Double Parton Distributions of the Pion

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One needs double hard interactions for calculating the background of new physics discovery at LHC.

Naive description:

\[ d\sigma_{DPS} = \frac{d\sigma_{SPS} d\sigma_{SPS}}{\sigma_{\text{eff}}} \]

⇒ Calculate DPS contributions, test to what extent the naive ansatz above works.
Assume factorization into soft and hard part [arXiv:1111.0910]:

\[
\frac{d\sigma}{dx_1 d\bar{x}_1 dx_2 d\bar{x}_2} = \sum_{\text{polarization}} \sum_{\text{flavour}} \sigma_1 X_1 \bar{X}_1 \sigma_2 X_2 \bar{X}_2 \frac{C}{\mathcal{C}} \int d^2 y \ F_{X_1 X_2}(x_i, y) F_{\bar{X}_1 \bar{X}_2}(\bar{x}_i, y)
\]

+ \{\text{interference}\} + \{\text{higher twist}\}

with collinear Double Parton Distributions (DPDs):

\[
F_{X_1 X_2}(x_1, x_2, y) = \left[ \prod_{j=1}^{2} \int \frac{dz^-_i}{2\pi} e^{i x_j z^-_j p^+} \right] 2p^+ \int dy^- \langle h(p) | \mathcal{O}_{X_1}(0, z_2) \mathcal{O}_{X_2}(y, z_1) | h(p) \rangle |_{z_i=0}
\]

\[
\mathcal{O}_{X}(y, z) = \bar{q}(y - \frac{z}{2}) \Gamma_X q(y + \frac{z}{2}) \Big|_{z+y=0}
\]
DPDs: Mellin Moments and Decompositions

First Mellin Moment:

\[ M_{X_1 X_2}(y) = \int_0^1 dx_1 \int_0^1 dx_2 \left[ F_{X_1 X_2}(x_1, x_2, y) - a_{X_1} F_{\bar{X}_1 X_2}(x_1, x_2, y) \right. \]

\[ - a_{X_2} F_{X_1 \bar{X}_2}(x_1, x_2, y) + a_{X_1} a_{X_2} F_{\bar{X}_1 \bar{X}_2}(x_1, x_2, y) \]

\[ = 2(p^+)^{-1} \int dy \left\langle h(p) \right| \mathcal{O}_{X_1}(0) \mathcal{O}_{X_2}(y) | h(p) \rangle \bigg|_{y^+ = 0} \]

\[ \Rightarrow M^p_{X_1 X_2} \text{ can be obtained from the lattice.} \]

Can decompose DPD matrix elements into invariant functions e.g.:

\[ M^p_{SS/PP} = 2m_h^2 A_{SS/PP}(py, y^2) \]

\[ \mathcal{T} M^p,\{\mu\nu\} = \left[ 2p^\mu p^\nu - \frac{1}{2} g^{\mu\nu} p^2 \right] A_{VV/AA}(py, y^2) \]

\[ + m_h^2 \left[ 2p^{\{\mu} y^{\nu\}} - \frac{1}{2} g^{\mu\nu} py \right] B_{VV/AA}(py, y^2) \]

\[ + m_h^4 \left[ 2y^{\mu} y^{\nu} - \frac{1}{2} g^{\mu\nu} y^2 \right] C_{VV/AA}(py, y^2) \]

\[ \text{+components} \Rightarrow \mathcal{T} M^p,++_{VV/AA} = 2(p^+)^2 A_{VV/AA}(py, y^2) \]

Relation between Mellin moments and invariant functions, e.g.:

\[ M_{q\Delta q\Delta q}(y) = \int d(py) A_{VV/AA}(py, y^2) \]
Consider DPD Mellin moments, start with first moment

\[ O_X(y) = \bar{q}(y) \Gamma_X q(y), \text{ no Wilson lines, no derivatives (first moment)} \]

Go to Euclidean space \((y = (y, y^4) \text{ with } y = (y^1, y^2, y^3)), y^4 = iy^0:\)

\[ \Rightarrow \text{ operators must be at the same time, } y^4 = 0. \]

