QCD with Flavored Minimally Doubled Fermions

J. H. Weber[1]

[1]Physik Department, Technische Universität München, Garching,

34th International Symposium on Lattice Field Theory,
University of Southampton, UK, 07/25/2016

preliminary work!

What are Minimally Doubled Fermions?

Minimally Doubled Fermions (MDF) are characterized by ...

- a **standard chiral symmetry** for finite $a \Rightarrow$ **no $\mathcal{O}(am)$ corrections!**
- an **ultra-local** (only next-neighbor interactions) Dirac operator $\Rightarrow$ **fast!**
- a **pair of chiral fermions** (cf. N.-N. theorem [H. Nielsen, M. Ninomiya 1981])
  $\Rightarrow$ **even number of flavors**, spectral doubling, **taste breaking!**
- **broken hyper-cubic symmetry** $\Rightarrow$ **anisotropic mixing & counterterms!**
- **violation of symmetry under charge conjugation and some reflection!**

Outline

- Overview & introduction
- Minimally Doubled Fermions (MDF) with flavor
- QCD vacuum with (flavored) Minimally Doubled Fermions
- Flavored mesons with (flavored) Minimally Doubled Fermions
- Summary

Flavored Minimally Doubled Fermions

Flavored Minimally Doubled Fermion action

- **Degenerate** MDF Dirac operator is derived from KW Fermions

$$\widehat{D} = \widehat{D}^N + m_0 + r\widehat{D}^W + 3r\widehat{D}^3$$

- Operators due to **Karsten-Wilczek term** (Wilczek parameter r, $r^2 > \frac{1}{4}$)

$$\widehat{D}^W = -\frac{i}{a_\tau}\gamma_0\left(3 + a_\sigma^2 \boldsymbol{D}\cdot\boldsymbol{D}^*\right), \quad \widehat{D}^3 = \frac{i}{a_\tau}\gamma_0$$

- **Karsten-Wilczek term** in momentum space

$$\widehat{D}^W = -\frac{i}{a_\tau}\gamma_0\sum_{j=1}^{3}\cos(a_\sigma k_j)$$

- **Minimal number** of doublers

$$\widehat{D}^W = \frac{1}{a_\tau}\begin{cases} -3 & k_j = 0 \ \forall\, j \\ +3 & k_j = \frac{\pi}{a_\sigma} \ \forall\, j \\ \pm 1 & \text{other Fermi points} \end{cases}$$

Flavored Minimally Doubled Fermion action

- **Degenerate** MDF Dirac operator is derived from KW Fermions

$$\widehat{D} = \widehat{D}^N + m_0 + r\widehat{D}^W + (3r + c_3)\widehat{D}^3 + d_4\widehat{D}^4$$

- Operators due to **Karsten-Wilczek term** (Wilczek parameter r, $r^2 > \frac{1}{4}$)

$$\widehat{D}^W = -\frac{i}{a_\tau}\gamma_0\left(3 + a_\sigma^2 \boldsymbol{D}\cdot\boldsymbol{D}^*\right), \quad \widehat{D}^3 = \frac{i}{a_\tau}\gamma_0, \quad \widehat{D}^4 = D_0\gamma_0$$

- The MDF action is combined with Wilson gauge action

$$S^g[U] = \sum_m \beta_\sigma \sum_{j<k} \frac{1}{N_c}\mathrm{Re}\,\mathrm{tr}\left(\mathbf{1} - U_m^{jk}\right) + \beta_\tau(1 + d_p)\sum_j \frac{1}{N_c}\mathrm{Re}\,\mathrm{tr}\left(\mathbf{1} - U_m^{j0}\right)$$

