S. Simula
INFN - Roma Tre

XXXIV International Symposium on
Lattice Field Theory
Southampton, UK, July 24-30, 2016

Electromagnetic corrections to the leptonic decay rates of charged pseudoscalar mesons: lattice results

in collaboration with:

V. Lubicz, G. Martinelli, C.T. Sachrajda, F. Sanfilippo, N. Tantalo, C. Tarantino

aim of the talk

- to show that the methodology recently proposed in **PRD91 (2015) 074506** to calculate ***QED corrections to hadronic processes***, although very challenging, is within the reach of present ***lattice technologies***
- to present the ***first (preliminary) lattice results*** on the electromagnetic effects in the leptonic decay rates $\boldsymbol{\pi}^+ \to \mu^+\nu[\gamma]$ and $K^+ \to \mu^+\nu[\gamma]$

basic steps of the procedure [PRD91 (2015) 074506]

1) the emission of virtual photons at leading order in the e.m. coupling is evaluated on the lattice
2) the subtraction of the infrared divergence is computed for a point-like meson using the finite lattice volume as the infrared regulator
3) the emission of virtual+real photons from a point-like meson is added using a photon mass for the infrared regularization

$$\Gamma = \left[\Gamma_0^{lattice}(L) - \Gamma_0^{pt}(L)\right] + \left[\Gamma_0^{pt}(m_\gamma) + \Gamma_1^{pt}(m_\gamma)\right]$$

master formula for the leptonic decay rate

$$\Gamma\left(PS \to \ell\nu[\gamma]\right) = \Gamma^{(tree)}\left(PS \to \ell\nu\right) \cdot R_{PS}\left(\Delta E_\gamma\right)$$

tree level: $\Gamma^{(tree)}\left(PS \to \ell\nu\right) = \frac{G_F^2}{8\pi}\left|V_{q_1 q_2}\right|^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{M_{PS}^2}\right)^2 \left[f_{PS}^{(0)}\right]^2 M_{PS}$

$$f_{PS}^{(0)} \equiv \frac{p_{PS}^\mu}{M_{PS}^2}\langle 0|\bar{q}_2\gamma_\mu\gamma_5 q_1|PS\rangle$$

$$M_{PS} = M_{PS}^{(0)} + \alpha_{em}\delta M_{PS} + (m_d - m_u)\delta_{IB}M_{PS}$$

$$R_{PS}\left(\Delta E_\gamma\right) = 1 + \alpha_{em}\left\{\frac{2}{\pi}\log\left(\frac{M_Z}{M_W}\right) + 2\delta\left[\frac{A_{PS}}{f_{PS}^{(0)}M_{PS}}\right] + \delta\Gamma^{pt}\left(\Delta E_\gamma\right)\right\}$$

short-distance e.w. correction not included in G_F (μ lifetime)

virtual photon emissions calculated on the lattice (using the lattice volume as IR regulator)

e.m. correction (virtual + real photons up to energy ΔE_γ) for a point-like PS meson (using a photon mass as IR regulator)

* δA_{PS} and $\delta\Gamma^{pt}\left(\Delta E_\gamma\right)$ are separately IR finite and indepedent on the specific IR regularization

calculation of $\delta\Gamma^{pt}(\Delta E_\gamma)$

$$\delta\Gamma^{pt}(\Delta E_\gamma) = \delta\Gamma_0^{pt} + \delta\Gamma_1^{pt}(\Delta E_\gamma)$$

the sum is IR finite (Bloch-Nordsieck mechanism)

FIG. 8. One loop diagrams contributing to the wave-function renormalization of a pointlike pion.

* virtual photons $\delta\Gamma_0^{pt}$

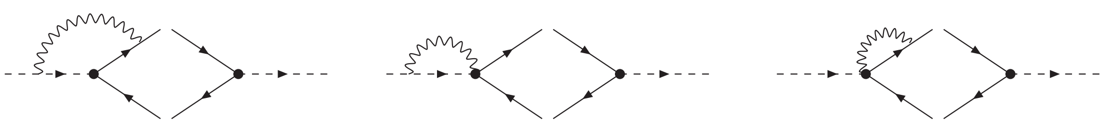


FIG. 9. Radiative corrections to the pion-lepton vertex. The diagrams represent $O(\alpha)$ contributions to $\Gamma_0^{\rm pt}$. The left part of each diagram represents a contribution to the amplitude and the right part the tree-level contribution to the Hermitian conjugate of the amplitude. The corresponding diagrams containing the radiative correction on the right-hand side of each diagram are also included.

* real photons $\delta\Gamma_1^{pt}(\Delta E_\gamma)$

N. CARRASCO *et al.* PHYSICAL REVIEW D **91,** 074506 (2015)

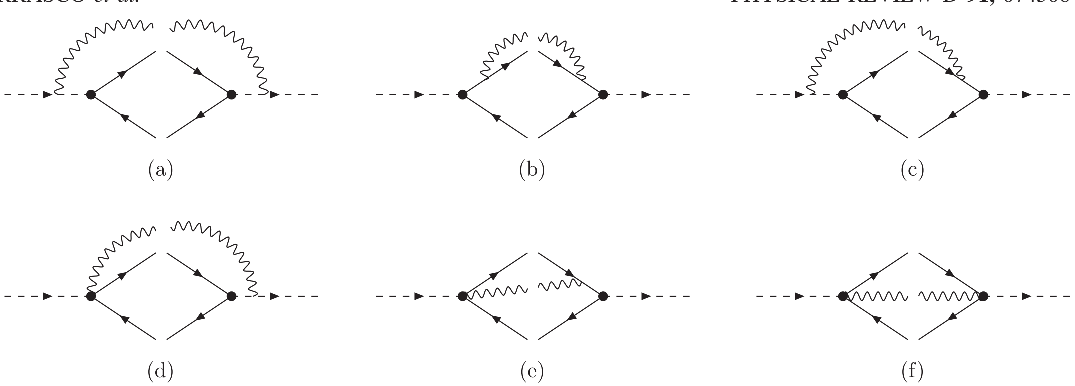


FIG. 10. Diagrams contributing to $\Gamma_1(\Delta E)$. For diagrams (c), (d) and (e) the "conjugate" contributions in which the photon vertices on the left and right of each diagram are interchanged are also to be included. The labels (a)–(f) are introduced to identify the individual diagrams when describing their evaluation in the text.

[PRD91 (2015) 074506]

$$\begin{aligned}
\delta\Gamma^{pt}(\Delta E_\gamma) \;=\; & \frac{1}{4\pi}\Big\{3\log(M_{PS}^2/M_W^2) - 3 + \log(r_\ell^2) \\
& - 4\log(r_E^2) + \frac{2-10r_\ell^2}{1-r_\ell^2}\log(r_\ell^2) \\
& - 2\frac{1+r_\ell^2}{1-r_\ell^2}\log(r_\ell^2)\log(r_E^2) \\
& - 4\frac{1+r_\ell^2}{1-r_\ell^2}\mathrm{Li}_2(1-r_\ell^2) \\
& + \frac{3+r_E^2-6r_\ell^2-4r_E(1-r_\ell^2)}{(1-r_\ell^2)^2}\log(1-r_E) \\
& + r_E\frac{4-r_E-4r_\ell^2}{(1-r_\ell^2)^2}\log(r_\ell^2) \\
& - r_E\frac{28r_\ell^2+3r_E-22}{2(1-r_\ell^2)^2} \\
& - 4\frac{1+r_\ell^2}{1-r_\ell^2}\mathrm{Li}_2(r_E)\Big\}
\end{aligned}$$

$$r_\ell = m_\ell / M_{PS}\,, \quad r_E = 2\Delta E_\gamma / M_{PS}$$

$\Delta E_\gamma \sim$ 10-20 MeV

for the point-like assumption to be valid

calculation of δA_{PS}

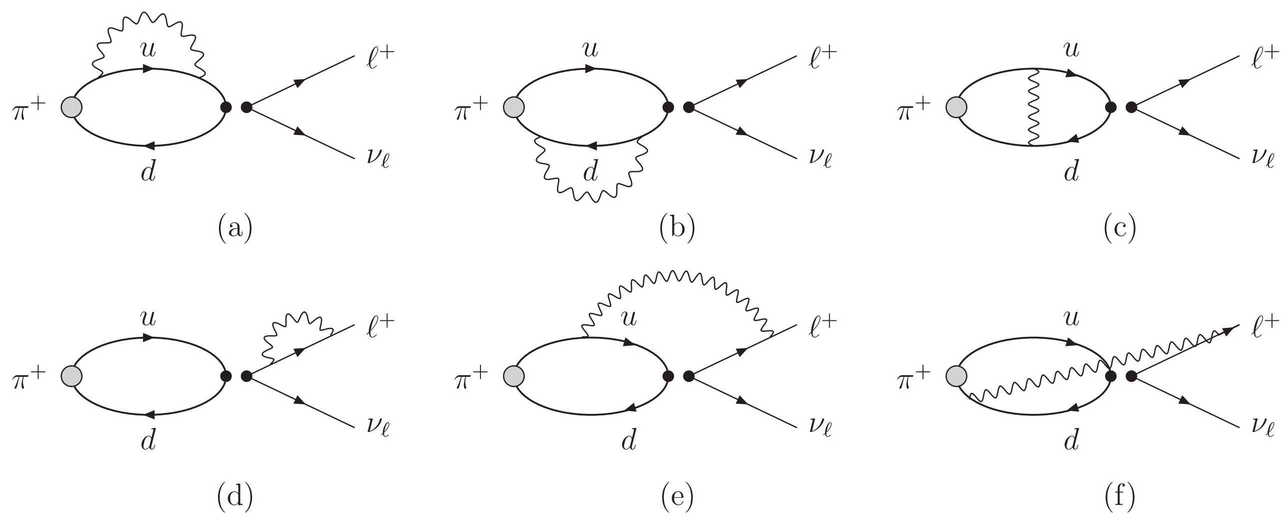


FIG. 5. Connected diagrams contributing at $O(\alpha)$ to the amplitude for the decay $\pi^+ \to \ell^+\nu_l$. The labels (a)–(f) are introduced to identify the individual diagrams when describing their evaluation in the text.

virtual photons between quarks and/or lepton

connected diagrams

N. CARRASCO *et al.* PHYSICAL REVIEW D **91,** 074506 (2015)

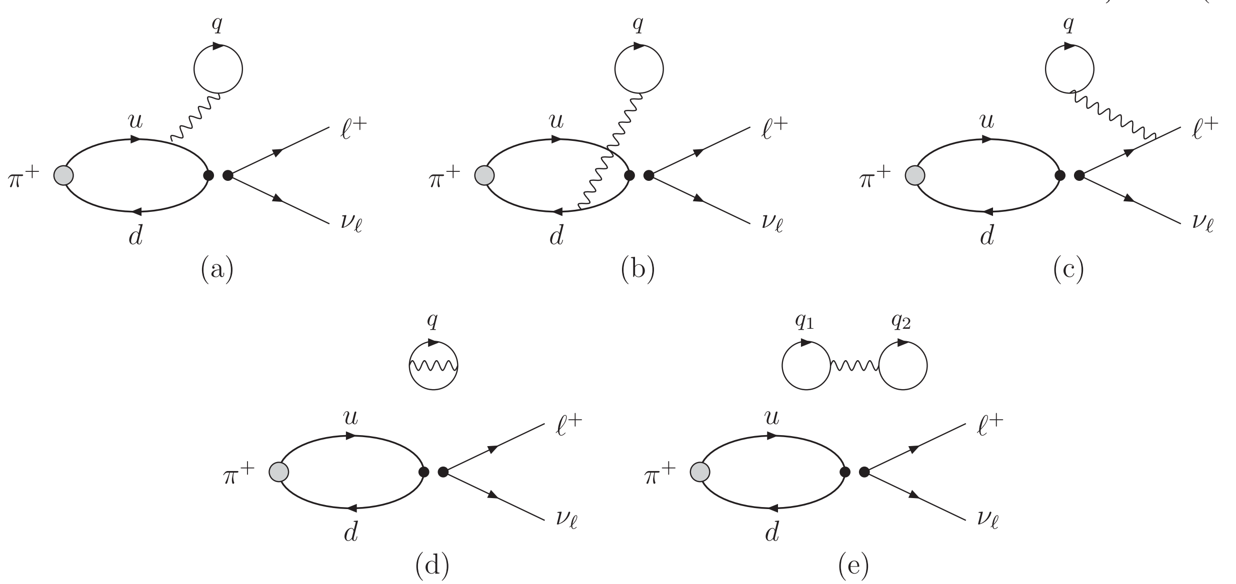


FIG. 6. Disconnected diagrams contributing at $O(\alpha)$ to the amplitude for the decay $\pi^+ \to \ell^+\nu_l$. The curly line represents the photon, and a sum over quark flavors q, q_1 and q_2 is to be performed. The labels (a)–(e) are introduced to identify the individual diagrams when describing their evaluation in the text.

disconnected diagrams

<u>quenched QED</u>

$e_f^{sea} = 0$

adopted in this work

- Wilson twisted-mass action for sea and valence up/down quarks, Osterwalder-Seiler action for valence strange (and charm) quark

- Iwasaki action for the gluons

- maximal twist guarantees an automatic O(a)-improvement for the above non-unitary setup

gauge ensembles from the European Twisted Mass Collaboration (ETMC)

N_f = 2+1+1 dynamical sea quarks

three values of the lattice spacing: a ~ 0.0885 (36), 0.0815 (30), 0.0619 (18) fm

lattice sizes from 1.8 to 3 fm
$3 < M_\pi L < 6$

pion masses from 225 to 500 MeV

the strange quark mass at each β is calculated using the physical m_s mass and Z_m obtained by ETMC in NPB 887 (2014)

