Six-dimensional regularization of chiral gauge theories on a lattice I

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1. Introduction

History of lattice chiral symmetry

1981 Nielsen-Ninomiya's no go theorem

1992 Domain-wall fermion (Kaplan)

1997 Fixed point action (Hasenfratz)

1998 Overlap fermion (Neuberger)

1999-2001 Luescher's proof for existence of U(1) chiral gauge invariant regularization

Kikukawa-Nakayama: SU(2)xU(1) also O.K.

2015 Grabowska & Kaplan : manifestly gauge invariant construction of chiral gauge theory

What's new in Grabowska-Kaplan, PRL 116 (2016) no.21 211602 ?

[Grabowska Theory Wed, Kaplan, plenary Sat]

Before GK (Neuberger, Luescher, Kikukawa, Suzuki…):

1. Break gauge sym. explicitly,
2. Find a counter-term if anomaly free,
3. (Mainly) studied w/ 4D overlap fermions.

Grabowska & Kaplan PRL 116 (2016) no.21 211602 :

1. Keep gauge sym. explicitly,
2. If not anomaly free, no 4D local action,
3. 5D construction is essential.

The key is gradient flow (again).

They put

$$U_\mu(x,t) = \begin{cases} \text{flow time } t \text{ configuration} & (\mu = 1,2,3,4) \\ 1 & (\mu = 5) \end{cases}$$

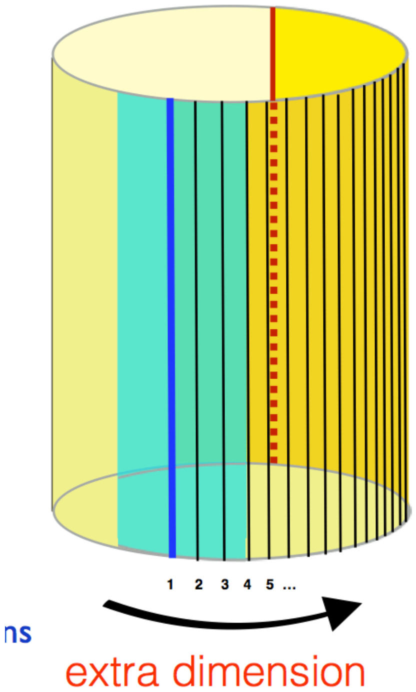


Figure by Kaplan

$$S_{DW} = \int d^4x dt \bar{\Psi}(D^{5\text{D}}_{\text{Wilson}} - \Lambda\epsilon(t))\Psi$$

$$\epsilon(t) = \begin{cases} +1 & (t \geq 0) \\ -1 & (t < 0) \end{cases}.$$

Gauge d.o.f. do not flow !

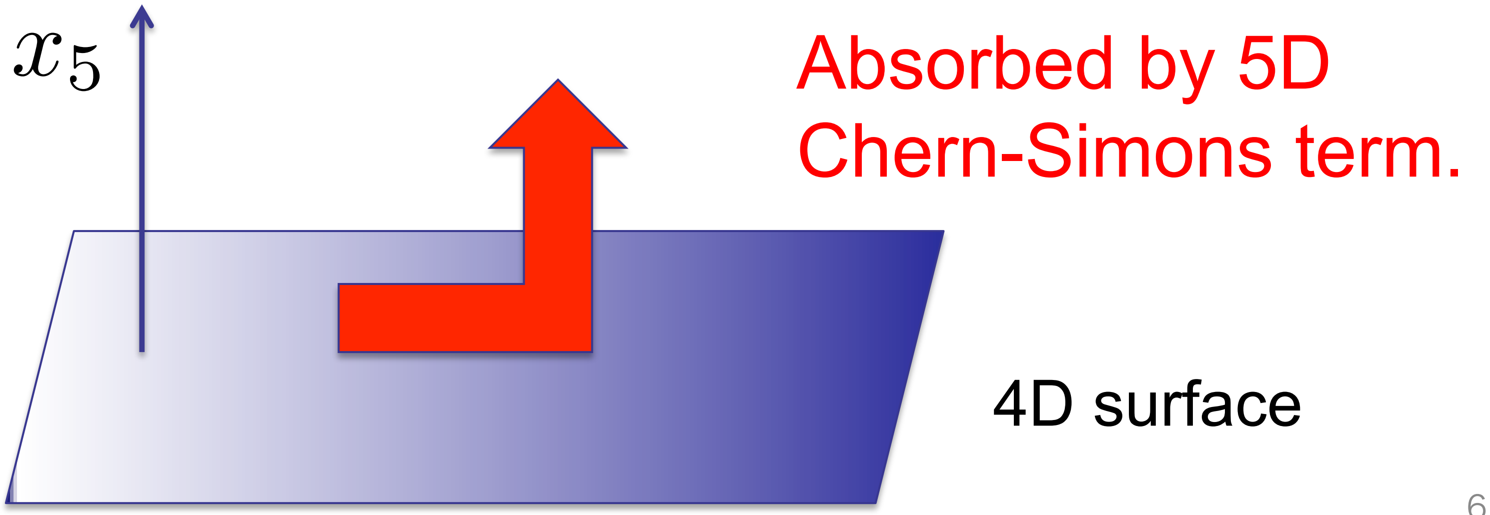
Links at different t have the same 4D gauge invariance.

Extra-dimension is essential.

They successfully reproduced a picture:

gauge anomaly = gauge current missing in extra dim, [Callan & Harvey 1985]

keeping total gauge invariance in 5D.