- **Coefficients** are known [S. Capitani, M. Creutz, H. Wittig, JHW, 2010; JHW, 2015]

d_4^{1L}/g_0^2	d_p^{1L}/g_0^2	c_3^{1L}/g_0^2	c_3^{np}
-0.00106	-0.0893	-0.249	$\frac{c_3^{1L}+n_2 g_0^4}{1+d_1 g_0^2}\ \begin{cases} n_2 = 0.216(1) \\ d_1 = -0.926(2)\end{cases}$

d_4 and d_p are **even functions** of r, c_3 is an **odd function** of r

Flavored Minimally Doubled Fermion action

- **Flavored** MDF Dirac operator is derived from KW Fermions

$$\widehat{D} = \widehat{D}^N + m_0 + r\widehat{D}^W + (3r + c_3)\widehat{D}^3 + d_4\widehat{D}^4 + m_3\widehat{M}$$

- Operators due to **Karsten-Wilczek term** (Wilczek parameter r, $r^2 > \frac{1}{4}$)

$$\widehat{D}^W = -\frac{i}{a_\tau}\gamma_0\left(3 + a_\sigma^2 \boldsymbol{D}\cdot\boldsymbol{D}^*\right), \quad \widehat{D}^3 = \frac{i}{a_\tau}\gamma_0, \quad \widehat{D}^4 = D_0\gamma_0$$

- **Flavor symmetry** is broken by a second **mass operator**, i.e.

$$\widehat{M} = \left(1 + a_\tau^2 \boldsymbol{D}_0 D_0^*\right)$$

- Alternative form: $\widehat{\mathcal{M}} = (1 + a_\tau^2 \boldsymbol{D}_0 D_0^*)\,(3 + a_\sigma^2 \boldsymbol{D}\cdot\boldsymbol{D}^*)$

- MDF action invariant under parity transform $\widehat{\mathcal{P}}$ and cubic group (W_3)
- Invariance: **joint** charge conjugation ($\widehat{\mathcal{C}}$)/time reflection ($\widehat{\mathcal{T}}$)
- **Standard chiral symmetry** ($\widehat{\chi} = \gamma_5$) in the chiral limit ($m_3 = m_0 = 0$)

su(2) algebra

- Wanted: **su(2) algebra**, where **one generator leaves $m_3\widehat{M}$ invariant**
- Construct **su(2) algebra** from three different **shift transforms**

$$\widehat{\lambda} = \gamma_0(-1)^{n_\sigma}, \quad n_\sigma = \sum_{j=1}^{3} n_j,$$

$$\widehat{\tau} = i\gamma_0\gamma_5(-1)^{n_0},$$

$$\widehat{\vartheta} = \gamma_5(-1)^{\bar{n}}, \quad \bar{n} = \sum_{\mu=0}^{3} n_\mu$$

- Representation of **su(2) algebra**

$$\begin{pmatrix} \widehat{\lambda} \\ \widehat{\tau} \\ \widehat{\vartheta} \end{pmatrix} = \begin{pmatrix} \sigma^1 \times (-1)^{n_\sigma} \\ \sigma^2 \times (-1)^{n_0} \\ \sigma^3 \times (-1)^{\bar{n}} \end{pmatrix} \otimes \mathbf{1}_{2\times 2}$$

Commutator relations $[\widehat{\lambda}, \widehat{\tau}] = 2i\widehat{\vartheta}, \quad [\widehat{\tau}, \widehat{\vartheta}] = 2i\widehat{\lambda}, \quad [\widehat{\vartheta}, \widehat{\lambda}] = 2i\widehat{\tau}$

Broken symmetries for flavored Minimally Doubled Fermions

- Redefine parameters

$$a = a_\sigma, \quad \xi = \frac{a_\sigma}{a_\tau}(1+d_4), \quad \xi_\beta = \frac{\beta_\tau}{\beta_\sigma}(1+d_p),$$

$$m_3^{\text{old}} \to m_3^{\text{new}} = \frac{\xi m_3^{\text{old}}}{1+d_4}, \quad r^{\text{old}} \to r^{\text{new}} = \frac{\xi r^{\text{old}}}{1+d_4}, \quad \rho = \frac{\xi r^{\text{old}}}{1+d_4}(3+\frac{c_3}{r^{\text{old}}}),$$

  collect in **multi-index** $\kappa = \{m_0, m_3, r, \rho, \xi; \beta, \xi_\beta\}$
- Regroup operators wrt. **behavior under broken discrete symmetries**

$$\widehat{X} = \widehat{D}^N + d_4 \widehat{D}^4 = \sum_{j=1}^{3} D_j \gamma_j + \xi D_0 \gamma_0, \quad \widehat{Y} = \widehat{D}^3, \quad \widehat{Z} = \widehat{D}^W$$