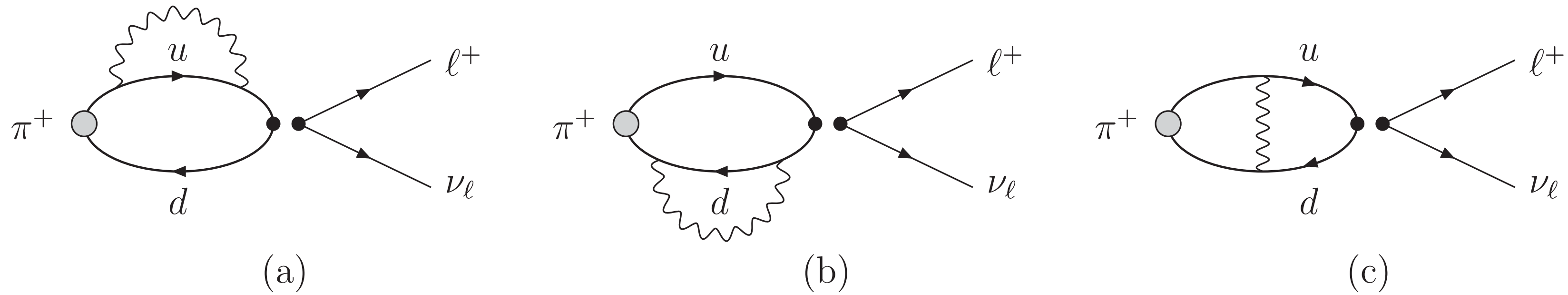
ensemble	β	V/a^4	$a\mu_{sea} = a\mu_\ell$	$a\mu_\sigma$	$a\mu_\delta$	N_{cfg}	$a\mu_s$	M_{π^+} (MeV)	M_{K^+} (MeV)	L (fm)	$M_\pi L$
$A30.32$	1.90	$32^3 \times 64$	0.0030	0.15	0.19	150	0.0236	278	564	2.9	4.0
$A40.32$			0.0040			100		318	573		4.6
$A50.32$			0.0050			150		351	581		5.1
$A40.24$		$24^3 \times 48$	0.0040			150		325	579	2.1	3.5
$A60.24$			0.0060			150		387	594		4.2
$A80.24$			0.0080			150		444	615		4.8
$A100.24$			0.0100			150		496	636		5.4
$A40.20$		$20^3 \times 48$	0.0040			150		331	583	1.8	3.0
$B25.32$	1.95	$32^3 \times 64$	0.0025	0.135	0.170	150	0.0209	261	542	2.6	3.5
$B35.32$			0.0035			150		304	551		4.1
$B55.32$			0.0055			150		377	574		5.0
$B75.32$			0.0075			80		438	596		5.8
$B85.24$		$24^3 \times 48$	0.0085			150		468	609	2.0	4.7
$D15.48$	2.10	$48^3 \times 96$	0.0015	0.12	0.1385	100	0.0161	226	526	3.0	3.4
$D20.48$			0.0020			100		257	529		3.9
$D30.48$			0.0030			100		313	546		4.8

all the relevant correlation functions calculated thanks to the

PRACE project **Pra10_2693**: “QED corrections to meson decay rates in LQCD”

18 Mcore-hours on the BG/Q system Fermi at Cineca (Italy), April 2015 - March 2016

* virtual photons between quarks: lattice calculation



$$\delta C^{(qq)}(t) = -\frac{1}{2}\sum_{\vec{x},x_1,x_2}\langle 0|T\left\{J^{\rho}_{ew}(0)\, j^{em}_{\mu}(x_1)\, j^{em}_{\mu}(x_2)\, \phi^{\dagger}_{PS}(\vec{x},-t)\right\}|0\rangle\, \Delta_{em}(x_1,x_2)\,\frac{p^{\rho}_{PS}}{M_{PS}}$$

(V-A) quark current — em quark current — PS interpolating field — photon propagator

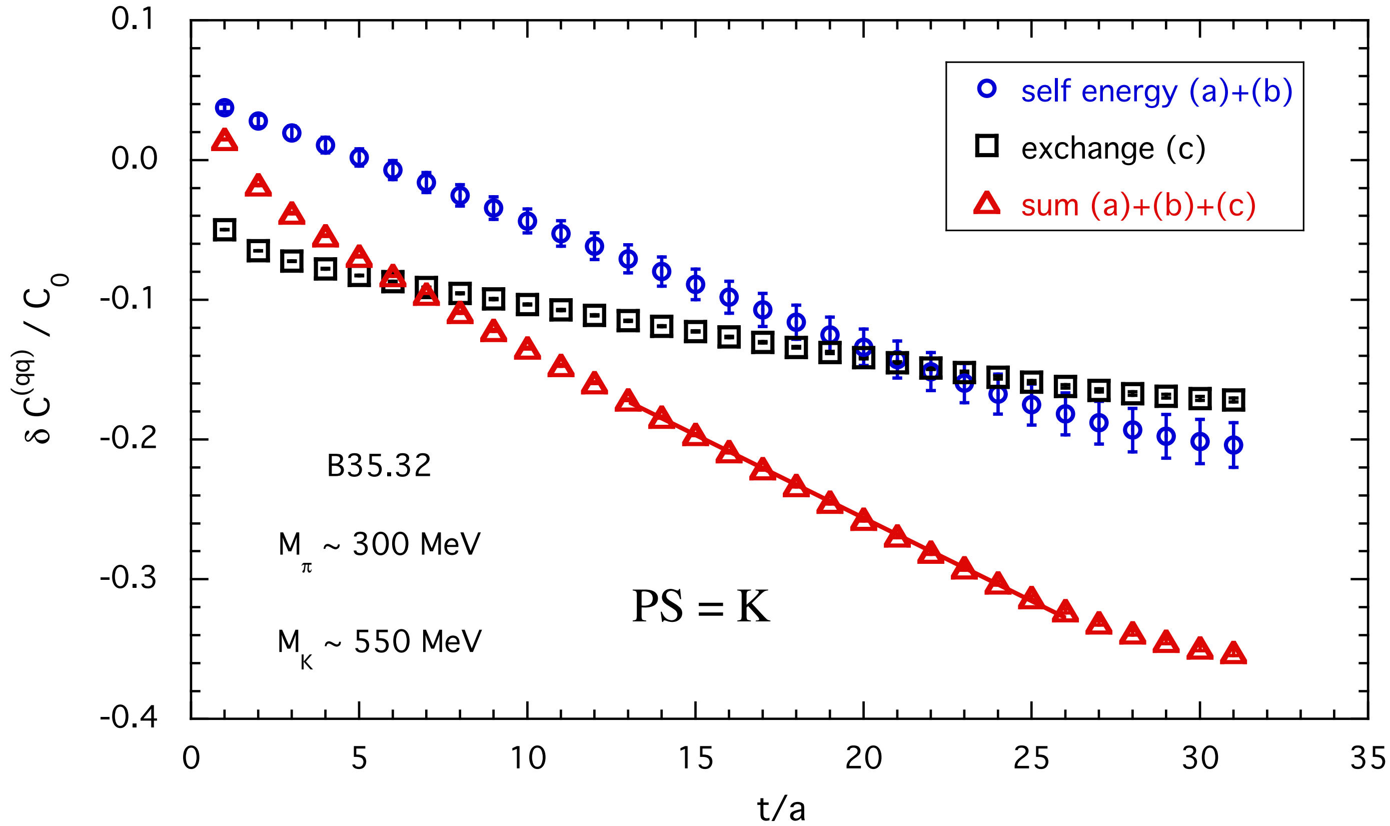
tree level: $C_0(t) = \sum_{\vec{x}}\langle 0|T\left\{J^{\rho}_{ew}(0)\,\phi^{\dagger}_{PS}(\vec{x},-t)\right\}|0\rangle\frac{p^{\rho}_{PS}}{M_{PS}}$

large time distances: $C_0(t)+\alpha_{em}\delta C^{(qq)}(t) \xrightarrow[t>>a]{} \frac{Z_{PS}A^{(qq)}_{PS}}{2M_{PS}}\left[e^{-M_{PS}t} - e^{-M_{PS}(T-t)}\right]$

$$M_{PS} = M^{(0)}_{PS} + \alpha_{em}\delta M_{PS}, \quad A^{(qq)}_{PS} = A^{(0)}_{PS} + \alpha_{em}\delta A^{(qq)}_{PS}, \quad Z_{PS} = Z^{(0)}_{PS} + \alpha_{em}\delta Z_{PS}$$

$$\frac{\delta C^{(qq)}(t)}{C_0(t)} \xrightarrow[t>>a]{} \frac{\delta\left[Z_{PS}A^{(qq)}_{PS}\right]}{Z^{(0)}_{PS}A^{(0)}_{PS}} + \frac{\delta M_{PS}}{M^{(0)}_{PS}} f(t) \qquad f(t) \equiv M^{(0)}_{PS}\left(\frac{T}{2}-t\right)\frac{e^{-M^{(0)}_{PS}t} + e^{-M^{(0)}_{PS}(T-t)}}{e^{-M^{(0)}_{PS}t} - e^{-M^{(0)}_{PS}(T-t)}} - 1 \approx -M^{(0)}_{PS}t$$

***** δM_{PS} from the slope and $\delta\left[Z_{PS}A^{(qq)}_{PS}\right]$ from the intercept *****



$\delta \mathrm{M}_{PS}$ from the slope

$\delta\left[Z_{PS} A_{PS}^{(qq)}\right]$ from the intercept

need to subtract δZ_{PS}

$$\delta C^{PS}(t) = -\frac{1}{2}\int d^3\vec{x}\, d^4x_1\, d^4x_2 \langle 0| T\left\{\phi_{PS}(0)\, j_\mu^{em}(x_1)\, j_\mu^{em}(x_2)\, \phi_{PS}^\dagger(\vec{x},-t)\right\}|0\rangle \Delta_{em}(x_1,x_2)$$

tree level: $C_0^{PS}(t) = \int d^3\vec{x} \langle 0| T\left\{\phi_{PS}(0)\, \phi_{PS}^\dagger(\vec{x},-t)\right\}|0\rangle$

$$\frac{\delta C^{PS}(t)}{C_0^{PS}(t)} \xrightarrow[t>>a]{} 2\frac{\delta\left[Z_{PS}\right]}{Z_{PS}^{(0)}} - \frac{\delta M_{PS}}{M_{PS}^{(0)}} + \frac{\delta M_{PS}}{M_{PS}^{(0)}} f^{PS}(t) \qquad f^{PS}(t) \equiv M_{PS}^{(0)}\left(\frac{T}{2}-t\right)\frac{e^{-M_{PS}^{(0)}t} - e^{-M_{PS}^{(0)}(T-t)}}{e^{-M_{PS}^{(0)}t} + e^{-M_{PS}^{(0)}(T-t)}} \approx -M_{PS}^{(0)}t$$

$\delta \mathrm{M}_{PS}$ from the slope and $\delta\left[Z_{PS}\right]$ from the intercept

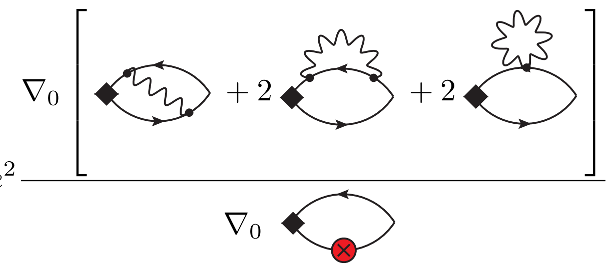
***** nice consistency between δM_{PS} extracted from δC and δC^{PS} *****

two further e.m. corrections due to Wilson (twisted-mass) fermions

- tadpole vertex: $$\sum_{f,\mu} e_f^2 T_\mu^f(x) = \sum_{f,\mu} e_f^2 \left[\bar{q}_f(x) \frac{i\gamma_5\tau_3 - \gamma_\mu}{2} U_\mu(x) q_f(x+\mu) + \bar{q}_f(x+\mu) \frac{i\gamma_5\tau_3 + \gamma_\mu}{2} U_\mu^\dagger(x) q_f(x) \right]$$

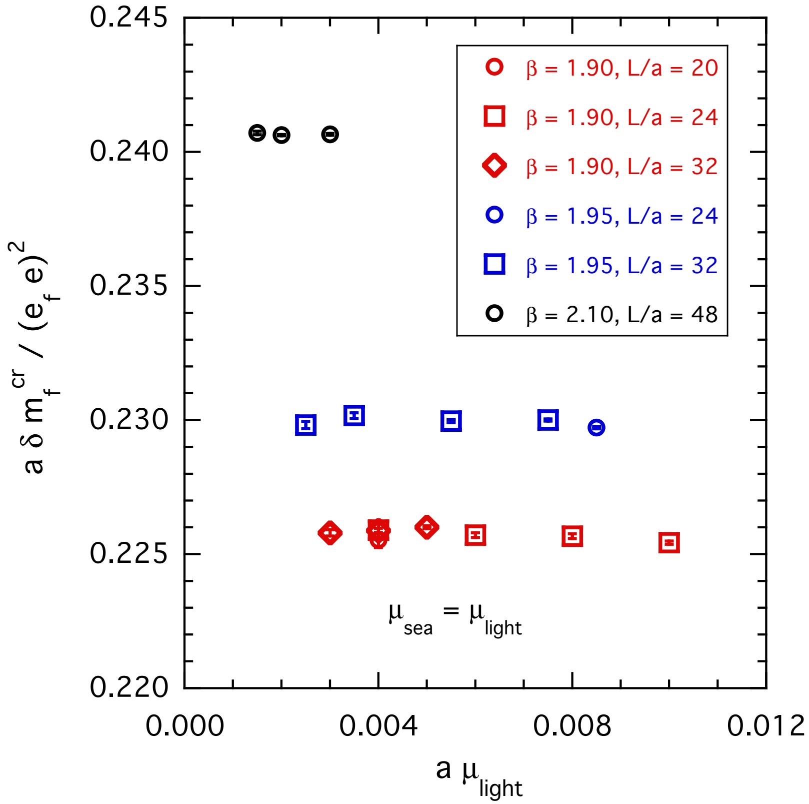
- shift of the critical mass: $\delta m_f^{cr} \bar{q}_f(x) i\gamma_5 \tau_3 q_f(x)$

