{inel} t} \]

with \( \delta E_{el} - \Delta E_{BB} = 50 \text{ MeV} \sim \mathcal{O}(1/L^2) \), \( \delta E_{inel} - \Delta E_{BB} = 500 \text{ MeV} \sim \mathcal{O}(\Lambda_{QCD}) \)

- g.s. saturation around \( t \rightarrow 10 \text{ fm} \)
- fake plateau around \( t \sim 1 \text{ fm} \)

![Graphs showing \( \Delta E_{BB}(t) - \Delta E_{BB} \) values for different parameters.](image)
\[ \Xi \Xi ({}^1S_0) \]

\[ \Delta E_{\Xi \Xi}(t) \text{ [MeV]} \]

\[ L = 32 \]

\[ \Xi \Xi ({}^1S_0) \] smeared src.

\[ \Xi \Xi ({}^1S_0) \] wall src.

relativistic op. and non-rela. op. (NR)
$\Xi\Xi(3S_1)$

\begin{align*}
\Delta E_{\Xi\Xi}(t) &\left[ \text{MeV} \right] \\
L = 32 &\quad \text{smeared src. } \Xi\Xi(3S_1) \\
&\quad \text{wall src. } \Xi\Xi(3S_1) \\
L = 40 &\quad \text{smeared src. } \Xi\Xi(3S_1) \\
&\quad \text{wall src. } \Xi\Xi(3S_1) \\
L = 48 &\quad \text{smeared src. } \Xi\Xi(3S_1) \\
&\quad \text{wall src. } \Xi\Xi(3S_1) \\
L = 64 &\quad \text{smeared src. } \Xi\Xi(3S_1) \\
&\quad \text{wall src. } \Xi\Xi(3S_1)
\end{align*}

\begin{align*}
\Delta E_{\Xi\Xi} &\left[ \text{MeV} \right] \\
1/L^3 &\quad \text{smeared src. } \Xi\Xi(3S_1) \\
&\quad \text{smeared src. NR } \Xi\Xi(3S_1)
\end{align*}

relativistic op. and non-rela. op. (NR)
$NN(1S_0)$

![Graphs showing $\Delta E_{\text{eff}}(t)$ for different values of $L$.](image)

- For $L = 32$ MeV, smeared src. $NN(1S_0)$
- For $L = 40$ MeV, wall src. $NN(1S_0)$
- For $L = 48$ MeV, smeared src. $NN(1S_0)$
- For $L = 64$ MeV, wall src. $NN(1S_0)$

![Graphs showing $\Delta E_{\text{eff}}(t)$ for different values of $L$.](image)

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- For $L = 48$ MeV, smeared src. $NN(1S_0)$
- For $L = 64$ MeV, wall src. $NN(1S_0)$

![Graphs showing $\Delta E_{\text{NN}}$ vs. $1/L^3$ for different values of $L$.](image)

- For $L = 32$ MeV, smeared src. $NN(1S_0)$
- For $L = 40$ MeV, wall src. $NN(1S_0)$
- For $L = 48$ MeV, smeared src. $NN(1S_0)$
- For $L = 64$ MeV, wall src. $NN(1S_0)$

**relativistic op. and non-rela. op. (NR)**
$NN(3S_1)$
Triton
Helium

![Graphs showing the energy difference $\Delta E_{4\text{He}}$ as a function of time $t$ and inverse length $1/L^3$. The graphs depict data for smeared source and wall source of $4\text{He}$, with and without relativistic corrections.](image)

The plots illustrate the energy difference $\Delta E_{4\text{He}}$ in MeV as a function of $1/L^3$ in $a^{-3} \times 10^{-5}$ for different lengths $L$. The graphs include data for smeared source and wall source of $4\text{He}$, with and without relativistic corrections (NR).
\[ \Delta E_{\text{eff}}(t) = E_{\Xi\Xi}^{\text{eff}}(t) - 2m_{\Xi}(t) : \text{Smeared Src. vs. Wall Src.} \]
$\Xi\Xi(1S_0)$ is Unbound at $m_\pi = 510$ MeV

$$k \cot \delta(k) = \frac{1}{\pi L} \sum_{n \in \mathbb{Z}^3} \frac{1}{|n|^2 - (kL/2\pi)^2},$$

$$\Delta E = 2\sqrt{m^2 + k^2 - 2m}$$

**Phase shift $\delta$**

- $40^3 \times 48$ at $t = 12$
- $48^3 \times 48$ at $t = 12$
- $64^3 \times 64$ at $t = 12$
- Luscher formula

**Volume dep. of $\Delta E_0$**

- $\alpha/L^3$-fit
- Eigenvalue

**Graphs:**
- $1/L^3 [a^{-3} \times 10^{-5}]$ vs $\Delta E_0$ in MeV
- $E_{CM}$ in MeV vs Phase shift $\delta$ in deg.
$t$-dependence of Potential