- Introduce signs $\{s, x, y, z\} = \pm 1$ to **capture common symmetry patterns**

$$\boxed{\widehat{D}_{\kappa_{s,x,y,z}}[U] = \widehat{X}[U] + x m_0 + szr\widehat{Z}[U] + sy\rho\widehat{Y} + xy m_3 \widehat{M}[U]}$$

- Operator mixing due to broken symmetries **restricted in loop functions**!

QCD vacuum with flavored MDF

- Physically 'most relevant' loop function: **quark determinant** det $D_{\kappa_{s,x,y,z}}$

- *Role model*: **Wilson twisted mass fermions** at maximal twist

$$\widehat{D}^{\mathrm{tm}} = \widehat{D}^{N} - \frac{r}{a} i\gamma_5 \left(a^2 \sum_{\mu=0}^{3} D_\mu D_\mu^* + (am_{cr}) \right) + m_0$$

three operators transform differently under two transforms: $\widehat{\chi}$ and $\widehat{\mathcal{P}}$

$\Rightarrow$ **parity is a symmetry of the vacuum**, no $\mathcal{O}(a)$ corrections to vacuum

- *Plan*: invoke the symmetry constraints for MDF quark determinant

$\Rightarrow$ analytically calculate dependence of det $D_{\kappa_{s,x,y,z}}$ on $\{s, x, y, z\}$,

$\Rightarrow$ derive properties of the (**non-perturbative**) **vacuum**

Eigenvalue equations

- **Eigenvalue equations** of the MDF Dirac operator

$$\bar{\phi}_{\omega}^{\kappa_{s,x,y,z}} \widehat{D}_{\kappa_{s,x,y,z}} \phi_{\omega}^{\kappa_{s,x,y,z}} = \omega_{\kappa_{s,x,y,z}}$$

A priori, $\phi_{\omega}^{\kappa_{s,x,y,z}}$ and $\omega_{\kappa_{s,x,y,z}}$ depend on **all parameters** $\kappa_{s,x,y,z}$.

- *Prototype*: γ_5 **hermiticity** $\Rightarrow$ **reality & positivity** of determinant, eigenvalues in complex conjugate pairs: $\omega_{\kappa_{s,x,y,z}} \Leftrightarrow \omega^*_{\kappa_{s,x,y,z}}$
- **Chiral symmetry & positivity** $\Rightarrow |\omega_{\kappa_{+x}}|^2 = |\omega_{\kappa_{-x}}|^2$
- **su(2) algebra** $\left(\widehat{\lambda} \text{ resp. } \widehat{\vartheta}\right) \Rightarrow \omega_{\kappa_{+y,+z}} = \omega_{\kappa_{-y,+z}} = \omega_{\kappa_{+y,-z}}$

$$\Rightarrow \ \det \widehat{D}_{\kappa_{s,x,y,z}} = d[m_0^2, m_3^2, r^2, \rho^2, s\rho m_3 m_0; \xi; \beta, \xi_\beta]$$

Invariance of the vacuum under all discrete symmetries is demonstrated for $m_3 = 0$ or $m_0 = 0$. **Arbitrary quark masses** require one further step…

Charge conjugated eigenvalue equation

- Transform with $\widehat{\mathcal{C}}$: cancel sign s ($\Rightarrow$ averaging quarks and antiquarks)

$$\omega_{\kappa_{s,x,y,z}} = \underbrace{\bar{\phi}^{\kappa_{s,x,y,z}}_{\omega}[U]\widehat{\mathcal{C}}}_{=-(\phi^{\kappa_{s,x,y,z}\,c}[U])^T} \quad \underbrace{\widehat{\mathcal{C}}\widehat{D}_{\kappa_{s,x,y,z}}[U]\widehat{\mathcal{C}}}_{=(\widehat{D}_{\kappa_{-s,x,y,z}}[U])^T} \quad \underbrace{\widehat{\mathcal{C}}\phi^{\kappa_{s,x,y,z}}_{\omega}[U]}_{=(\bar{\phi}^{\kappa_{s,x,y,z}\,c}_{\omega}[U])^T}$$