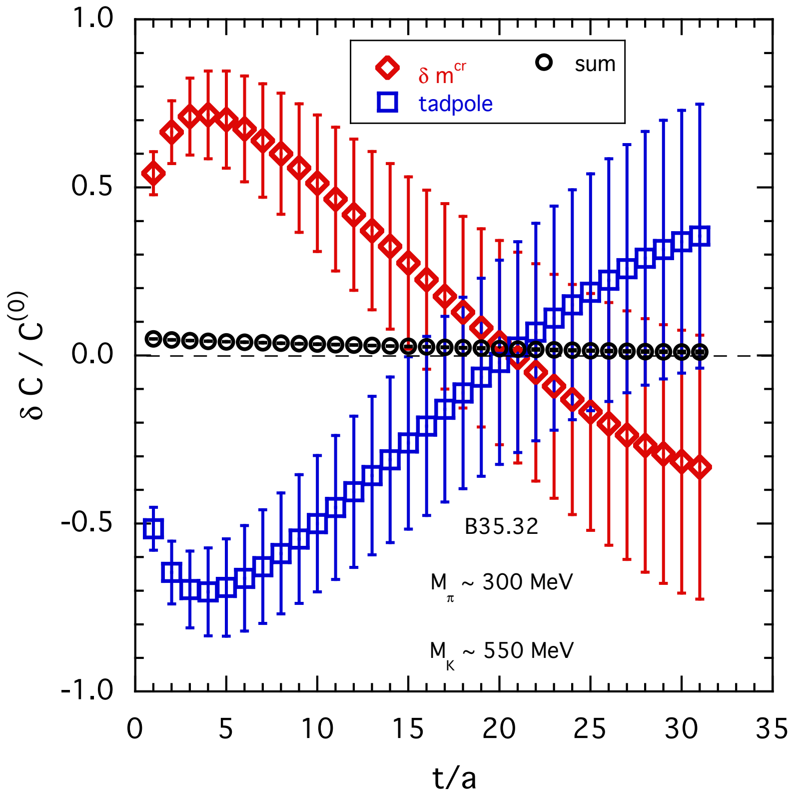
PRD 87 (2013) 114505

$\delta m_f^{cr} = -\frac{e_f^2}{2} e^2$ 

from vector WI

◆ $= \gamma_0$ ⊗ $= \gamma_5$





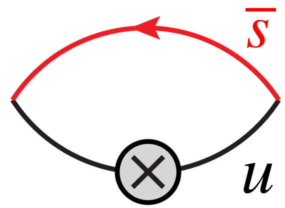
strong correlation between δm^{cr} and tadpole terms

the sum is well determined

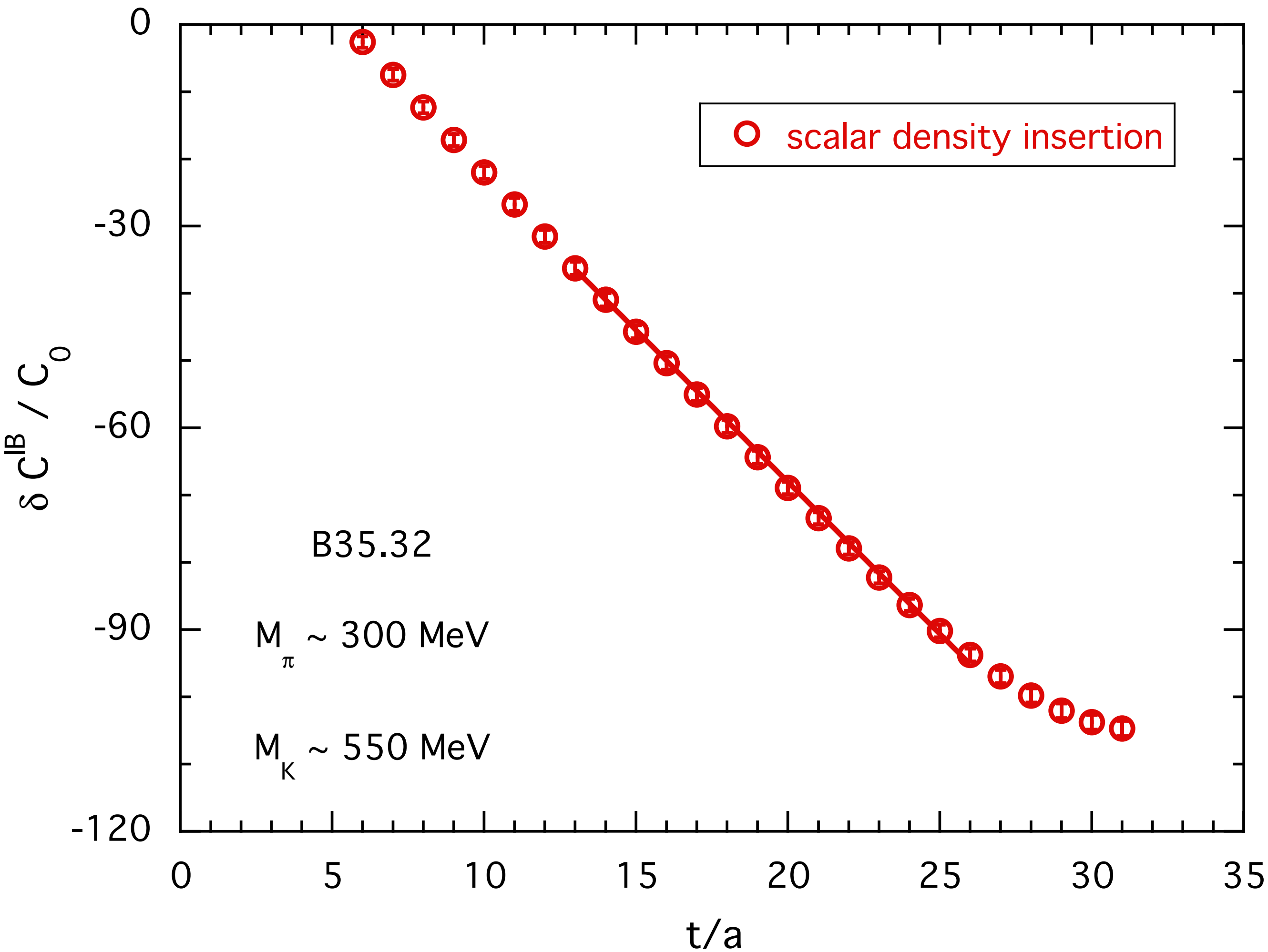
besides e.m. corrections at leading order in α_{em} , we adopt the RM123 approach to evaluate **the slope of the leading IB corrections due to $m_d \neq m_u$**, based on the insertion of the (isovector) scalar density in the isospin symmetric QCD limit

JHEP 04 (2012) 124
PRD 87 (2013) 114505

K+:



$$\otimes = \bar{d}(x)d(x) - \bar{u}(x)u(x)$$



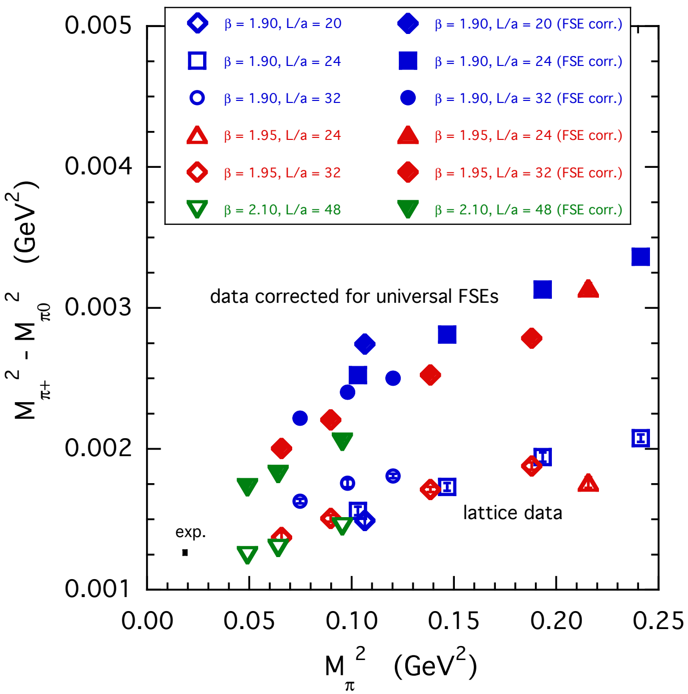
$$M_{PS} = M_{PS}^{(0)} + (m_d - m_u)\,\delta_{IB} M_{PS}$$

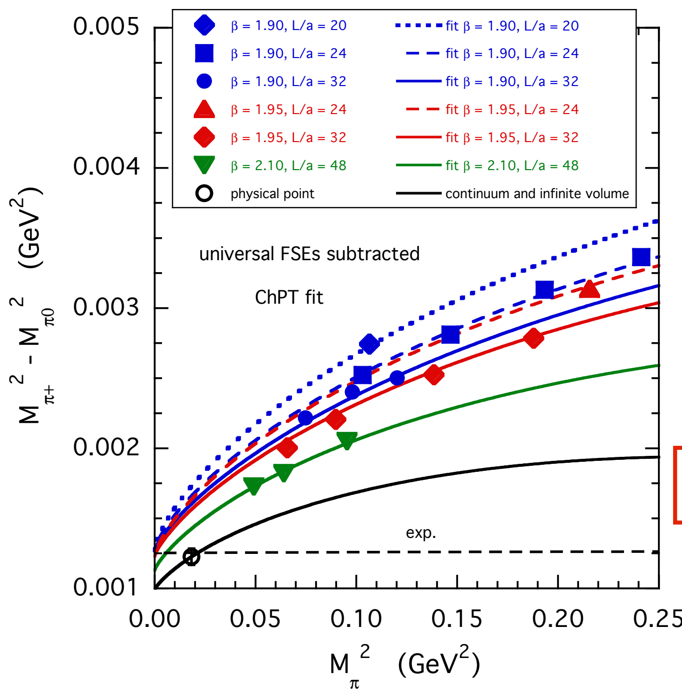
$$A_{PS} = A_{PS}^{(0)} + (m_d - m_u)\,\delta_{IB} A_{PS}$$

$\delta_{IB} M_{PS}$ from the slope

$\delta_{IB} A_{PS}$ from the intercept(s)

preliminary results for PS meson masses





pion

universal FSE: $-\alpha_{em}\kappa\left(2+M_\pi L\right)/L^2$

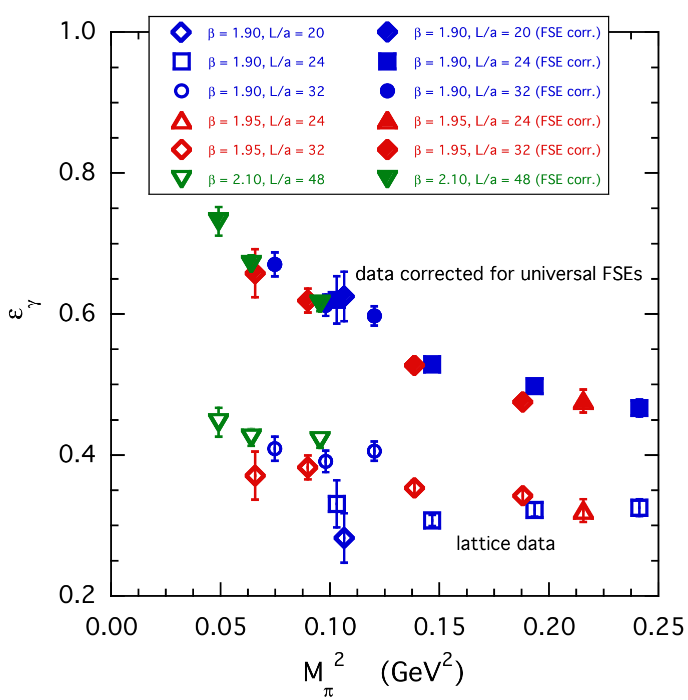
Hayakawa&Uno [PTP '08]

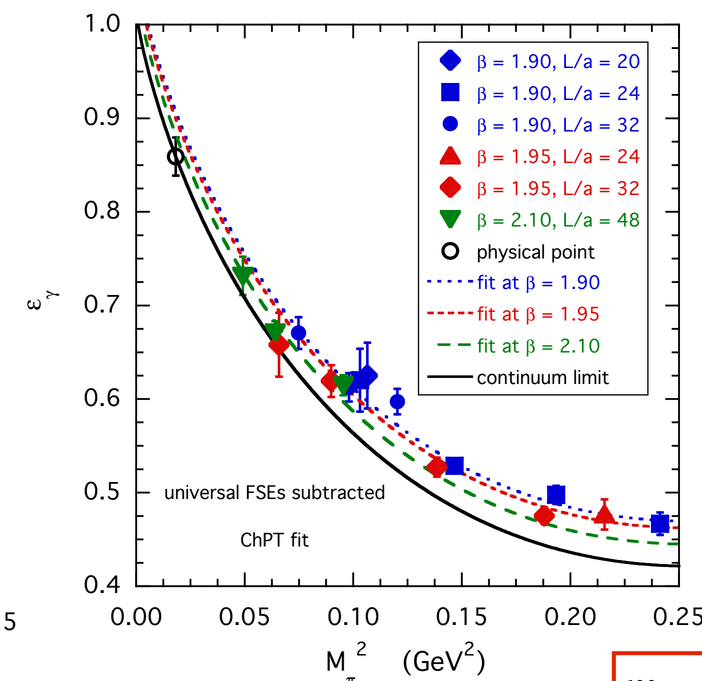
BMW: QED$_L$ on T^4 [Science '15]

residual **Finite Size Effects** still visible

$$\left[M_{\pi^+}^2 - M_{\pi^0}^2\right]^{phys} = 1.226\,(58)_{stat}\,(96)_{syst}\;10^{-3}\;\mathrm{GeV}^2$$

$$\left[M_{\pi^+}^2 - M_{\pi^0}^2\right]^{exp} = 1.2612\,(1)\,10^{-3}\;\mathrm{GeV}^2$$





kaon

$$\varepsilon_\gamma = \frac{\left[M_{K^+}^2 - M_{K^0}^2 + M_{\pi^0}^2 - M_{\pi+}^2\right]^{QED}}{M_{\pi^+}^2 - M_{\pi^0}^2}$$

$$\varepsilon_\gamma^{phys} = 0.833\,(22)_{stat}\,(28)_{syst}\,(\ldots)_{qQED}$$

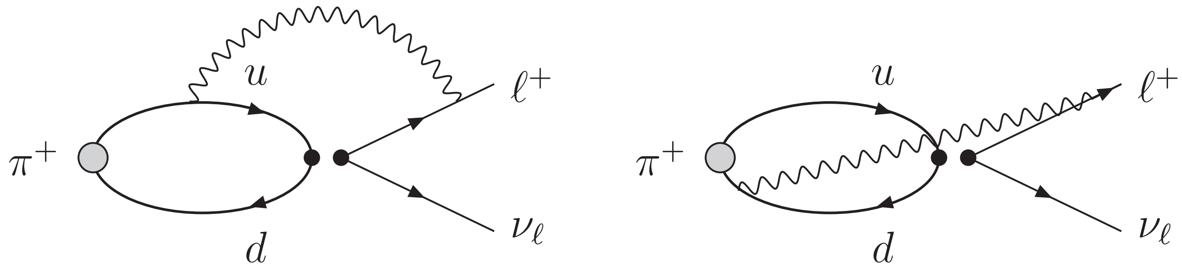
$$\varepsilon_\gamma^{FLAG} = 0.7\,(3)$$ [arXiv:1607.00299]

$$\delta_{IB} M_K = -4.66\,(6)_{stat}\,(22)_{syst}$$

$\overline{MS}(2\,\mathrm{GeV})$

$$m_d - m_u = 2.69\,(5)_{stat}\,(13)_{syst}\,(\ldots)_{qQED}\;\mathrm{MeV}$$

* **virtual photons between quarks and final lepton: lattice calculation**



(e) (f)

[times the tree-level leptonic part]

$$\delta C^{(q\ell)}(t) = -\sum_{\vec{x}, x_1, x_2} \langle 0 | T\left\{J^{\rho}_{ew}(0)\, j^{em}_{\mu}(x_1)\, \phi^{\dagger}_{PS}(\vec{x},-t)\right\}|0\rangle\, \Delta_{em}(x_1,x_2)\, e^{E_\ell t_2 - i\vec{p}_\ell\cdot\vec{x}_2}$$

$$\cdot\, \bar{u}(p_\nu)\gamma_\rho(1-\gamma_5) S^{\ell}(0,x_2)\gamma_\mu v(p_\ell)\left[\bar{v}(p_\ell)\gamma_\sigma(1-\gamma_5)u(p_\nu)\frac{p^{\sigma}_{PS}}{M_{PS}}\right]$$