$t$-dependence of Wall Src. potential is stable

$64^3 \ t = 12 - 17$

$\Sigma(1S_0)$ potential [MeV]

$2 \times m_{\Sigma}$ or $E^{\Sigma}_{eff}$ [MeV], $L = 64$

wall src. $\Sigma$

wall src. $\Sigma \Sigma(1S_0)$
Time-dependent HAL QCD Method

space-time correlation function

\[ R(\vec{r}, t) \equiv \langle 0| T\{B(\vec{x} + \vec{r}, t)B(\vec{x}, t)\} \tilde{J}(0)|0\rangle /\{G_B(t)\}^2 \]

\[ = \sum_n A_n \psi_n(\vec{r}) e^{-(E_n - 2m_B)t} + O(e^{-(E_{th} - 2m_B)t}) \]

each \( \psi_n(\vec{r}) e^{-E_n t} \equiv \langle 0| T\{B(\vec{x} + \vec{r}, t)B(\vec{x}, t)\}|2B, n\rangle \) satisfies

\[ \left[ \frac{k_n^2}{m_B} - H_0 \right] \psi_n(\vec{r}) = \int d\vec{r}' U(\vec{r}, \vec{r}') \psi_n(\vec{r}') \]

with non-local interaction kernel \( U(\vec{r}, \vec{r}') \)

\( R \)-corr. satisfies \( t \)-dep. Schrödinger-like eq. with elastic saturation

\[ \left[ \frac{1}{4m_B} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0 \right] R(\vec{r}, t) = \int d\vec{r}' U(\vec{r}, \vec{r}') R(\vec{r}', t) \]

“potential” using velocity expansion \( U(\vec{r}, \vec{r}') \simeq V(\vec{r}) \delta(\vec{r} - \vec{r}') \)

\[ V(\vec{r}) = \frac{1}{4m_B} \left( \frac{\partial}{\partial t} \right)^2 R(\vec{r}, t) - \frac{(\partial / \partial t) R(\vec{r}, t)}{R(\vec{r}, t)} - \frac{H_0 R(\vec{r}, t)}{R(\vec{r}, t)} \]

This method does not require the ground state saturation.
HAL: Wave Function and $\Xi \Xi (1S_0)$ Potential $V_c(\vec{r})$

- **wall src.** — weak $t$-dep.
- **smeared src.** — strong $t$-dep.
- Contribution of excited states
- Time-dep. HAL method works well
- $O(100)$ MeV of cancellation

$$V_c(\vec{r}) = -\frac{H_0 R}{R} - \frac{(\partial / \partial t) R}{R} + \frac{(\partial / \partial t)^2 R}{4mR}$$
Next Leading Order of Derivative Expansion

Derivative expansion: \( U(r, r') = \{V_0(r) + V_1(r) \nabla^2\} \delta(r - r') \) (for \(^1S_0\))

\[
\left[ \frac{1}{4m} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0 \right] R(r, t) = \int d^3r' U(r, r') R(r', t)
\]

\[
\cdot \frac{1}{4m} \frac{\partial^2}{\partial t^2} R - \frac{\partial}{\partial t} \frac{H_0 R}{R} = V_0(r) + V_1(r) \frac{\nabla^2 R(r, t)}{R(r, t)} \equiv \tilde{V}_{\text{eff}}(r, t)
\]

\( R^{\text{smear}} \) and \( R^{\text{wall}} \)

\[
\begin{align*}
V_0(r) + V_1(r) \nabla^2 R^{\text{smear}} / R^{\text{smear}} &= \tilde{V}_{\text{eff}}^{\text{smear}}(r, t_{\text{smear}}) \\
V_0(r) + V_1(r) \nabla^2 R^{\text{wall}} / R^{\text{wall}} &= \tilde{V}_{\text{eff}}^{\text{wall}}(r, t_{\text{wall}}),
\end{align*}
\]

\( \triangleright \) LO \( V_0(r) \) and NLO \( V_1(r) \) potentials are given by

\[
V_1(r) = \frac{\tilde{V}_{\text{eff}}^{\text{smear}}(r, t_{\text{smear}}) - \tilde{V}_{\text{eff}}^{\text{wall}}(r, t_{\text{wall}})}{\nabla^2 R^{\text{smear}} / R^{\text{smear}} - \nabla^2 R^{\text{wall}} / R^{\text{wall}}}
\]

\[
V_0(r) = \tilde{V}_{\text{eff}}^{\text{smear}}(r, t_{\text{smear}}) - V_1(r) \frac{\nabla^2 R^{\text{smear}}}{R^{\text{smear}}}.
\]