$$= \left(\bar{\phi}^{\kappa_{s,x,y,z}\,c}[U]\widehat{D}_{\kappa_{-s,x,y,z}}[U]\phi^{\kappa_{s,x,y,z}\,c}[U]\right)^T = \omega_{\kappa_{-s,x,y,z}}$$

same spectrum for $\Rightarrow$ $\widehat{D}_{\kappa_{-s,x,y,z}}$ and $\widehat{D}_{\kappa_{s,x,y,z}}$ (but different eigenvectors)

- Eigenvalues depend only on x: $\omega_{\kappa_{s,x,y,z}} \equiv \omega_{\kappa_x}$ & $\left|\omega_{\kappa_x}\right|^2 \equiv \Omega_\kappa$

$$\boxed{\det \widehat{D}_{\kappa_{s,x,y,z}} = \prod_{0\le\arg(\omega_{\kappa_x})<\pi} \Omega_\kappa = d[m_0^2, m_3^2, r^2, \rho^2;\xi;\beta,\xi_\beta] \ge 0}$$

Invariance of the MDF quark determinant under all discrete symmetries is manifest for arbitrary bare parameters and for arbitrary gauge fields.

Properties of the QCD vacuum

Invariance of the quark determinant implies invariance of the vacuum.

- If the counterterm coefficients are tuned non-perturbatively, the quark determinant's leading cutoff effects are of $\mathcal{O}(a^2)$.

$\Rightarrow$ **The QCD vacuum with Minimally Doubled Fermions receives no $\mathcal{O}(a)$ corrections and is an even function of the two bare quark masses.**

- NB: Another flavored MDF field does not alter the symmetries.
- NB: $\det \widehat{D}_{\kappa_{s,x,y,z}}$ is an even function of ξ, which follows directly from its m_3^2 dependence and a hopping parameter expansion.
- NB: $\det \widehat{D}_{\kappa_{s,x,y,z}}$ does not have to be an even function of the counterterm coefficients c_3 and d_4.

The QCD vacuum is fully ignorant of $\{s, x, y, z\}$. Hence, valence quarks with different choices for $\{s, x, y, z\}$ have exactly the same QCD vacuum.

Flavored mesons with flavored MDF

Generalized even-odd structure of quark propagators

- Introduce **su(2) projectors** $\widehat{P}^{\pm}_{\tau} = \frac{1}{2}(\mathbf{1} \pm \widehat{\tau})$

→ Sort quark matrix and propagator by $\widehat{P}^{\pm}_{\tau}$ **even-odd** structures

$$\widehat{D}_{\kappa_{s,x,y,z}} = \widehat{E} + \widehat{O}, \qquad \widehat{E} = \widehat{X} + x m_0, \quad \widehat{O} = xy m_3 \widehat{M} + sy\rho \widehat{Y} + szr\widehat{Z}$$

$$\widehat{S}_{\kappa_{s,x,y,z}} = \widehat{S}^E + \widehat{S}^O, \quad \widehat{S}^E = \left(\widehat{E} - \widehat{O}\widehat{E}^{-1}\widehat{O}\right)^{-1}, \quad \widehat{S}^O = -\widehat{S}^E \widehat{O}\widehat{E}^{-1}$$

- Introduce **charge conjugation projectors** $\widehat{P}^{\pm}_{\mathcal{C}} = \frac{1}{2}\left(\mathbf{1} \pm \widehat{\mathcal{C}}\right)$,