$S^{\ell}(0,x) =$ free twisted-mass lepton propagator $\quad E_\ell = \sqrt{m_\ell^2 + \vec{p}_\ell^{\,2}}, \quad E_\ell + E_\nu = M^{(0)}_{PS} \quad \vec{p}_\ell$ injected via non-periodic b.c.

tree-level: $C^{(q\ell)}_0(t) = C_0(t)\, Tr(p_\ell, p_{PS})$

leptonic trace: $Tr(p_\ell, p_{PS}) = \bar{u}(p_\nu)\gamma_\rho(1-\gamma_5)v(p_\ell)\bar{v}(p_\ell)\gamma_\sigma(1-\gamma_5)u(p_\nu)\dfrac{p^{\rho}_{PS}}{M_{PS}}\dfrac{p^{\sigma}_{PS}}{M_{PS}}$

* expanding the (V-A) structure of the quark e.w. current:

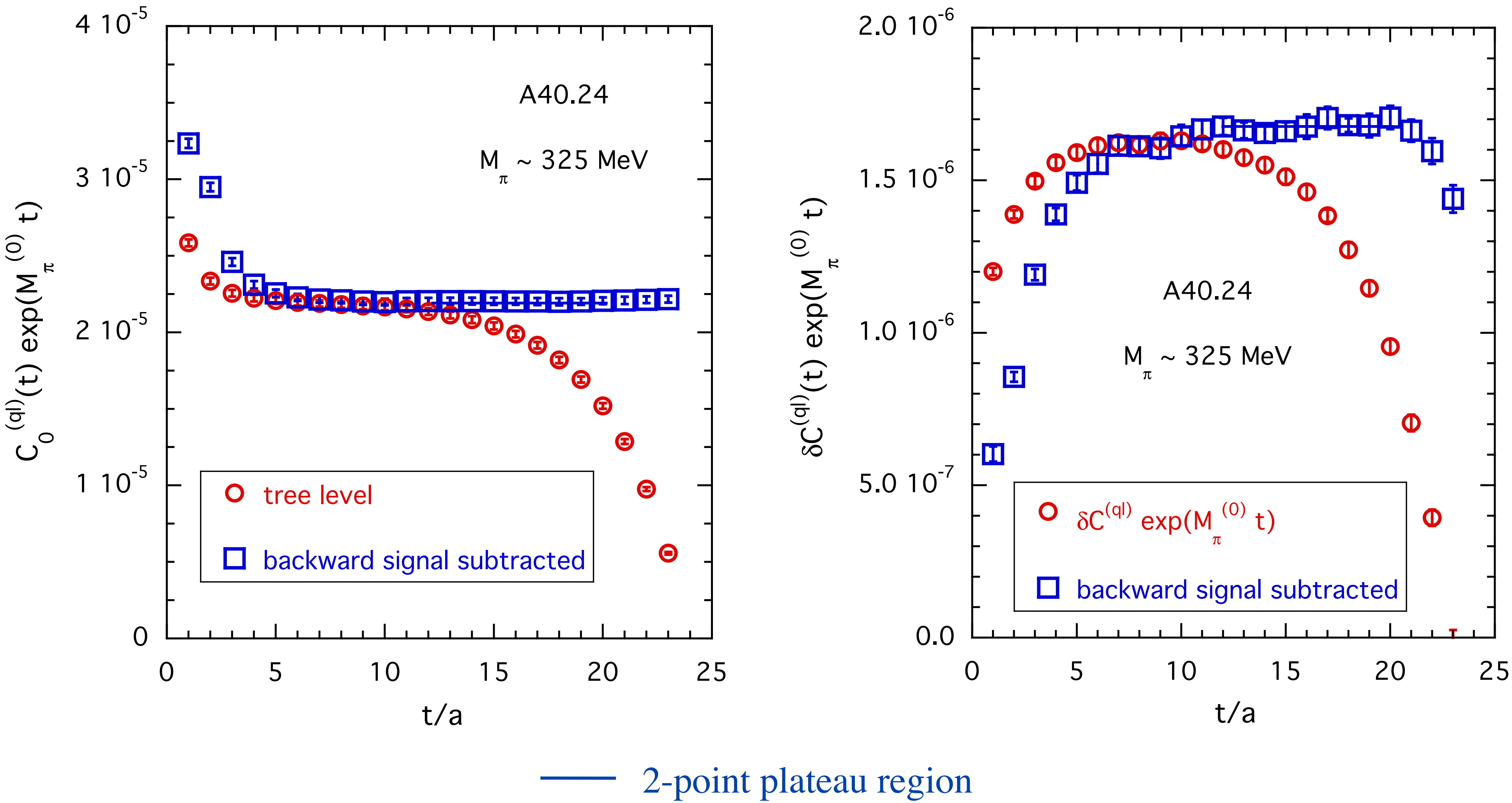
$$\delta C^{(q\ell)}(t) = Z_A\left[\delta C^{(V_0)}(t) + \delta C^{(V_k)}(t)\right] + Z_V\left[\delta C^{(A_0)}(t) + \delta C^{(A_k)}(t)\right]$$

(twisted-mass renormalization)

$$\delta C^{(q\ell)}(t) \xrightarrow[t>>a]{} \frac{Z^{(0)}_{PS}}{2M^{(0)}_{PS}}\, \delta A^{(q\ell)}_{PS}\, Tr(p_\ell, p_{PS})\left[e^{-M^{(0)}_{PS} t} \pm \text{ backward signals}\right]$$

depending on the time/spatial components

* subtraction of backward signals: $\bar{C}(t)e^{M_{PS}^{(0)}t} \equiv \frac{1}{2}\left[C(t)+\frac{C(t-1)-C(t+1)}{e^{M_{PS}^{(0)}}-e^{-M_{PS}^{(0)}}}\right]e^{M_{PS}^{(0)}t} \xrightarrow[t>>a]{} const.$



* after subtraction of backward signals: $\frac{\delta\bar{C}^{(q\ell)}(t)}{\bar{C}_0^{(q\ell)}(t)} \xrightarrow[t>>a]{} \frac{\delta A_{PS}^{(q\ell)}}{A_{PS}^{(0)}}$

chirality mixing

* e.m. corrections to the four-fermion effective theory generate UV divergencies that can be regularized by multiplying the photon propagator by $M_W^2/(M_W^2 - k^2)$ (W-regularization)
* on the lattice a perturbative matching has been calculated at LO in α_{em} [PRD 91 (2015) 074506] for lattice formulations breaking chiral symmetry

$$O_1^{W-reg} = O_1^{bare} + \alpha_{em} \sum_{i=1,5} Z_i O_i^{bare}$$

$$O_1^{bare} = \bar{q}_2 \gamma_\mu (1-\gamma_5) q_1 \bar{\nu} \gamma^\mu (1-\gamma_5) \ell$$

$$O_2^{bare} = \bar{q}_2 \gamma_\mu (1+\gamma_5) q_1 \bar{\nu} \gamma^\mu (1-\gamma_5) \ell, \quad O_3^{bare} = \bar{q}_2 (1-\gamma_5) q_1 \bar{\nu} (1+\gamma_5) \ell$$

$$O_4^{bare} = \bar{q}_2 (1+\gamma_5) q_1 \bar{\nu} (1+\gamma_5) \ell, \quad O_5^{bare} = \bar{q}_2 \sigma_{\mu\rho} (1+\gamma_5) q_1 \bar{\nu} \sigma^{\mu\rho} (1-\gamma_5) \ell$$

Wilson fermions:

$$Z_1 = \frac{1}{4\pi} \left[\frac{5}{2} \log\left(a^2 M_W^2\right) - 5.506 \right] Z_1^{QCD}$$

$$Z_2 = \frac{1}{4\pi} [0.323] Z_2^{QCD}, \qquad Z_3 = \frac{1}{4\pi} [0.969] Z_3^{QCD}$$

$$Z_4 = \frac{1}{4\pi} [-1.938] Z_4^{QCD}, \qquad Z_5 = \frac{1}{4\pi} [-0.485] Z_5^{QCD}$$

Z_i^{QCD} = non-perturbative QCD corrections $O(\alpha_s)$

* Wilson twisted-mass fermions (rotation to the physical basis) $\left[\langle 0 | O_5^{bare} | PS \rangle = 0 \right]$

$$\left[O_1^{bare}\right]_{phys}^{W-reg} = \left[O_1^{bare}\right]_{phys} + \alpha_{em} \left\{ Z_1 \left[O_1^{bare}\right]_{phys} - Z_2 \left[O_2^{bare}\right]_{phys} - r\, Z_3 \left[O_3^{bare}\right]_{phys} - r\, Z_4 \left[O_4^{bare}\right]_{phys} \right\}$$

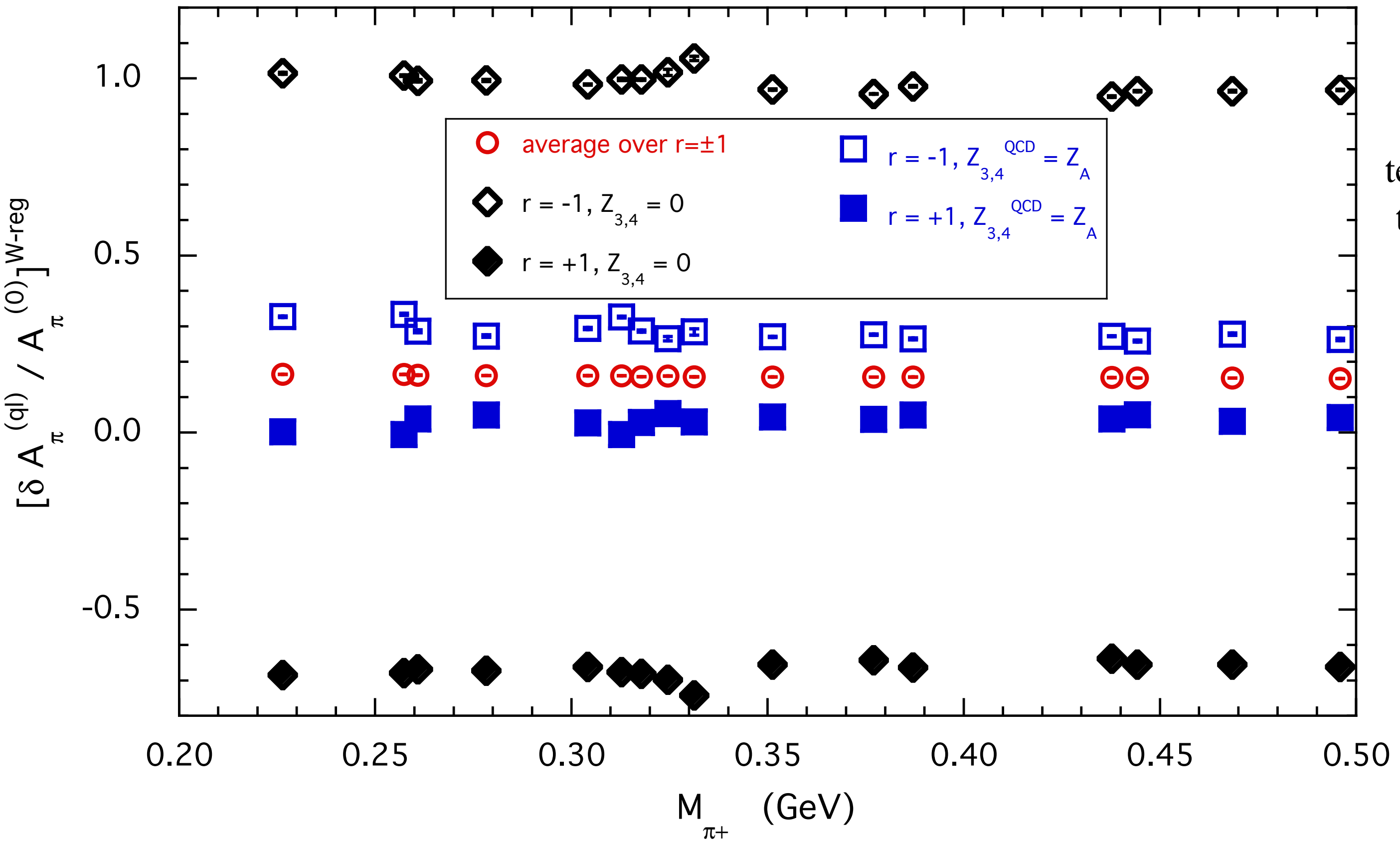
Wilson r-parameters: $r \equiv r_{q_1} \, r_\ell \qquad \left(r_{q_2} = -r_{q_1} \right)$

to keep discretization errors on M_{PS} of order $O(a^2 m)$

* $r = \pm 1$, but physical quantities cannot depend on r

$$\left[\frac{\delta A_{PS}}{A_{PS}^{(0)}}\right]^{W-reg} = \frac{\delta A_{PS}}{A_{PS}^{(0)}} + \frac{Z_1 + Z_2}{Z_V} + r\left(Z_3 - Z_4\right)\frac{M_{PS}^{(0)}}{m_\ell}\frac{Z_{PS}^{(0)}}{A_{PS}^{(0)}}$$

$$\langle 0|O_1^{bare}|PS\rangle = -\langle 0|O_2^{bare}|PS\rangle = A_{PS}^{(0)}$$
$$\langle 0|O_3^{bare}|PS\rangle = -\langle 0|O_4^{bare}|PS\rangle = Z_{PS}^{(0)}$$



mixings with O_1, O_2 don't depend on r
mixings with O_3, O_4 depend on r

terms due to axial current don't depend on r
terms due to vector current do depend on r

$$Z_1^{QCD} = Z_2^{QCD} = Z_V$$
$$Z_3^{QCD} = Z_4^{QCD} = Z_A$$

"factorization approximation"
between QED and QCD
vertex corrections

mixings with O_3 and O_4 can be exactly cancelled out by averaging over r = ±1

similar result can be obtained using
$Z_3^{QCD} = Z_4^{QCD} \sim 1.15 - 1.20\ Z_A$

15 - 20 % violation of the
"factorization approximation"

subleading effect ($\sim 10^{-3}$)
in pion decay and absent
in the decay ratio K/**π**

* the non-perturbative determination of Z_1^{QCD} and Z_2^{QCD} is in progress

subtraction of IR divergence and of universal FSEs

$$\frac{\delta A_{PS}}{A_{PS}^{(0)}} \to \frac{\delta A_{PS}(L)}{A_{PS}^{(0)}} - \frac{\delta A^{(pt)}(L)}{A_{PS}^{(0)}}$$

virtual photon emission from a point-like meson using the lattice volume as IR regulator

from Tantalo's talk: $\frac{\delta A^{(pt)}(L)}{A_{PS}^{(0)}} = b_{IR} \log(M_{PS}L) + b_0 + b_1 \frac{1}{M_{PS}L} + b_2 \frac{1}{(M_{PS}L)^2} + b_3 \frac{1}{(M_{PS}L)^3}$