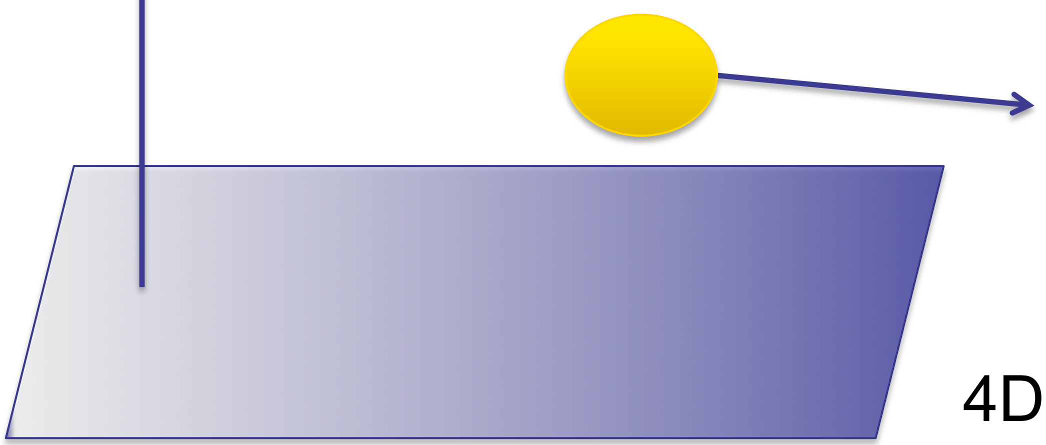
How about global anomaly ?

Global anomaly [Witten 1982]:

Gauge anomaly SU(2) = mod 2 index of 5D Dirac operator w/ 5-th direction

= gauge non-invariant path

$$A_\mu(x_\mu, x_5) = (1 - x_5)A_\mu(x) + x_5 A^g_\mu(x_\mu)$$



Extra-dim. is essential for global anomaly, too.

Witten's claim at Strings 2015:
We need "extension" of global anomaly.

Not only for "mapping torus":

$$A_\mu(x_\mu, x_5) = (1 - x_5)A_\mu(x) + x_5 A^g_\mu(x_\mu)$$

but also for ANY D+1 manifold with D-dim. boundary Weyl fermions, if the determinant has a phase $\exp(i\pi\eta) \neq 1,$
then the theory has a global anomaly.

Anomaly cannot be understood within 4-dim !

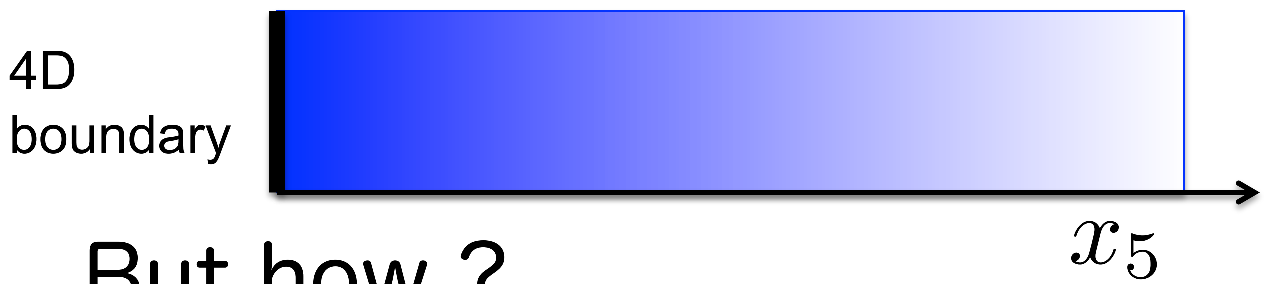
We need extra dimension(s) !

We want combine them.

1. Grabowska-Kaplan's 5D

Gauge inv. gradient flow → cannot detect global anomaly.