→ Sort quark matrix and propagator by $\widehat{P}^{\pm}_{\mathcal{C}}$ **even-odd** structures

$$\widehat{D}_{\kappa_{s,x,y,z}} = \widehat{\eta} + s\,\widehat{\omega}, \qquad \widehat{\eta} = \widehat{X} + x m_0 + xy m_3 \widehat{M}, \qquad \widehat{\omega} = y\rho\widehat{Y} + zr\widehat{Z},$$

$$\widehat{S}_{\kappa_{s,x,y,z}} = \widehat{S}^{\eta} + s\,\widehat{S}^{\omega}, \quad \widehat{S}^{\eta} = \left(\widehat{\eta} - \widehat{\omega}\widehat{\eta}^{-1}\widehat{\omega}\right)^{-1}, \quad \widehat{S}^{\omega} = -\widehat{S}^{\eta}\widehat{\omega}\widehat{\eta}^{-1}$$

$\widehat{\mathcal{C}}$-odd structures due to **Karsten-Wilczek term**: $\widehat{\omega}$, $\widehat{S}^{\omega} \sim \mathcal{O}(a)$

- **Nested even-odd structure** for different broken discrete symmetries

$$\widehat{S}_{\kappa_{s,x,y,z}} = \widehat{S}^{\eta E} + m_3 \widehat{S}^{\eta O} + s m_3 \widehat{S}^{\omega E} + s\widehat{S}^{\omega O}$$

linear dependence on **flavor non-singlet mass** parameter m_3 exposed

Generic meson correlation functions

- Meson two-point function has **quark-disconnected & -connected** parts

$$C_{\widehat{\Gamma}_a,\widehat{\Gamma}_b}(n_0) = \sum_{n} \left[\left\langle \left(\bar{\psi}_0 \widehat{\Gamma}_a \psi_0\right) \left(\bar{\psi}_n \widehat{\Gamma}_b \psi_n\right) \right\rangle\right]_U$$
$$= \sum_{n} \left[\left\langle \widehat{S}_{0,0} \widehat{\Gamma}_a \right\rangle \left\langle \widehat{S}_{n,n} \widehat{\Gamma}_b \right\rangle\right]_U - \left[\left\langle \widehat{S}_{n,0} \widehat{\Gamma}_a \widehat{S}_{0,n} \widehat{\Gamma}_b \right\rangle\right]_U$$

$\left[\ldots\right]_U \rightarrow$ gauge average, $\left\langle \ldots \right\rangle \rightarrow$ combined Dirac and color trace

- Sort **kernels** $\widehat{\Gamma}_{a,b}$ **of the interpolating operators** by symmetries, e.g.

$\{\widehat{\Gamma}^\eta\} = \{\mathbf{1}, \gamma_5, i\gamma_\mu\gamma_5\}, \qquad \{\widehat{\Gamma}^\omega\} = \{\gamma_\mu, i\gamma_\mu\gamma_\nu\}, \quad$ point-split operators

- **Quark propagators are linked to kernels** through even-odd structures

$$\left\langle \widehat{S}\widehat{\Gamma}_a \right\rangle = \left\langle \widehat{S}^\eta \widehat{\Gamma}_a^\eta + \mathsf{s}\widehat{S}^\omega \widehat{\Gamma}_a^\omega \right\rangle = \left\langle \widehat{S}^\eta \widehat{\Gamma}_a^\eta \right\rangle + \mathsf{s} \left\langle \widehat{S}^\omega \widehat{\Gamma}_a^\omega \right\rangle$$

- **Terms** $\propto \mathsf{s}$ vanish for sufficiently **symmetric** $\widehat{\Gamma}_{a,b} \Rightarrow$ **no** $\widehat{\mathcal{C}}$ **or** $\widehat{\mathcal{T}}$ **violation**
- **Vacuum is ignorant** of $\mathsf{s} = \pm 1 \Rightarrow$ may average $\mathsf{s} = \pm 1$ **in valence sector**

Nested $\widehat{P}_{\mathcal{T}}^{\pm}$ and $\widehat{P}_{\mathcal{C}}^{\pm}$ projections

- Apply $\widehat{P}_{\mathcal{C}}^{\pm}$ projections $\longrightarrow$ average $\pm$s $\longrightarrow$ apply $\widehat{P}_{\mathcal{T}}^{\pm}$ projections