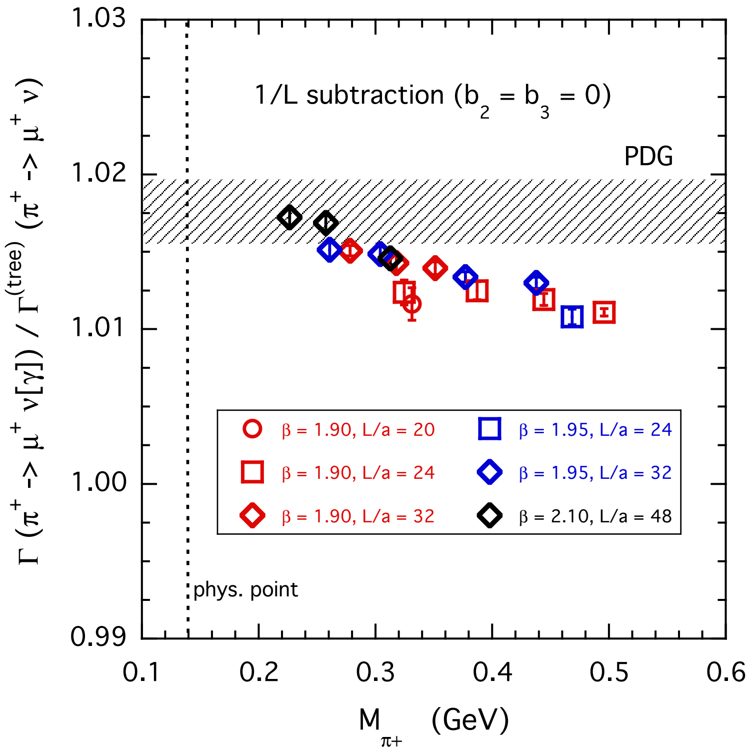
$b_i = b_i(r_\ell, \vec{p}_\ell)$

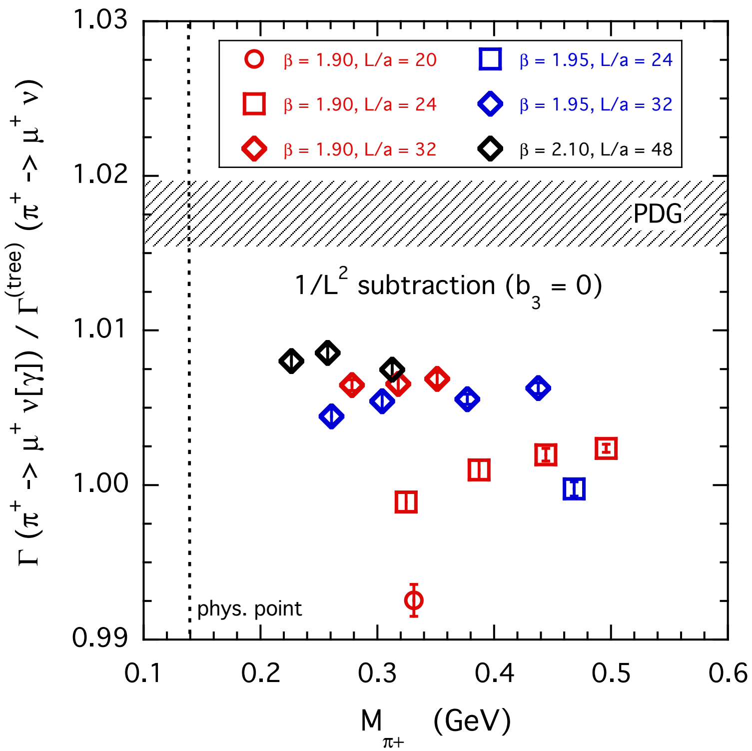
$r_\ell = m_\ell / M_{PS}$

* structure-dependent FSEs start at order $(1/L)^2$ → compare

* up to $1/L$ subtraction: $b_2 = b_3 = 0$
* up to $1/L^2$ subtraction: $b_3 = 0$

$$\pi^+ \to \mu^+ \nu[\gamma]$$





PDG '16

$R_\pi(\Delta E_\gamma^{max}) = 1.0176\,(21)$

used to get

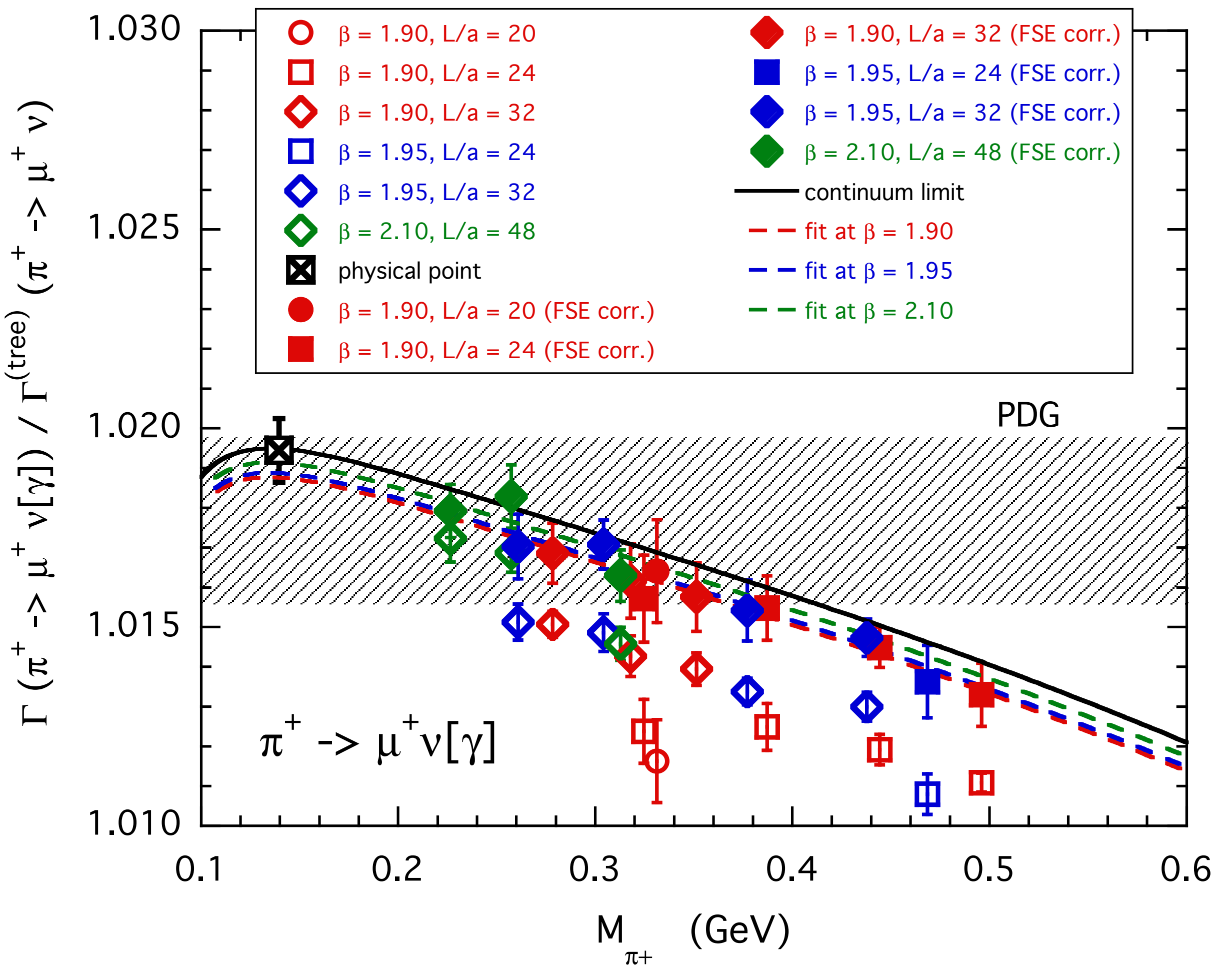
$|V_{ud}|\, f_{\pi^-} = 127.13\,(2)_{exp}(13)_{R_\pi}$ MeV

residual (structure-dependent) FSEs still visible

* **chiral extrapolation** [Knecht et al., EPJC 12 (2000) 469]

$$R_\pi\left(\Delta E_\gamma^{\max}\right) = 1 + \alpha_{em}\left\{4\pi E(\mu) + \frac{3}{4\pi}\log\left(\frac{\xi}{\mu^2}\right) + A_1\xi + Da^2 + \delta\Gamma^{pt}\left(\Delta E_\gamma^{\max}\right) + K_\pi^{FSE}(L)\right\} \qquad \xi \equiv \frac{M_\pi^2}{(4\pi f_0)^2}$$

residual (structure-dependent) FSEs: $K_\pi^{FSE}(L) = \dfrac{K_2}{(M_\pi L)^2} + \dfrac{K_2^\ell}{(E_\ell L)^2}$ $\qquad E, A_1, D, K_2, K_2^\ell$: 5 free parameters



$\pi^+ \to \mu^+\nu[\gamma]$

$\Delta E_\gamma^{\max} \cong 29.6$ MeV

open markers: lattice data with subtraction of universal FSEs up to 1/L

full markers: lattice data with subtraction of both universal and structure-dependent FSEs

$\chi_{d.o.f.} \simeq 0.7$

$$R_\pi^{phys}\left(\Delta E_\gamma^{\max}\right) = 1.0210\,(8)_{stat+fit}\,(11)_{chiral}\,(6)_{FSE}\,(2)_{a^2}\,(\ldots)_{qQED}$$
$$= 1.0210\,(8)_{stat+fit}\,(13)_{syst}\,(\ldots)_{qQED} = 1.0210\,(15)(\ldots)_{qQED}$$

$$\frac{R_\pi^{phys}\left(\Delta E_\gamma^{\max}\right)}{R_\pi^{PDG}\left(\Delta E_\gamma^{\max}\right)} = 1.0033\,(26)(\ldots)_{qQED}$$

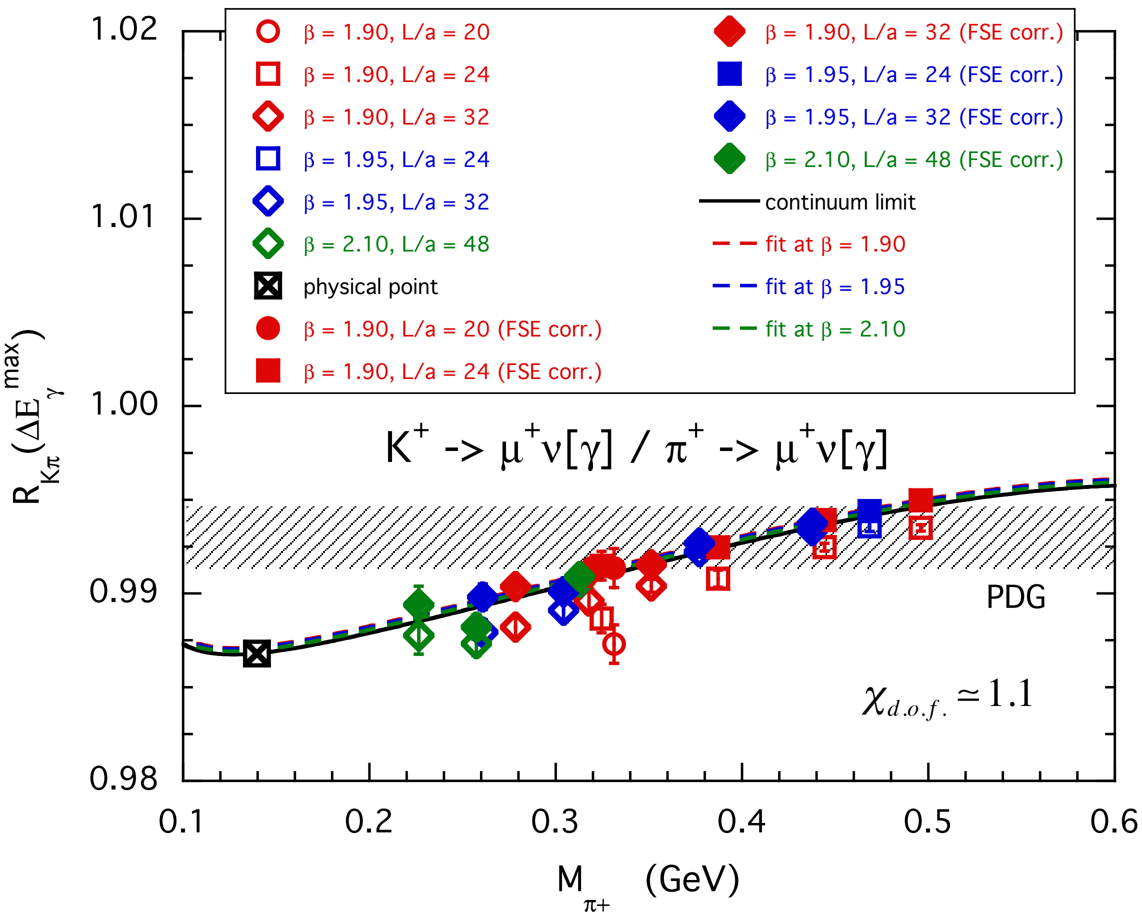
preliminary

* **K/π ratio:** $R_{K\pi}\left(\Delta E_\gamma\right) = 1 + R_K\left(\Delta E_\gamma\right) - R_\pi\left(\Delta E_\gamma\right)$

$$R_{K\pi}\left(\Delta E_\gamma^{\max}\right) = 1 + \alpha_{em}\left\{\tilde{A}_0 - \frac{3}{4\pi}\log\left(\frac{M_\pi^2}{M_K^2}\right) + \tilde{A}_1\xi + \tilde{A}_2\xi^2 + \tilde{D}a^2 + \delta\Gamma_K^{pt}\left(\Delta E_\gamma^{\max}\right) - \delta\Gamma_\pi^{pt}\left(\Delta E_\gamma^{\max}\right) + K_{K\pi}^{FSE}(L)\right\}$$

$$K_{K\pi}^{FSE}(L) = \frac{\tilde{K}_2}{(M_K L)^2} + \frac{\tilde{K}_2^\ell}{\left(E_\ell^{(K)}L\right)^2} - \frac{K_2}{(M_\pi L)^2} - \frac{K_2^\ell}{\left(E_\ell^{(\pi)}L\right)^2}$$

$\tilde{A}_0, \tilde{A}_1, \tilde{A}_2, \tilde{D}, \tilde{K}_2, \tilde{K}_2^\ell$: 6 free parameters



$$\frac{K^+ \to \mu^+\nu[\gamma]}{\pi^+ \to \mu^+\nu[\gamma]} \qquad \Delta E_\gamma^{\max} \cong 235.5 \text{ MeV}$$