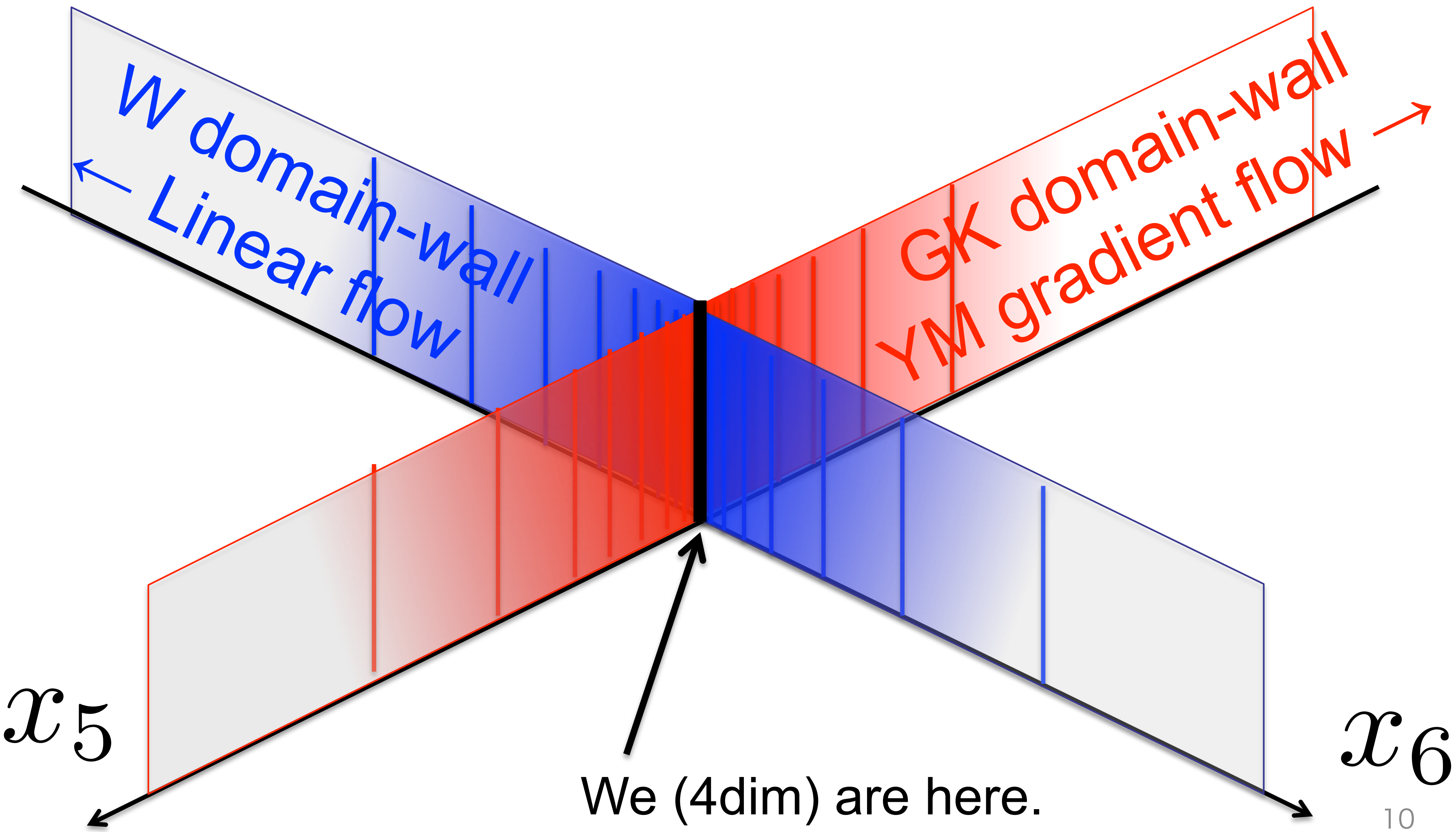
2. Witten's 5D



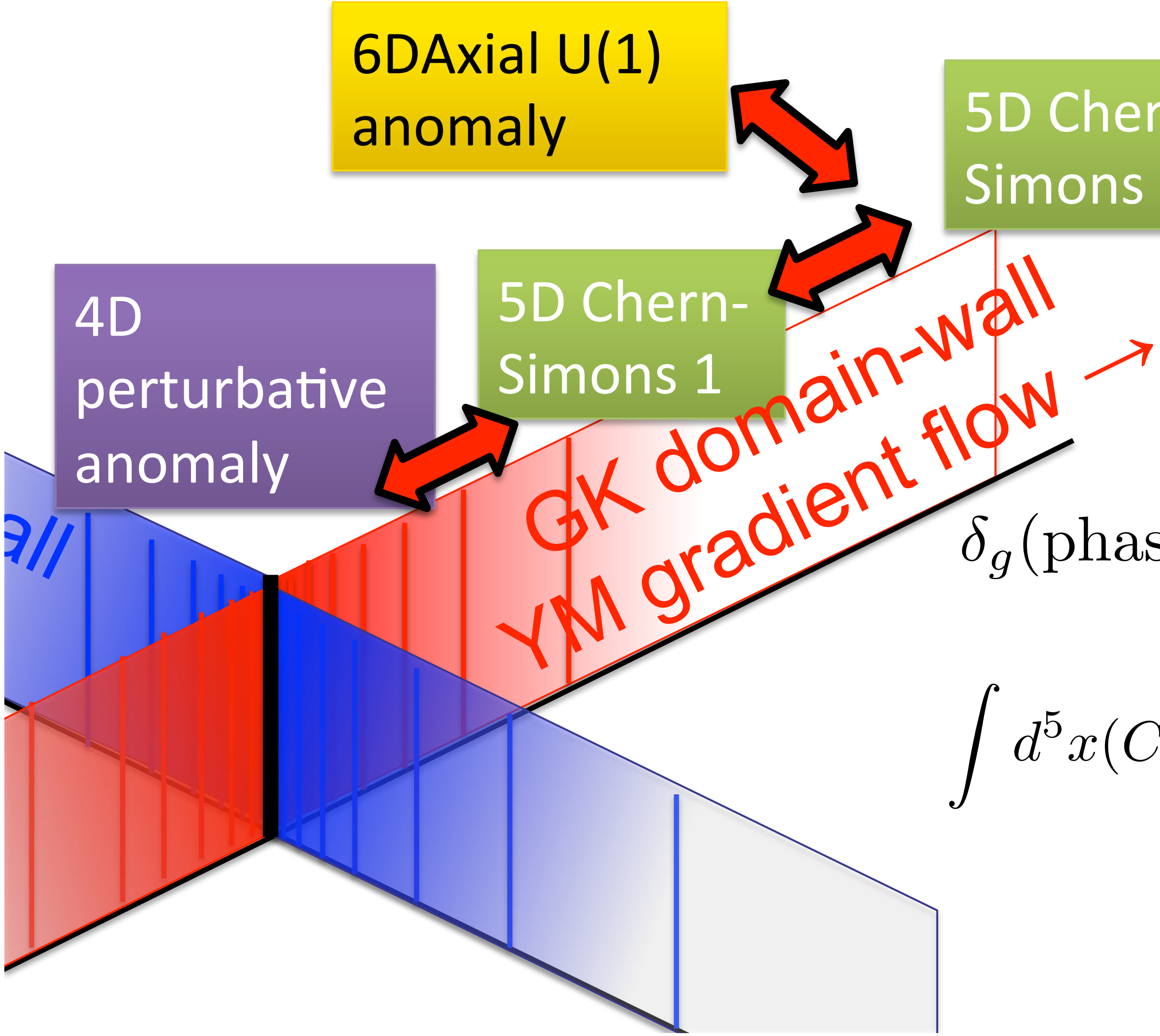
Gauge non-invariant flow → cannot keep gauge symmetry.