- Quark-disconnected contribution (**propagators intertwined with** $\widehat{\Gamma}_{a,b}$):

$$C^{\text{disc}}_{\widehat{\Gamma}_a,\widehat{\Gamma}_b}(n_0) = \sum_{\boldsymbol{n}} \left[\left\langle \widehat{S}^{\eta E}_{0,0} \widehat{\Gamma}^{\eta E}_a \right\rangle \left\langle \widehat{S}^{\eta E}_{n,n} \widehat{\Gamma}^{\eta E}_b \right\rangle + \left\langle \widehat{S}^{\omega O}_{0,0} \widehat{\Gamma}^{\omega O}_a \right\rangle \left\langle \widehat{S}^{\omega O}_{n,n} \widehat{\Gamma}^{\omega O}_b \right\rangle \right]_U$$
$$+ m_3^2 \left[\left\langle \widehat{S}^{\eta O}_{0,0} \widehat{\Gamma}^{\eta O}_a \right\rangle \left\langle \widehat{S}^{\eta O}_{n,n} \widehat{\Gamma}^{\eta O}_b \right\rangle + \left\langle \widehat{S}^{\omega E}_{0,0} \widehat{\Gamma}^{\omega E}_a \right\rangle \left\langle \widehat{S}^{\omega E}_{n,n} \widehat{\Gamma}^{\omega E}_b \right\rangle \right]_U$$

- Quark-connected contribution (**same pattern for any symmetric** $\widehat{\Gamma}^S_{a,b}$)

$$C^{\text{con}}_{\widehat{\Gamma}^S_a,\widehat{\Gamma}^S_b}(n_0) = -\sum_{\boldsymbol{n}} \left[\left\langle \widehat{S}^{\eta E}_{n,0} \widehat{\Gamma}^S_a \widehat{S}^{\eta E}_{0,n} \widehat{\Gamma}^S_b \right\rangle + \left\langle \widehat{S}^{\omega O}_{n,0} \widehat{\Gamma}^S_a \widehat{S}^{\omega O}_{0,n} \widehat{\Gamma}^S_b \right\rangle \right]_U$$
$$+ m_3^2 \left[\left\langle \widehat{S}^{\eta O}_{n,0} \widehat{\Gamma}^S_a \widehat{S}^{\eta O}_{0,n} \widehat{\Gamma}^S_b \right\rangle + \left\langle \widehat{S}^{\omega E}_{n,0} \widehat{\Gamma}^S_a \widehat{S}^{\omega E}_{0,n} \widehat{\Gamma}^S_b \right\rangle \right]_U$$

Parity partners and 'taste breaking'

- **Quark-connected correlation functions** for MDF receive contributions from **parity partners** (same as for Kogut-Susskind Fermions)

$$C^{\mathrm{con}}_{\widehat{\Gamma}^S_a,\widehat{\Gamma}^S_b}(n_0) = -\sum_{\boldsymbol{n}}\left[\left\langle \widehat{S}^E_{n,0}\widehat{\tau}^2\widehat{\Gamma}^S_a\widehat{S}^E_{0,n}\widehat{\Gamma}^S_b\right\rangle + \left\langle \widehat{S}^O_{n,0}\widehat{\tau}^2\widehat{\Gamma}^S_a\widehat{S}^O_{0,n}\widehat{\Gamma}^S_b\right\rangle\right]_U$$

$$= -\sum_{\boldsymbol{n}}\left[\left\langle \widehat{S}^E_{n,0}\tau_0\widehat{\Gamma}^S_a\widehat{S}^E_{0,n}\widehat{\Gamma}^S_b\tau_{n_0}\right\rangle - \left\langle \widehat{S}^O_{n,0}\tau_0\widehat{\Gamma}^S_a\widehat{S}^O_{0,n}\widehat{\Gamma}^S_b\tau_{n_0}\right\rangle\right]_U$$

where $\tau_{n_0} = i\gamma_0\gamma_5(-1)^{n_0}$ is the matrix of the **temporal shift transform**