PDG '16

$R_{K\pi}\left(\Delta E_\gamma^{\max}\right) = 0.9931\,(17)$

used to get

$|V_{us}|\, f_{K^-} / |V_{ud}|\, f_{\pi^-} = 0.27599\,(29)_{\exp}(24)_{R_{K\pi}}$

open markers: lattice data with subtraction of universal FSEs up to 1/L

full markers: lattice data with subtraction of both universal and structure-dependent FSEs

$$R_{K\pi}^{phys}\left(\Delta E_\gamma^{\max}\right) = 0.9863\,(11)_{stat+fit}\,(6)_{chiral}\,(1)_{FSE}\,(1)_{a^2}\,(\ldots)_{qQED}$$
$$= 0.9863\,(11)_{stat+fit}\,(6)_{syst}\,(\ldots)_{qQED} = 0.9863\,(13)(\ldots)_{qQED}$$

preliminary

$$\frac{R_{K\pi}^{phys}\left(\Delta E_\gamma^{\max}\right)}{R_{K\pi}^{PDG}\left(\Delta E_\gamma^{\max}\right)} = 0.9931\,(21)(\ldots)_{qQED}$$

CONCLUSIONS AND PERSPECTIVES

* the methodology proposed in **PRD91 (2015) 074506** to calculate ***QED corrections to hadronic processes***, although very challenging, **is within the reach of present lattice technologies**

* we have presented the **first lattice results** on the electromagnetic effects in the leptonic decay rates $\pi^+ \to \mu^+\nu[\gamma]$ and $K^+ \to \mu^+\nu[\gamma]$

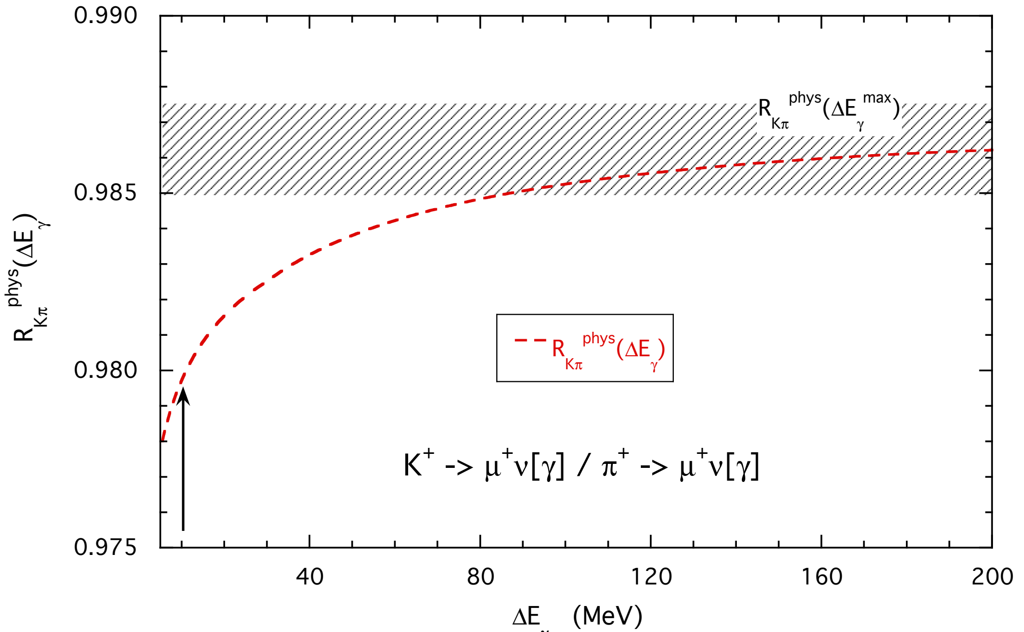
preliminary

$$R_\pi^{phys}\left(\Delta E_\gamma^{\max}\right) = 1.0210\,(15)(\ldots)_{qQED}$$

$$R_{K\pi}^{phys}\left(\Delta E_\gamma^{\max}\right) = 0.9863\,(13)(\ldots)_{qQED}$$

$$\frac{R_\pi^{phys}\left(\Delta E_\gamma^{\max}\right)}{R_\pi^{PDG}\left(\Delta E_\gamma^{\max}\right)} = 1.0033\,(26)(\ldots)_{qQED}$$

$$\frac{R_{K\pi}^{phys}\left(\Delta E_\gamma^{\max}\right)}{R_{K\pi}^{PDG}\left(\Delta E_\gamma^{\max}\right)} = 0.9931\,(21)(\ldots)_{qQED}$$



the point-like approximation for real photon emission is reliable at small values of $\Delta E_\gamma \sim 10$ - 20 MeV

~ OK for the pion case ($\Delta E_\gamma^{max} \sim 30$ MeV)
NOT for the kaon case ($\Delta E_\gamma^{max} \sim 235$ MeV)

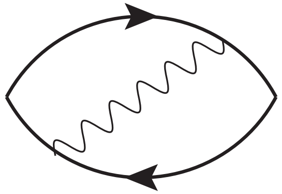
exp. cuts in the photon energy should be (re)considered

* improvements may be expected from a better theoretical understanding of structure-dependent FSEs;
* the inclusion of disconnected diagrams is mandatory for removing the quenched QED approximation;
* extensions to leptonic heavy-light meson decays and semileptonic $K_{\ell 3}$ decays are being targeted.

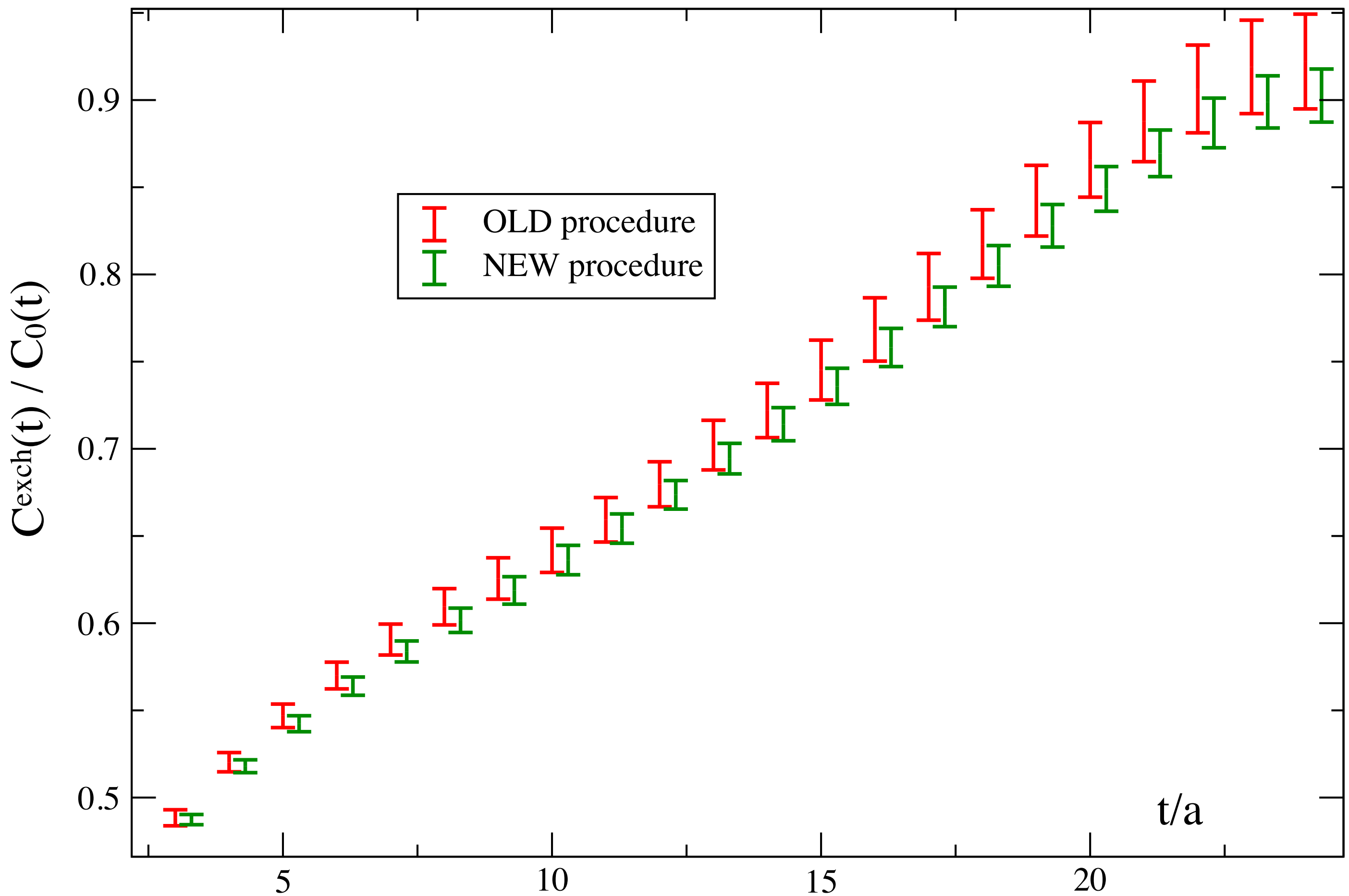
BACKUP SLIDES

stochastic evaluation of the photon propagator

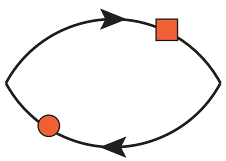
"exchange" diagram



JHEP 04 (2012) 124
PRD 87 (2013) 114505



OLD procedure

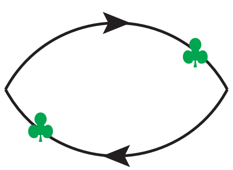


● $= \eta(x)$

■ $= \varphi(x) = \Delta_\gamma(x-y)\eta(y)$

this work

NEW procedure



♣ $= \rho(x) = \int dk e^{ikx} \sqrt{\Delta_\gamma(k)}\tilde{\eta}(k)$

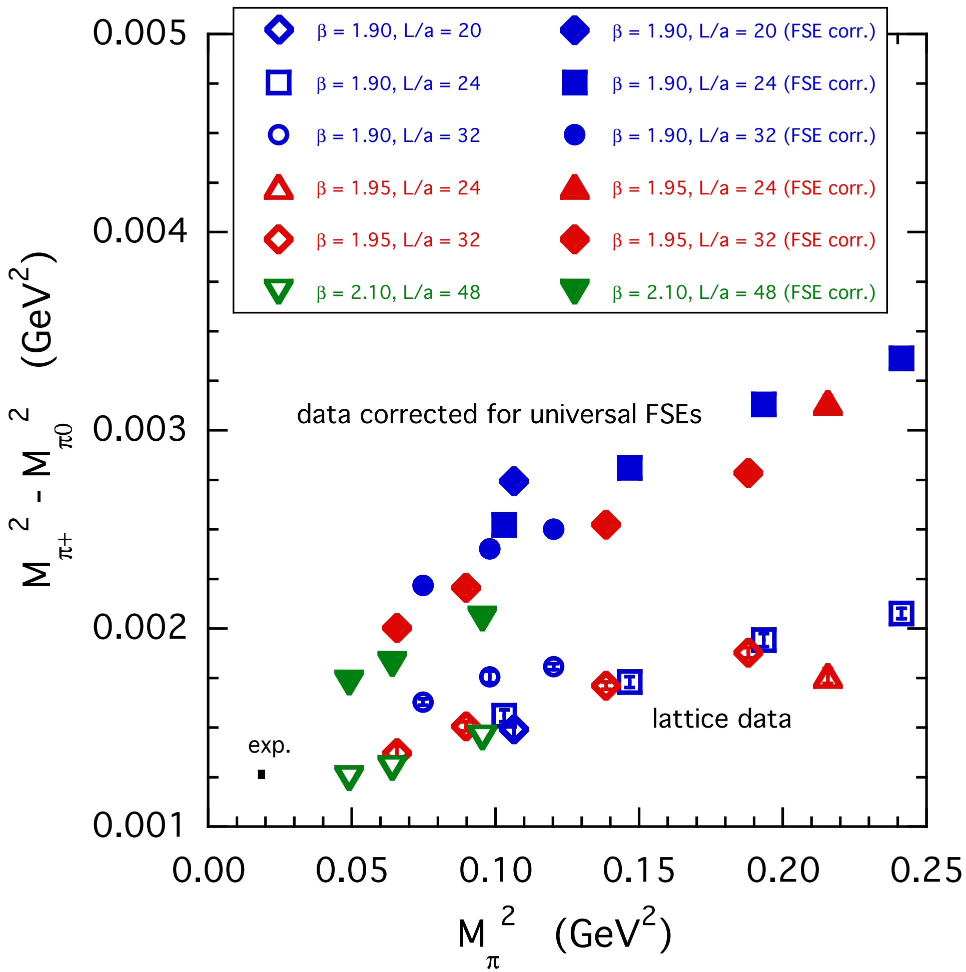
* computation of ϱ(x) (via FFT) is expensive as the one of φ(x)

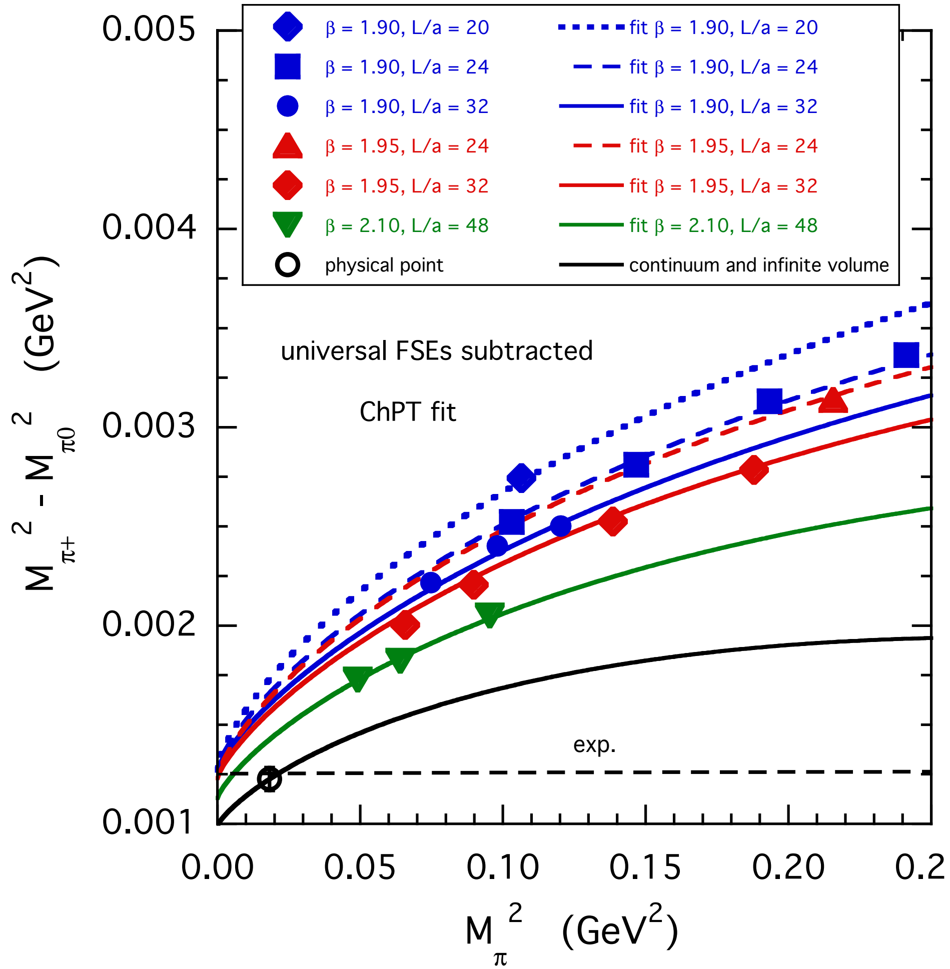
* the new procedure requires 1 inversion less