But how ?

Our proposal = 6D with 2 different domain-walls



GK domain-wall contains Stora-Zumino anomaly descent equations



$$\delta_g(\text{phase}) = \int_{x_5<0} d^5x \delta_g(CS)$$

$$\int d^5x (CS) = \int d^6x F \wedge F \wedge F$$

[Stora 1983, Zumino 1983, Alvarez-Gaume & Ginsparg 1984,Sumitani 1984]

W domain-wall mediates global anomaly inflow.

SU(2) example

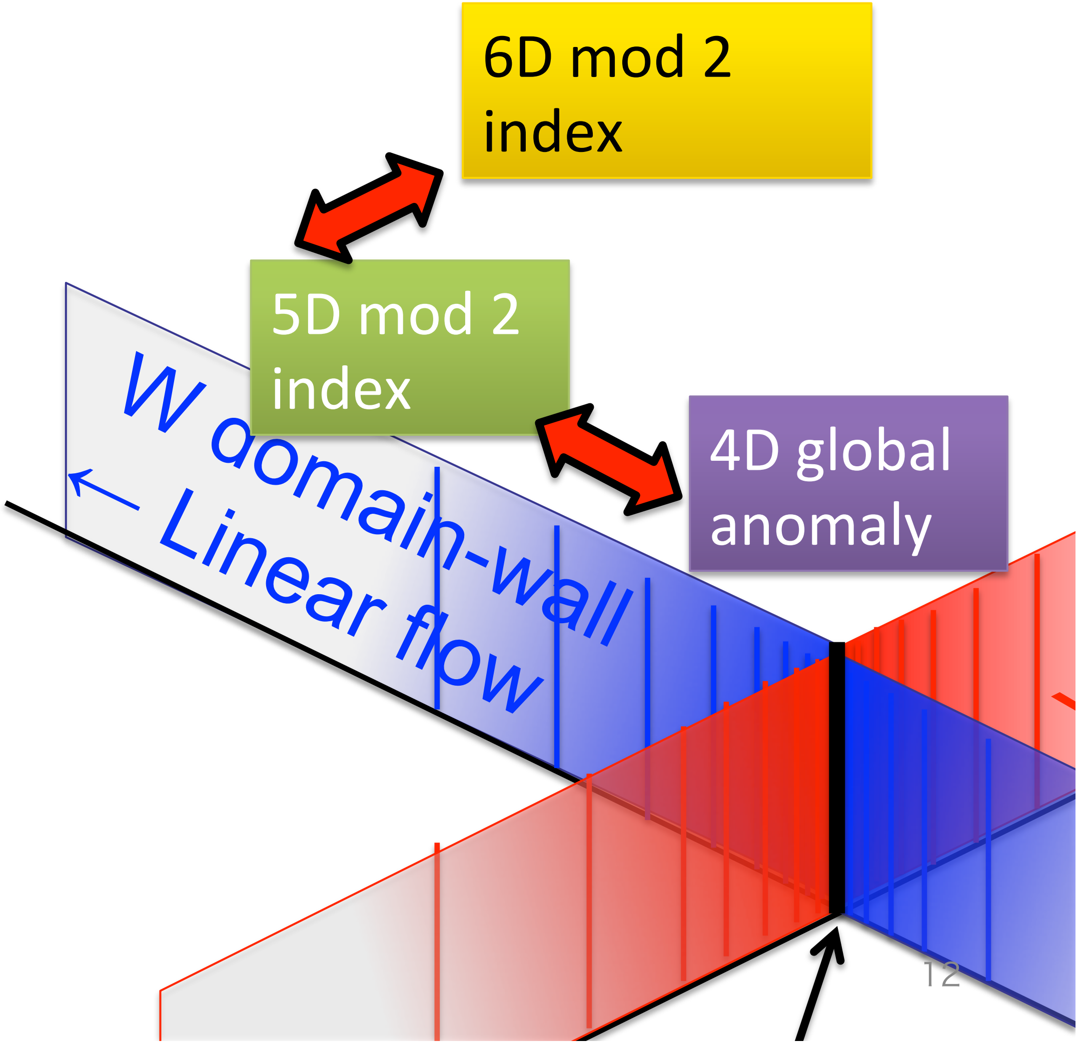
$$\text{global anomaly}$$

$$\leftrightarrow$$

$$\pi_4(SU(2)) = \mathbb{Z}_2$$

$$\leftrightarrow$$

$$\pi_5(SU(2)) = \mathbb{Z}_2$$



Our 6-dim formulation has

1. Stora-Zumino anomaly ladder:
   6D $U(1)_A$ index → gauge anomaly.
2. Global anomaly ladder (new finding):
   6D *exotic* index → global anomaly.
3. Anomaly free condition = sign-problem free condition in 6D:

→ If anomaly free, 6D determinant is real positive. → Monte Carlo is O.K. !