For a pilot study start with pion at zero momentum.

\[ \Rightarrow \text{ obtain } M_{X_1X_2} = \langle \pi^+(0) | O_{X_1}(0) O_{X_2}(y) | \pi^+(0) \rangle \text{ from the lattice for several channels:} \]

\[ \Gamma_X \in \{ 1, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu} \} \]

Ground state matrix element:

\[ \langle \pi^+ | O_{X_1}(0) O_{X_2}(y) | \pi^+ \rangle = \frac{C^{X_1X_2 4pt}(t, \tau, y)}{2m_{\pi} C^{2pt}(t)} \bigg|_{t \gg \tau \gg 0} \]

with 4pt function \( C^{X_1X_2 4pt}(t, \tau, y) = \langle O^p_{\pi^+}(t) O_{X_1}(0) O_{X_2}(y) O^\dagger_{\pi^+}(0) \rangle \)

and 2pt function \( C^{2pt}(t) = \langle O^p_{\pi^+}(t) O^\dagger_{\pi^+}(0) \rangle \)

Pion interpolators:

\[ |\pi^+_p(t)\rangle + \cdots = O^p_{\pi^+}(t)|\Omega\rangle = \frac{1}{V} \sum_x e^{ip \cdot x} \bar{u}(x) \gamma_5 d(x) |\Omega\rangle = O^-_{\pi}(t)|\Omega\rangle \]
Obtain six independent Wick contractions:

\[ C1 \]
\[
\pi_p^+(0) \quad O_1(\tau, 0) \quad \pi_p^+(t) \\
\begin{array}{ccc}
  u & u & u \\
  d & d & d \\
\end{array} \\
O_2(\tau, y)
\]

\[ C2 \]
\[
\pi_p^+(0) \quad O_1(\tau, 0) \quad \pi_p^+(t) \\
\begin{array}{ccc}
  u & u & u \\
  d & d & d \\
\end{array} \\
O_2(\tau, y)
\]

\[ A \]
\[
\pi_p^+(0) \quad O_1(\tau, 0) \quad \pi_p^+(t) \\
\begin{array}{ccc}
  u & u & u \\
  d & d & d \\
\end{array} \\
O_2(\tau, y)
\]

\[ D \]
\[
\pi_p^+(0) \quad O_1(\tau, 0) \quad \pi_p^+(t) \\
\begin{array}{ccc}
  u & u & u \\
  d & d & d \\
\end{array} \\
O_2(\tau, y)
\]

Choose \( t = 15a \), expect plateau at \( 6a \lesssim \tau \lesssim 9a \) 
\[ \Rightarrow \text{fit or average} \]
Lattice Setup and Simulation Details

Used gauge ensemble \((N_f = 2)\) Wilson-Clover fermions, c.f. [arXiv:1412.7336]:

<table>
<thead>
<tr>
<th>Ensemble</th>
<th>(\beta)</th>
<th>(a[\text{fm}])</th>
<th>(\kappa)</th>
<th>(V)</th>
<th>(m_\pi[\text{GeV}])</th>
<th>(N(N_{4\text{pt}}))</th>
<th>(N_{\text{sm}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>5.29</td>
<td>0.071</td>
<td>0.13632</td>
<td>(40^3 \times 64)</td>
<td>0.2888(11)</td>
<td>2025(984)</td>
<td>400</td>
</tr>
</tbody>
</table>

Renormalization (conversion to \(\overline{\text{MS}} (2\text{GeV})\)) [arXiv:1003.5756]:

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>P</th>
<th>V</th>
<th>A</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Z)</td>
<td>0.4577(18)</td>
<td>0.3538(92)</td>
<td>0.7365(48)</td>
<td>0.76487(64)</td>
<td>0.9141(26)</td>
</tr>
<tr>
<td>(Z_{\text{conv}})</td>
<td>1.3543</td>
<td>1.3543</td>
<td>1</td>
<td>1</td>
<td>0.93313</td>
</tr>
</tbody>
</table>

Further Details:

- use stochastic \(\mathbb{Z}_2 \otimes \mathbb{Z}_2\) sources (see e.g. [arXiv:0804.1501]) \(\Rightarrow\) One-End-Trick / Two-Hand-Trick
Results for $M_{SS} (L = 40)$

Results for $M_{PP} (L = 40)$

Results for $M_{V0V0} (L = 40)$

Results for $M_{A0A0} (L = 40)$
Invariant functions $A_{SS}(py = 0, y^2)$ and $A_{VV}(py = 0, y^2)$

**PRELIMINARY**

Results for $A_{SS}(L = 40)$

Results for $A_{VV}(L = 40)$
Insert complete set of states; assume that the pion states dominate:

$$\langle \pi^+(p) | O_1 O_2 | \pi^+(p) \rangle = \sum_Y \langle \pi^+(p) | O_1 | Y \rangle \langle Y | O_2 | \pi^+(p) \rangle$$

$$\approx \int \frac{d^4p'}{(2\pi)^4} \langle \pi^+(p) | O_1 | \pi^+(p') \rangle \langle \pi^+(p') | O_2 | \pi^+(p) \rangle \delta(p'^2 - m^2_{\pi})$$

**Test 1**: Factorize $\langle \pi^+(p) | O_{1uu}^{uu}(z_1,0) O_{2dd}^{dd}(z_2,y) | \pi^+(p) \rangle$

$\Rightarrow$ can relate invariant function $A_{X_1 X_2}$ to pion form factors $F_X(t)$:

$$(t(\zeta, r^2) = -\frac{\zeta^2 m^2_{\pi} + r^2}{1 - \zeta}) :$

$$A_{VV/SS}^{VV/SS}(p_y = 0, y^2) \approx \frac{\eta_C^{V/S}}{\pi} \int_0^1 d\zeta \left(1 - \frac{\zeta}{2}\right) \int \frac{d^2r}{(2\pi)^2} e^{-iy \cdot r} F_{V/S}^2(t(\zeta, r^2))$$

**Test 2**: Factorize local Matrix element ($t(r) = 2m^2 - 2mE_r$) :

$$\langle \pi^+(p) | O_{1uu}^{uu} V_0^0(0) O_{2dd}^{dd} V_0^0(y) | \pi^+(p) \rangle \approx -\frac{1}{4\pi^2 |y|} \int_0^\infty d(r^2) \frac{\sin(|y||r|)(m + E_r)^2}{2E_r} F_V^2(t(r^2))$$

$$\langle \pi^+(p) | O_{Suu}^{uu} S_0^0(0) O_{Sdd}^{dd}(y) | \pi^+(p) \rangle \approx \frac{1}{4\pi^2 |y|} \int_0^\infty d(r^2) \frac{\sin(|y||r|)}{2E_r} F_S^2(t(r^2))$$

Both tests trivially fail for the pseudoscalar and axialvector case, since $F_P = F_A = 0$ for the pion in contrast to our lattice results for 4pt-functions.
The Pion Form Factor

The Pion form factor:

\[ F_S(Q^2) = \langle \pi(p + Q) | O_S | \pi(p) \rangle \quad F_V(Q^2)(2p + Q) = \langle \pi(p + Q) | O_V^\mu | \pi(p) \rangle \]

\[ \Rightarrow \text{obtain FF from 3pt-functions, at first neglect disconnected contributions} \]

\[ \Rightarrow \text{use momenta } |p| \leq \frac{2\pi}{N_S} \sqrt{3} \]

Fit data on the parametrization:

\[ F(t) = \frac{F_0}{\left(1 + \frac{t}{M^2}\right)^p}, \quad t = Q^2 \]

For the vector FF can fix \( F_0 = 1 \) (charge conservation)

Fit Result (correlated fit):