* noise is reduced, in particular for the exchange diagram

ensemble	β	V/a^4	$a\mu_\ell$	M_π (MeV)	M_K (MeV)	L (fm)	$M_\pi L$	$a\ \delta M_{\pi^+}/e^2$ $(P_5 A_0)$	$a\ \delta M_{\pi^+}/e^2$ $(P_5 P_5)$
$A30.32$	1.90	$32^3 \times 64$	0.0030	275	577	2.84	3.96	0.01835(12)	0.01835(09)
$A40.32$			0.0040	315	588		4.53	0.01815(18)	0.01816(14)
$A50.32$			0.0050	350	595		5.04	0.01781(11)	0.01778(09)
$A40.24$		$24^3 \times 48$	0.0040	324	594	2.13	3.50	0.01556(48)	0.01606(14)
$A60.24$			0.0060	388	610		4.19	0.01628(28)	0.01617(15)
$A80.24$			0.0080	438	624		4.73	0.01676(24)	0.01665(12)
$A100.24$			0.0100	497	650		5.37	0.01732(11)	0.01730(05)
$A40.20$		$20^3 \times 48$	0.0040	329	587	1.77	2.95	0.01474(42)	0.01504(29)
$B25.32$	1.95	$32^3 \times 64$	0.0025	259	553	2.61	3.43	0.01538(19)	0.01523(18)
$B35.32$			0.0035	300	562		3.97	0.01521(17)	0.01515(08)
$B55.32$			0.0055	377	587		4.99	0.01532(14)	0.01522(08)
$B75.32$			0.0075	437	608		5.78	0.01546(14)	0.01544(09)
$B85.24$		$24^3 \times 48$	0.0085	463	617	1.96	4.60	0.01448(18)	0.01452(13)
$D15.48$	2.10	$48^3 \times 96$	0.0015	224	538	2.97	3.37	0.01225(21)	0.01192(13)
$D20.48$			0.0020	255	541		3.84	0.01162(10)	0.01129(08)
$D30.48$			0.0030	310	554		4.67	0.01114(06)	0.01115(05)

Table 2: *Values of the simulated pion and kaon masses, of the lattice size L, of the product $M_\pi L$ and of the e.m. correction to the charged pion mass extracted from the correlators (15) and (18), respectively, for the gauge ensembles used in our project. The values of M_K correspond to a renormalized strange quark mass equal to the physical value $m_s = 99.6(4.3)$ MeV determined in Ref. [2].*

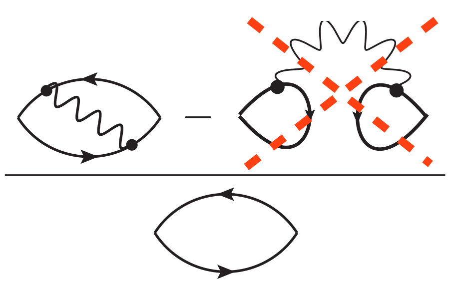




ChPT fit: Hayakawa&Uno [PTP '08]

$$\left(M_{\pi^+}^2 - M_{\pi^0}^2\right) - \alpha_{em}\frac{\kappa}{L^2}\left(2+M_\pi L\right) = \alpha_{em}4\pi f_0^2 C\left\{1-\left(4+\frac{3}{C}\right)\frac{M_\pi^2}{(4\pi f_0)^2}\left[\log\left(\frac{M_\pi^2}{\mu^2}\right)+B(\mu)\right]+Da^2+\frac{K}{L^3}\right\}$$

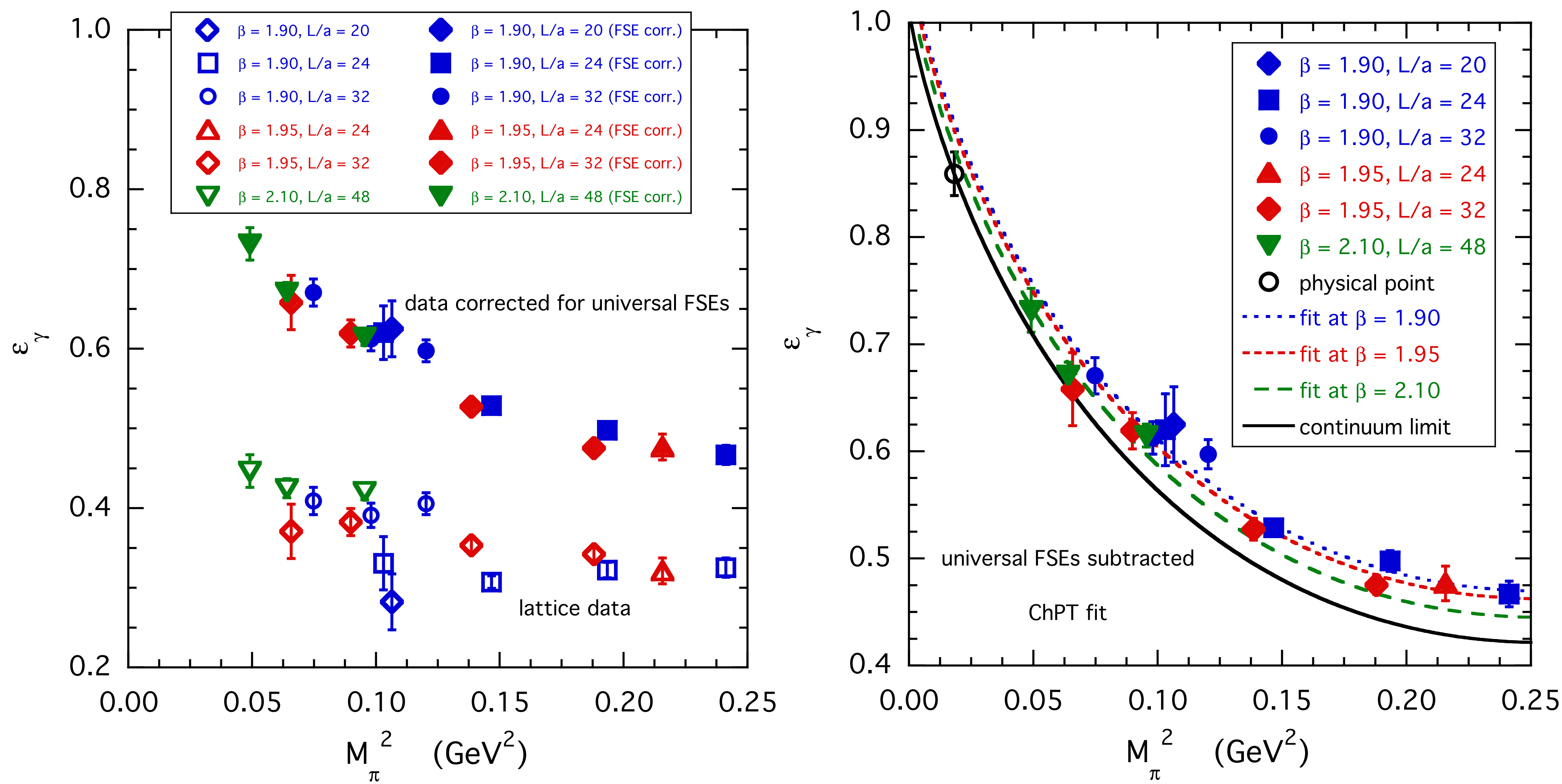
$$M_{\pi^+} - M_{\pi^0} = \frac{(e_u-e_d)^2}{2}e^2\partial_t$$



$\propto M_\pi^2$

$$\boxed{\left[M_{\pi^+}^2 - M_{\pi^0}^2\right]^{phys} = 1.226\,(58)_{stat}\,(96)_{syst}\,10^{-3}\ \mathrm{GeV}^2}$$

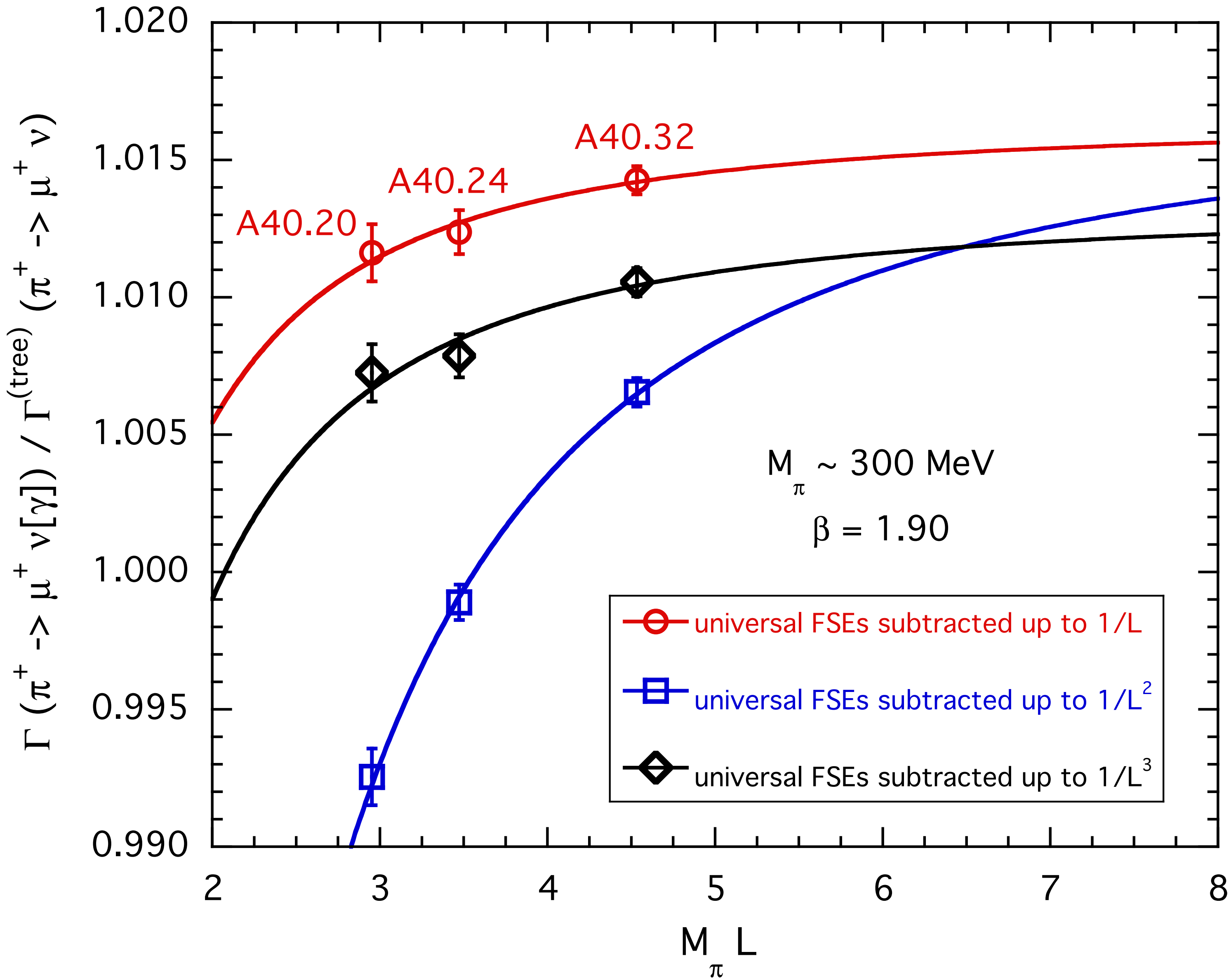
$$\left[M_{\pi^+}^2 - M_{\pi^0}^2\right]^{\exp} = 1.2612\,(1)\,10^{-3}\ \mathrm{GeV}^2$$



ChPT fit: Hayakawa&Uno [PTP '08]

$$\varepsilon_\gamma - \frac{\kappa}{L}\frac{M_K - M_\pi}{4\pi f_0^2 C} = \left(\frac{4}{3} + 2Q^{sea} + \frac{3}{C}\right)\left\{\tilde{A} + \frac{M_\pi^2}{(4\pi f_0)^2}\left[\log\left(\frac{M_\pi^2}{\mu^2}\right) + \tilde{B}(\mu)\right]\right\} + \tilde{D}a^2 + \frac{\tilde{K}}{L^3} \qquad \left[Q^{sea} = 0\right]$$