Contents

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2. “Parity” and axial U(1) anomalies in 6D

Two anomalous symmetries

1. Axial U(1) symmetry

$$\psi \to e^{i\alpha\gamma_7}\psi, \quad \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_7}$$

2. Parity' symmetry

(reflection in 5th direction)

$$P'\psi(x_{1,\cdots 4}, x_5, x_6) = i\gamma_5 R_5 \psi(x_{1,\cdots 4}, x_5, x_6)$$

$$= i\gamma_5 \psi(x_{1,\cdots 4}, -x_5, x_6)$$

$$P'^2 = -1$$

Mass-term is not invariant

Parity anomaly

Cf. Usual parity (only in even-dim.) :

$$P\psi(x_1, x_{2,\cdots 6}) = \gamma_1\psi(x_1, -x_{2,\cdots 6})$$

mass is allowed since $P^2 = 1$

P' (in any dim.) has anomaly:

massless fermion action is invariant,

but (zero-mode part of) fermion measure is NOT :

$$D\bar{\psi}_0 P' D P' \psi_0 = -D\bar{\psi}_0 D\psi_0.$$

Two mass terms

$M\bar{\psi}\psi$:U(1)$_A$ and P' asymmetric.

$\mu\bar{\psi}(i\gamma_6\gamma_7 R_5 R_6)\psi$:odd in P' but U(1)$_A$ invariant.

⇒ Dirac fermion w/ periodic boundary

$$\det\left(\frac{D^{\mathrm{6D}} - M - i\mu\gamma_6\gamma_7 R_5 R_6}{D^{\mathrm{6D}} + M + i\mu\gamma_6\gamma_7 R_5 R_6}\right) = (-1)^{\mathcal{P}+\mathcal{I}}$$

$\mathcal{P}$: $U(1)_A$ index ($\rightarrow$ perturbative anomaly)

$\mathcal{I}$: exotic index ($\rightarrow$ global anomaly)

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3. Two domain-walls

Two domain-walls

Let's consider a 6D Dirac fermion

GK DW ↓ W DW ↓ $\epsilon(x) = x/|x|$

$$\det\left(\frac{D^{6\mathrm{D}} + M\epsilon(x_6) + i\mu\epsilon(x_5)\gamma_6\gamma_7 R_5 R_6}{D^{6\mathrm{D}} + M + i\mu\gamma_6\gamma_7 R_5 R_6}\right)$$

where we assume $M > 0, \quad \mu > 0$

$$A_5 = A_6 = 0,$$

$A_{\mu=1,\cdots 4}(x)$ is symmetric under $x_5 \to -x_5,$ $x_6 \to -x_6$

(* later, gauge field is given by gradient & linear flows)

Fermion determinant is still real !

$$\det\left(\frac{D^{6\mathrm{D}}+M\epsilon(x_6)+i\mu\epsilon(x_5)\gamma_6\gamma_7R_5R_6}{D^{6\mathrm{D}}+M+i\mu\gamma_6\gamma_7R_5R_6}\right)\propto(-1)^{\mathcal{P}+\mathcal{I}}$$

determinant has $\gamma_5 R_5$ Hermiticity.

Indices become non-trivial

[Atiyah-Patodi-Singer 1975]

$\mathcal{P}$: APS index through GK domain-wall

$\rightarrow$ Perturbative anomaly in 4D

$\mathcal{I}$: APS index through W domain-wall

$\rightarrow$ global anomaly in 4D

Massless Weyl fermion appears !

Dirac equation

$$(D^{6\mathrm{D}} + M\epsilon(x_6) + i\mu\epsilon(x_5)\gamma_6\gamma_7 R_5 R_6)\psi(x) = 0$$

has a localized solution at $x_5 = x_6 = 0$

as $$\psi(x) = e^{-M|x_6|}e^{-\mu|x_5|}\phi(\bar{x}),$$

$$D^{4\mathrm{D}}\phi(\bar{x}) = 0, \qquad \bar{x} = (x_1, x_2, x_3, x_4)$$

$$\gamma_6\phi(\bar{x}) = \phi(\bar{x}),$$
$$i\gamma_5\gamma_6\gamma_7 R_5 R_6\phi(\bar{x}) = \phi(\bar{x})$$
$$\rightarrow \begin{pmatrix} \bar{\gamma}_5 & 0 \\ 0 & 0 \end{pmatrix}\phi(\bar{x}) = +\phi(\bar{x})$$

* Opposite chiral mode appears if $M < 0, \mu < 0$

4. Anomaly ladder through GK domain-wall

Bulk/edge decomposition

Simple example without W domain-wall

$$\mu = 0$$

$$\det\left(\frac{D^{6\mathrm{D}} + M\epsilon(x_6)}{D^{6\mathrm{D}} + M}\right) \quad [\propto (-1)^{\mathfrak{I}}]$$

Bulk

$$= \det\left(\frac{D^{6\mathrm{D}} + M\epsilon(x_6) + iM_2\gamma_6\gamma_7 R_6}{D^{6\mathrm{D}} + M}\right) \quad [\propto \exp(i\phi_{6D})]$$

edge

$$\times \det\left(\frac{D^{6\mathrm{D}} + M\epsilon(x_6)}{D^{6\mathrm{D}} + M\epsilon(x_6) + iM_2\gamma_6\gamma_7 R_6}\right) \quad [\propto \exp(i\phi_{5D})]$$

where we assume $M \gg M_2 \gg 0$

Imaginary part → $\pi\mathfrak{I} = \phi_{6D} + \phi_{5D}$

Atiyah-Patodi-Singer index

6D bulk → Axial U(1) anomaly

$$\phi_{6D} = \pi \int d^6x \frac{1-\epsilon(x_6)}{2} \frac{1}{6(4\pi)^3} \epsilon^{\mu_1\cdots\mu_6} \mathrm{tr}[F_{\mu_1\mu_2}F_{\mu_3\mu_4}F_{\mu_5\mu_6}]$$