<table>
<thead>
<tr>
<th>#</th>
<th>quantity</th>
<th>( F_0 )</th>
<th>( M[\text{GeV}] )</th>
<th>( p )</th>
<th>( \chi^2/\text{DOF} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( F_{em} )</td>
<td>1(fixed)</td>
<td>0.777(12)</td>
<td>1(fixed)</td>
<td>6.010</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1(fixed)</td>
<td>0.872(16)</td>
<td>1.173(69)</td>
<td>4.400</td>
</tr>
<tr>
<td>3</td>
<td>( F_{scal} )</td>
<td>2.222(19)GeV</td>
<td>1.314(39)</td>
<td>1(fixed)</td>
<td>7.886</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>2.212(19)GeV</td>
<td>2.023(50)</td>
<td>2(fixed)</td>
<td>9.877</td>
</tr>
</tbody>
</table>
Test 1 for $A_{VV}$:

- Preliminary

Comparison for $A_{VV}^{ud}$: 3pt+3pt to 4pt ($L = 40$)

- Good agreement for large distances of the two scattering quarks
- Clear discrepancy for scattering distances smaller than $m_{\pi}^{-1}$

$\Rightarrow$ Naive factorization might be a good assumption if the scattering quarks are far apart from each other
Test 2 for $M_{V_4 V_4}$:

PRELIMINARY

No region where the two curves match

$\Rightarrow$ Bad assumption, only order of magnitude is roughly the same for large distances
Factorization Test Results

Test 1 for $A_{SS}$ (PRELIMINARY):

- Test 1: Agreement worse than for $A_{VV}$ (disconnected contributions?)

Test 2 for $M_{SS}$ (PRELIMINARY):

- Test 2: Bad agreement as for $M_{V4V4}$
Achievements:

- Calculated all 4pt-graphs for the $\pi^+$ at $p = 0$
- Good quality of the data (except for $S_2$ and $D$)
- Tested naive factorization into convolution of two form factors:
  - Test 1: ok if distances are large
    - Clear deviation for distances smaller than the pion wave length
  - Test 2: worse (order of magnitude correct)
  ⇒ Calculation of two-current matrix elements essential!

To do:

- Get better signals for $S_2$ and $D$ -graph
- Take derivatives into account
- Go to non-zero momenta
- Investigate other particles: $\rho$-meson, nucleon (LHC experiments)
Thank you for your attention!
$S_2$ results:

**PRELIMINARY**

Results for $M_{PP} (L = 40)$: $S_2$-graph

Results for $M_{A_4A_4} (L = 40)$: $S_2$-graph
Depending on the quark flavor of the insertion operators, a certain set of 4pt-graphs must be summed up.

Isovector operators:

\[ \mathcal{O}_X^1 = \bar{u}\Gamma_X d + \bar{d}\Gamma_X u \]
\[ \mathcal{O}_X^2 = i\bar{u}\Gamma_X d - i\bar{d}\Gamma_X u \]
\[ \mathcal{O}_X^3 = \bar{u}\Gamma_X u - \bar{d}\Gamma_X d \]

Matrix Elements in terms of 4pt-graphs:

\[ \langle \pi^+ | \mathcal{O}_X^1 \mathcal{O}_X^1 | \pi^+ \rangle = \langle \pi^+ | \mathcal{O}_X^2 \mathcal{O}_X^2 | \pi^+ \rangle \sim 4C_2^{X_1X_2} - 2A^{X_1X_2} - 2S_2^{X_1X_2} \]
\[ \langle \pi^+ | \mathcal{O}_X^3 \mathcal{O}_X^3 | \pi^+ \rangle \sim 4C_2^{X_1X_2} - 2C_1^{X_1X_2} - 2S_2^{X_1X_2} \]

Neglect contributions of \( S_2 \).
Isospin Matrix Elements: Results

Preliminary

Results for $M^{ii} = \langle \pi^{+}|\mathcal{O}^{i}(0)\mathcal{O}^{i}(y)|\pi^{+}\rangle$ ($L = 40$, $i = 1, 2$)

Double Parton Distributions
Isospin Matrix Elements: Results

PRELIMINARY

Results for $M^{33} = \langle \pi^+ | O^3(0) O^3(y) | \pi^+ \rangle$ ($L = 40$)

Results for $M^{33} = \langle \pi^+ | O^3(0) O^3(y) | \pi^+ \rangle$ ($L = 40$)