leptonic decay rate: subtraction of universal FSEs



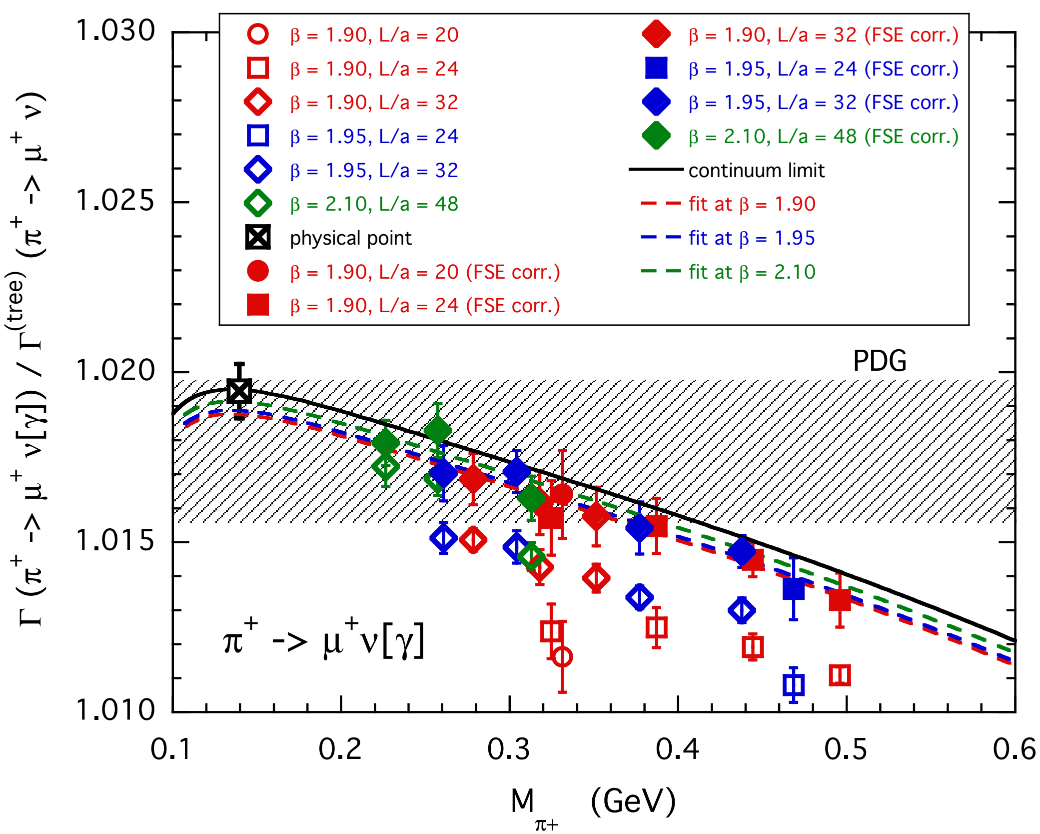
solid lines: $A + B/(M_\pi L)^2$ fit

$$\pi^+ \to \mu^+ \nu[\gamma]$$

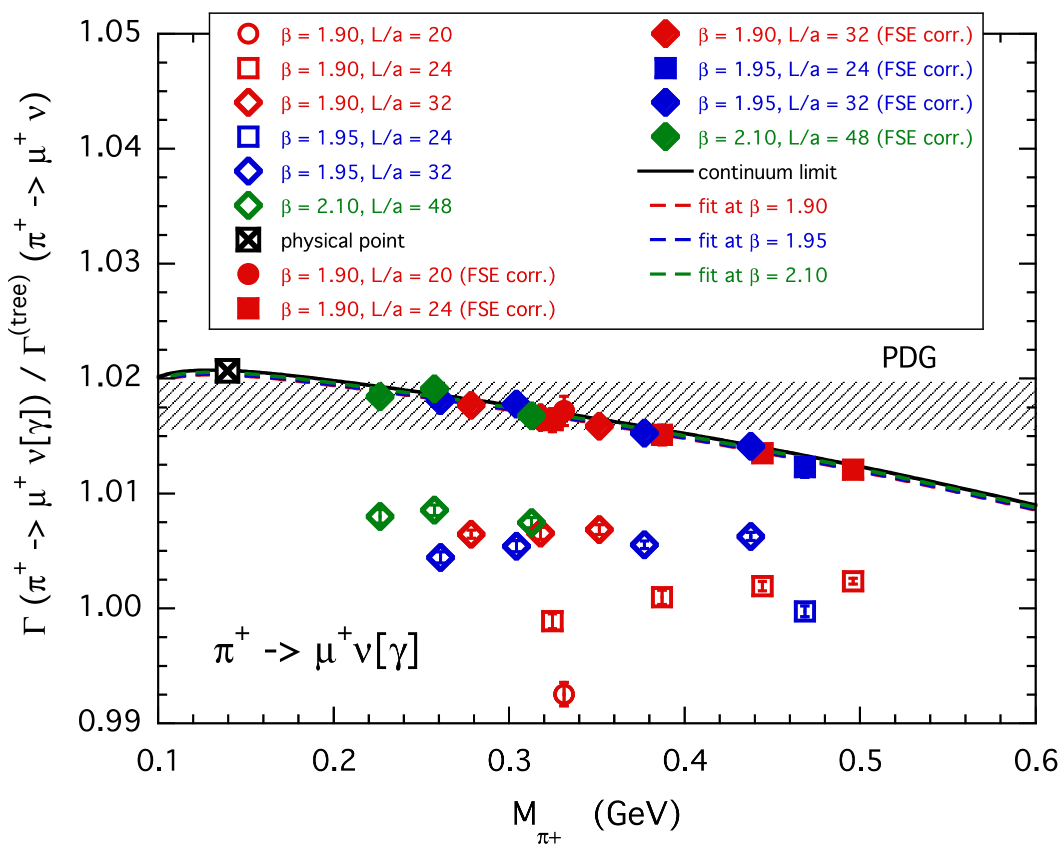
$$R_\pi\left(\Delta E_\gamma^{\max}\right) = 1 + \alpha_{em}\left\{4\pi E(\mu) + \frac{3}{4\pi}\log\left(\frac{\xi}{\mu^2}\right) + A_1\xi + Da^2 + \delta\Gamma^{pt}\left(\Delta E_\gamma^{\max}\right) + K_\pi^{FSE}(L)\right\}$$

$$K_\pi^{FSE}(L) = \frac{K_2}{(M_\pi L)^2} + \frac{K_2^\ell}{(E_\ell L)^2}$$

subtraction of universal FSEs up to $1/L$



subtraction of universal FSEs up to $1/L^2$



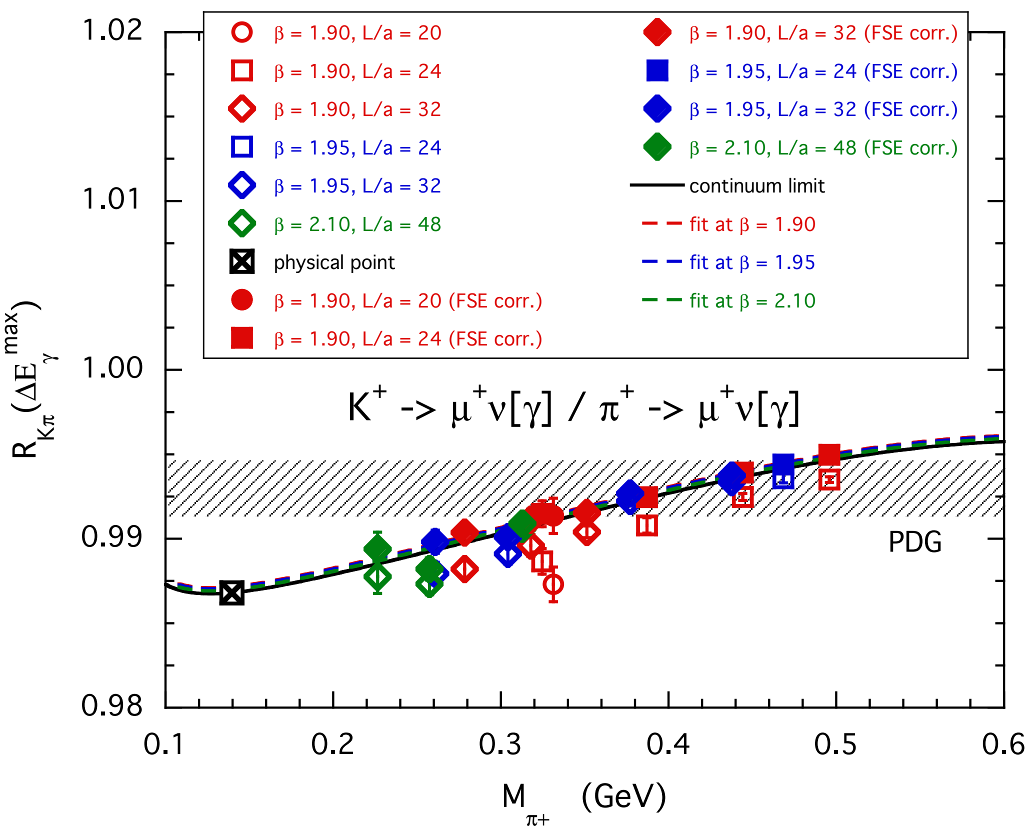
***** FSE subtraction under good control *****

$$K^+ \to \mu^+ \nu[\gamma] / \pi^+ \to \mu^+ \nu[\gamma]$$

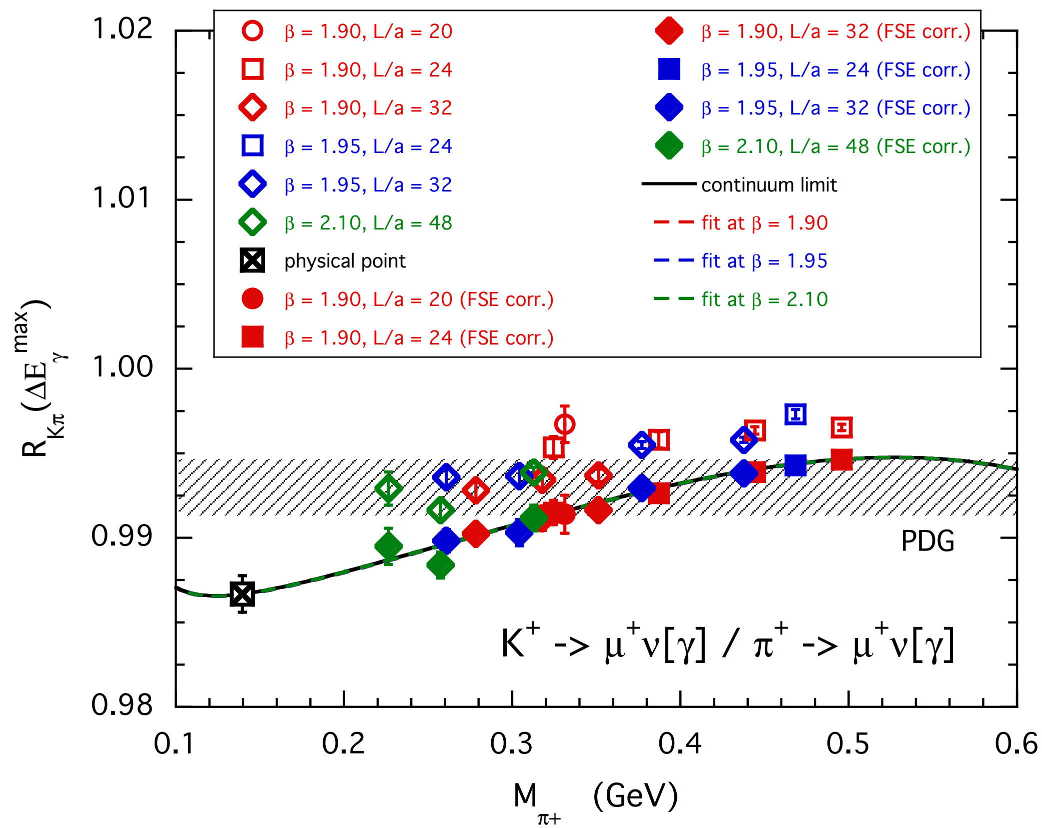
$$R_{K\pi}\left(\Delta E_\gamma^{\max}\right) = 1 + \alpha_{em}\left\{\tilde{A}_0 - \frac{3}{4\pi}\log\left(\frac{M_\pi^2}{M_K^2}\right) + \tilde{A}_1\xi + \tilde{A}_2\xi^2 + \tilde{D}a^2 + \delta\Gamma_K^{pt}\left(\Delta E_\gamma^{\max}\right) - \delta\Gamma_\pi^{pt}\left(\Delta E_\gamma^{\max}\right) + K_{K\pi}^{FSE}(L)\right\}$$

$$K_{K\pi}^{FSE}(L) = \frac{\tilde{K}_2}{(M_K L)^2} + \frac{\tilde{K}_2^\ell}{\left(E_\ell^{(K)} L\right)^2} - \frac{K_2}{(M_\pi L)^2} - \frac{K_2^\ell}{\left(E_\ell^{(\pi)} L\right)^2}$$

subtraction of universal FSEs up to $1/L$



subtraction of universal FSEs up to $1/L^2$



***** FSE subtraction under good control *****

pion and kaon/pion analyses

$$\pi^+ \to \mu^+ \nu [\gamma]$$

$$\frac{K^+ \to \mu^+ \nu [\gamma]}{\pi^+ \to \mu^+ \nu [\gamma]}$$

data set	chiral log	a^2-term	χ^2/d.o.f.	R_π^{phys}
$b_2 = b_3 = 0$	yes	yes	0.72	1.0195 (8)
	no	yes	0.75	1.0217 (8)
	yes	no	0.74	1.0192 (7)
	no	no	0.77	1.0213 (7)
$b_3 = 0$	yes	yes	1.00	1.0207 (8)
	no	yes	0.99	1.0229 (8)
	yes	no	0.95	1.0204 (7)
	no	no	0.94	1.0227 (7)

data set	chiral log	a^2-term	χ^2/d.o.f.	$R_{K\pi}^{phys}$
$b_2 = b_3 = 0$	yes	yes	1.07	0.9868 (5)
	no	yes	1.04	0.9856 (8)
	yes	no	0.96	0.9870 (5)
	no	no	0.93	0.9858 (10)
$b_3 = 0$	yes	yes	1.18	0.9867 (11)
	no	yes	1.14	0.9855 (13)
	yes	no	1.14	0.9871 (17)
	no	no	1.04	0.9857 (11)

$$R_\pi^{phys}\left(\Delta E_\gamma^{\max}\right) = 1.0210\,(8)_{stat+fit}\,(11)_{chiral}\,(6)_{FSE}\,(2)_{a^2}\,(\ldots)_{qQED}$$
$$= 1.0210\,(8)_{stat+fit}\,(13)_{syst}\,(\ldots)_{qQED} = 1.0210\,(15)(\ldots)_{qQED}$$

$$R_{K\pi}^{phys}\left(\Delta E_\gamma^{\max}\right) = 0.9863\,(11)_{stat+fit}\,(6)_{chiral}\,(1)_{FSE}\,(1)_{a^2}\,(\ldots)_{qQED}$$
$$= 0.9863\,(11)_{stat+fit}\,(6)_{syst}\,(\ldots)_{qQED} = 0.9863\,(13)(\ldots)_{qQED}$$

J. Rosner, S. Stone and R. Van der Water, arXiv:1509.02220 [minireview for PDG '16]

$$R_{\pi}\left(\Delta E_{\gamma}^{\max}\right) = 1.0176\,(21)$$
$$R_{K\pi}\left(\Delta E_{\gamma}^{\max}\right) = 0.9931\,(17)$$

... It includes the universal short-distance electroweak correction obtained by Sirlin [18], the universal long-distance correction for a point-like meson from Kinoshita [19], and corrections that depend on the hadronic structure [20]. We evaluate [it] using the latest experimentally-measured meson and lepton masses and coupling constants from the Particle Data Group [3], and taking the low-energy constants (LECs) that parameterize the hadronic contributions from Refs. [17], [21], [22]. The finite non-logarithmic parts of the LECs were estimated within the large-N_C approximation assuming that contributions from the lowest-lying resonances dominate ...

... The uncertainty is dominated by that from theoretical estimate of the hadronic structure-dependent radiative corrections, which include next-to-leading order contributions of $O(e^2 p^2_{\pi,K})$ in chiral perturbation theory [17] ...

17. V. Cirigliano and I. Rosell, JHEP 10, 005 (2007).
18. A. Sirlin, Nucl. Phys. B196, 83 (1982).
19. T. Kinoshita, Phys. Rev. Lett. 2, 477 (1959).
20. M. Knecht et al., Eur. Phys. J. C12, 469 (2000).
21. B. Ananthanarayan and B. Moussallam, JHEP 06, 047 (2004).
22. S. Descotes-Genon and B. Moussallam, Eur. Phys. J. C42, 403 (2005).

pion decay rate: dependence on the photon energy

$$R_\pi^{phys}\left(\Delta E_\gamma\right) - R_\pi^{phys}\left(\Delta E_\gamma^{\max}\right) = \alpha_{em}\left[\delta\Gamma^{pt}\left(\Delta E_\gamma\right) - \delta\Gamma^{pt}\left(\Delta E_\gamma^{\max}\right)\right]$$