Fujikawa's method ↑

$$= \pi \mathcal{P}^{6D}_{x_6<0} + \pi CS$$

$$CS \equiv -\int_{x_6=0} d^5x \frac{2}{3(4\pi)^3} \epsilon^{\mu_1\cdots\mu_5} \mathrm{tr}\left[\frac{1}{2}A_{\mu_1}F_{\mu_2\mu_3}F_{\mu_4\mu_5} - \frac{i}{2}A_{\mu_1}A_{\mu_2}A_{\mu_3}F_{\mu_4\mu_5} - \frac{1}{5}A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}A_{\mu_5}\right]$$

2nd determinant → 5D Dirac fermion

$$\lim_{M\to\infty} \det\left(\frac{D^{6\mathrm{D}} + M\epsilon(x_6)}{D^{6\mathrm{D}} + M\epsilon(x_6) + iM_2\gamma_6\gamma_7 R_6}\right) = \det\left(\frac{\bar{D}^{5\mathrm{D}}}{\bar{D}^{5\mathrm{D}} + M_2}\right) = \left|\det\left(\frac{\bar{D}^{5\mathrm{D}}}{\bar{D}^{5\mathrm{D}} + M_2}\right)\right| e^{-i\pi\eta/2}$$

$$\Rightarrow \quad \mathfrak{I} = \mathcal{P}^{6D}_{x_6<0} + CS - \frac{\eta^{5\mathrm{D}}}{2}$$

[Atiyah-Patodi-Singer 1975]

Integer = non-integer + non-integer

Full 6D/5D/4D decomposition

With W domain-wall and $M \gg \mu \gg 0$

$$\det\left(\frac{D^{6\mathrm{D}}+M\epsilon(x_6)+i\mu\epsilon(x_5)\gamma_6\gamma_7R_5R_6}{D^{6\mathrm{D}}+M+i\mu\gamma_6\gamma_7R_5R_6}\right)\quad\left[\propto(-1)^{\mathfrak{I}}\right]$$

$$= \det\left(\frac{D^{6\mathrm{D}}+M\epsilon(x_6)+i\mu\gamma_6\gamma_7R_6}{D^{6\mathrm{D}}+M+i\mu\gamma_6\gamma_7R_6}\right)$$

$$\times \mathrm{Det}\left(\frac{\delta(x-x')(\bar{D}^{5\mathrm{D}}+\mu\epsilon(x_5)\epsilon(x_5-L_5)R)+\mu_2^{x_5,x_5'}}{\delta(x-x')(\bar{D}^{5\mathrm{D}}+\mu)}\right)$$

$$\times \mathrm{Det}\left(\frac{\delta(x-x')(\bar{D}^{5\mathrm{D}}+\mu\epsilon(x_5)\epsilon(x_5-L_5)R_5)}{\delta(x-x')(\bar{D}^{5\mathrm{D}}+\mu\epsilon(x_5)\epsilon(x_5-L_5)R_5)+\mu_2^{x_5,x_5'}}\right),$$

No change in 6D bulk.
(U(1)A cannot feel W-DW.)

$\propto \exp(i\pi(\mathcal{P}^{6D}_{x_6<0}+CS))$

$CS^{(x_5<0)}$

Weyl fermion !

* 5D/4D decomposition is (almost) the same as Grabowska & Kaplan.

Stora-Zumino anomaly ladder

To summarize what we have computed,

$$\mathfrak{I} = \mathcal{P} + \mathcal{I} = \mathcal{P}^{6D}_{x_6<0} + CS - \frac{\eta^{5\mathrm{D}}}{2}$$

(integer)

6D $U(1)_A$ anomaly → 5D parity anomaly

$$\frac{1}{2}\eta^{5\mathrm{D}} = CS^{(x_5<0)} - \frac{\phi^{anom}}{\pi} + \text{gauge invariant phase}$$

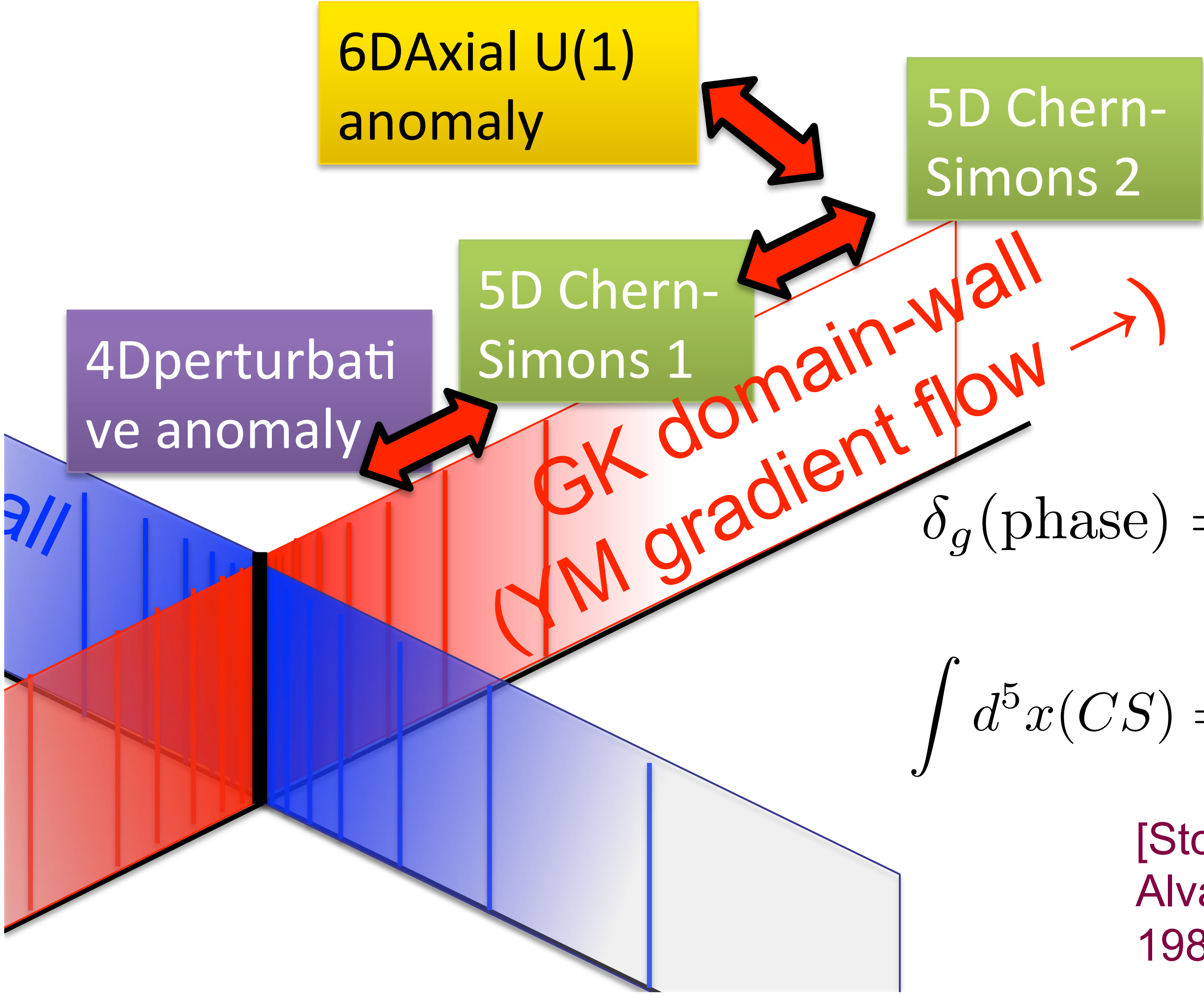
→ 4D gauge anomaly

↓

$\mathcal{I}$ is hidden.
(Next talk)

[Stora 1983, Zumino 1983, Alvarez-Gaume & Ginsparg 1984, Sumitani 1984]

Stora-Zumino anomaly ladder



$$\delta_g(\mathrm{phase}) = \int_{x_5<0} d^5x \delta_g(CS)$$

$$\int d^5x(CS) = \int d^6x F \wedge F \wedge F$$

[Stora 1983, Zumino 1983, Alvarez-Gaume & Ginsparg 1984,Sumitani 1984]

Summary of part I

Our 6D determinant w/ 2-different DWs

$$\det\left(\frac{D^{6\mathrm{D}}+M\epsilon(x_6)+i\mu\epsilon(x_5)\gamma_6\gamma_7R_5R_6}{D^{6\mathrm{D}}+M+i\mu\gamma_6\gamma_7R_5R_6}\right)\ \left[\propto(-1)^{\mathfrak{I}}\right]$$

1. is real,

$$\epsilon(x)=x/|x|$$

2. has a Weyl fermion at 4D junction,
3. gauge anomaly originates from 6D $U(1)_A$ index [Stora-Zumino anomaly ladder].

$$\pi\mathfrak{I}=\phi_{6D}+\phi_{5D}+\phi_{4D}$$

6D $U(1)_A$ anomaly→ 5D parity anomaly→ 4D gauge anomaly

Next talk

My talk

- ✔ 1. Introduction
- ✔ 2. "Parity" and axial U(1) anomalies in 6D
- ✔ 3. Two domain-walls
- ✔ 4. Anomaly ladder through GK domain-wall

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5. Anomaly ladder through W domain-wall
6. Anomaly free condition
7. Lattice regularization
8. Summary and discussion

Next talk by Ryo Yamamura

From Witten's slide "anomaly revisited" at Strings 2015

Claiming anomaly free ⇔ sign problem free.

Although I do not claim a complete proof, I believe that there is a general answer for when a theory with fermions is completely consistent and anomaly-free, meaning that the path integral on a general manifold can be defined in a way that is anomaly-free and consistent with all principles of unitarity, locality and cutting and pasting. The condition is just that

$$e^{i\pi\eta} = 1$$

for all $D+1$-manifolds Y, not just for mapping tori. Anomaly cancellation gives the same condition just for mapping tori.

Back-up slides

Possible applications

- 4D→2D : Doubly gapped (M and μ) topological insulator can exist ?
- Higgs : → *definition* of standard model ?
- Higher dim theory : Our world is really 6D ?

What are really essential ?

- 6D : Yes. Stora-Zumino's solution for consistent anomaly is *unique*.
- Two domain-walls : Yes. At least, need to distinguish $U(1)_A$ and P'
- Gradient flow : we don't know. no imaginary part even without it.
- Non-locality (R_5,R_6): probably no. but analysis is easier with them.

Phase of 5D determinant

$$\det\left(\frac{\bar{D}^{5\mathrm{D}}+\mu\epsilon(x_5)}{\bar{D}^{5\mathrm{D}}+\mu}\right)\propto\exp(-i\pi\eta^{5\mathrm{D}})$$

$$\pi\eta^{5\mathrm{D}}=\pi CS+\phi_{\text{gauge non invariant}}+\phi_{\text{gauge invariant}}$$

Perturbative anomaly

global anomaly (old definition)

global anomaly new def. by Witten 2015:
no local 4D action to express the phase.

Anomaly-free → $\eta^{5\mathrm{D}}$ must be zero !

Why 6D ?

$\eta^{\rm 5D}$ can be determined only relatively

(direct computation is ill-defined due to UV div.).

$$\eta^{\rm 5D} = \int_0^1 du \frac{d\eta^{\rm 5D}(u)}{du}$$ [Alvarez-Gaume et al. 1986]

u is our 6th coordinate ! → We need

5th direction to separate L/R chiral modes,

6th direction to determine $\eta^{\rm 5D}$

CP restoration

Complex phase of 5D determinant =
CP violating lattice artifact (w/o CKM)

Our 6D construction may be automatically giving a counter-term to keep the CP symmetry at finite lattice spacing.

[Fujikawa-Ishibashi-Suzuki 2002, Hasenfratz 2005]

Global anomaly classification

- SU(2) global anomaly : O.K.
- Other groups on 4-dim torus: Maybe. Index $\mathbb{Z}$, $\mathbb{Z}_2$ can be detected by P'.
- But higher dim: we don't know. For example, $\pi_6(SU(2)) = \mathbb{Z}_{12}$ may require quite non-trivial treatment.

Parity anomaly on a lattice

- P' has an anomaly.
- On the lattice, we may need Ginsparg-Wilson-type relation for P' symmetry.
- The $U(1)_A$ invariant mass term in the kernel of overlap Dirac operator ?

Anomaly free condition

1. Axial U(1) cancelation in 6D:

$$\sum_L \mathrm{tr} T_L^a \{T_L^b, T_L^c\} - \sum_R \mathrm{tr} T_R^a \{T_R^b, T_R^c\} = 0$$

cancels perturbative anomaly.

2. "Parity" anomaly cancelation :

fundamental rep. = even

cancels global anomaly.

⇒ Our determinant is real positive !

5D -> 4D

Together with anti-domain-wall, it becomes

$$\det\left(\frac{\bar{D}^{5\mathrm{D}}+\mu\epsilon(x_5)\epsilon(x_5-L_5)R_5}{\bar{D}^{5\mathrm{D}}+\mu}\right)$$

$$\mu_2^{x_5,x_5'}\equiv\mu_2\left[\delta(x_5)\delta(x_5'-L_5)+\delta(x_5-L_5)\delta(x_5')\right]$$

$$=\mathrm{Det}\left(\frac{\delta(x-x')(\bar{D}^{5\mathrm{D}}+\mu\epsilon(x_5)\epsilon(x_5-L_5)R)+\mu_2^{x_5,x_5'}}{\delta(x-x')(\bar{D}^{5\mathrm{D}}+\mu)}\right)\times\mathrm{Det}\left(\frac{\delta(x-x')(\bar{D}^{5\mathrm{D}}+\mu\epsilon(x_5)\epsilon(x_5-L_5)R_5}{\delta(x-x')(\bar{D}^{5\mathrm{D}}+\mu\epsilon(x_5)\epsilon(x_5-L_5)R_5)+\mu}\right.$$

↓ Another CS on 5D

$$-\pi\int_{x_6=0}d^5x\frac{4}{3(4\pi)^3}\frac{1-\epsilon(x_5)\epsilon(x_5-L_5)}{2}\epsilon^{\mu_1\cdots\mu_5}\mathrm{tr}\left[\frac{1}{2}A_{\mu_1}F_{\mu_2\mu_3}F_{\mu_4\mu_5}-\frac{i}{2}A_{\mu_1}A_{\mu_2}A_{\mu_3}F_{\mu_4\mu_5}-\frac{1}{5}A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}A_{\mu_5}\right]$$

↓ Weyl fermion !

$$\det\frac{\mathcal{D}}{\mathcal{D}+\mu_2}\times\underset{\text{bulk}}{\det}(R_5)$$

$$\mathcal{D}=P_-^5\bar{D}^{4\mathrm{D}}P_+^5+P_+^5\bar{\partial}^{4\mathrm{D}}P_-^5$$

$$\frac{1}{2}\eta^{5\mathrm{D}}=CS^{(x_5<0)}+\frac{1}{2}\eta^{4\mathrm{D}}-\frac{\phi^{anom}}{\pi}-\frac{\phi'}{\pi},$$

Massless Weyl fermion appears !

Dirac equation

$$(D^{6\mathrm{D}} + M\epsilon(x_6) + i\mu\epsilon(x_5)\gamma_6\gamma_7 R_5 R_6)\psi(x) = 0$$

has a localized solution at $x_5 = x_6 = 0$

as $$\psi(x) = e^{-M|x_6|}e^{-\mu|x_5|}\phi(\bar{x}),$$

$$D^{4\mathrm{D}}\phi(\bar{x}) = 0, \qquad \bar{x} = (x_1, x_2, x_3, x_4)$$

$$\left.\begin{aligned} \gamma_6\phi(\bar{x}) &= \phi(\bar{x}), \\ i\gamma_5\gamma_6\gamma_7 R_5 R_6\phi(\bar{x}) &= \phi(\bar{x}) \end{aligned}\right\} \quad \begin{pmatrix} \bar{\gamma}_5 & 0 \\ 0 & 0 \end{pmatrix}\phi(\bar{x}) = +\phi(\bar{x})$$

* Opposite chiral mode appears if $M < 0, \mu < 0$

Summary of part 1 and 2 : Our 6D formulation has

1. Stora-Zumino anomaly ladder:
   6D $U(1)_A$ index → gauge anomaly
2. Global anomaly ladder :
   6D *exotic* index → global anomaly.
3. Gradient flow in x_5 + linear interpolation in x_6 → mirror fermions are decoupled.
4. Anomaly free condition = sign-problem free condition in 6D:

Monte Carlo is O.K. !