- **States with opposite parities** in quark-connected correlation functions

$$C^{\mathrm{con}}_{\widehat{\Gamma}^S_a,\widehat{\Gamma}^S_b}(n_0) = -\frac{1}{2}\sum_{\boldsymbol{n}}\left\langle \left\langle \widehat{S}^E_{n,0}\widehat{\Gamma}^S_a\widehat{S}^E_{0,n}\widehat{\Gamma}^S_b\right\rangle^s + \left\langle \widehat{S}^O_{n,0}\widehat{\Gamma}^S_a\widehat{S}^O_{0,n}\widehat{\Gamma}^S_b\right\rangle^s\right\rangle_U$$

$$+(-1)^{n_0}\left\langle \left\langle \widehat{S}^E_{n,0}\tau_0\widehat{\Gamma}^S_a\widehat{S}^E_{0,n}\widehat{\Gamma}^S_b\tau_0\right\rangle^s - \left\langle \widehat{S}^O_{n,0}\tau_0\widehat{\Gamma}^S_a\widehat{S}^O_{0,n}\widehat{\Gamma}^S_b\tau_0\right\rangle^s\right\rangle_U$$

$\Rightarrow$ **the parity partner** has different cutoff effects – **'taste breaking'**

- Trade **taste breaking** between partners by $+s \rightarrow -s$ **on one propagator**

Pseudoscalar spectrum

Pseudoscalars with $\widehat{\Gamma}_{a,b} = \{\widehat{\gamma}_5, \gamma_0, \mathbf{1}, i\gamma_0\gamma_5\}$ (for $\gamma_0, \mathbf{1}$ as parity partners)

$$\mathbf{1}, i\gamma_0\gamma_5 \in \{\widehat{\Gamma}^{\eta E}\}, \quad \widehat{\gamma}_5 \in \{\widehat{\Gamma}^{\eta O}\}, \quad \gamma_0 \in \{\widehat{\Gamma}^{\omega O}\}$$

$\widehat{\gamma}_5$**-channel**:

$$\begin{aligned} C_{\widehat{\gamma}_5,\widehat{\gamma}_5}(n_0) &= C^{\text{con}}_{\widehat{\gamma}_5,\widehat{\gamma}_5}(n_0) + C^{\text{disc}}_{\widehat{\gamma}_5,\widehat{\gamma}_5}(n_0) \\ &= -\frac{1}{2}\sum_{\boldsymbol{n}} \left[\left\langle \widehat{S}^{\eta E}_{n,0}\widehat{\gamma}_5 \widehat{S}^{\eta E}_{0,n}\widehat{\gamma}_5 \right\rangle + \left\langle \widehat{S}^{\omega O}_{n,0}\widehat{\gamma}_5 \widehat{S}^{\omega O}_{0,n}\widehat{\gamma}_5 \right\rangle \right]_U \\ &\qquad + m_3^2 \left[\left\langle \widehat{S}^{\eta O}_{n,0}\widehat{\gamma}_5 \widehat{S}^{\eta O}_{0,n}\widehat{\gamma}_5 \right\rangle + \left\langle \widehat{S}^{\omega E}_{n,0}\widehat{\gamma}_5 \widehat{S}^{\omega E}_{0,n}\widehat{\gamma}_5 \right\rangle \right]_U \\ &\quad + \sum_{\boldsymbol{n}} m_3^2 \left[\left\langle \widehat{S}^{\eta O}_{0,0}\widehat{\gamma}_5 \right\rangle \left\langle \widehat{S}^{\eta O}_{n,n}\widehat{\gamma}_5 \right\rangle \right]_U \end{aligned}$$

- **Quark-disconnected part** at leading order $\mathcal{O}(m_3^2)$, **zero in isospin limit**
- Compare with e.g. SU(2) ChPT @ NLO [J. Gasser, H. Leutwyler, 1984]

$$l_7 \langle \chi_- \rangle^2 = l_7 \left[2B_0(m_d - m_u)\right]^2 \langle \tau^3 \mathrm{Im}\, U \rangle^2$$

- **Characteristic quark-disconnected term** $\Rightarrow$ ground state is **neutral pion**

Pseudoscalar spectrum

Pseudoscalars with $\widehat{\Gamma}_{a,b} = \{\widehat{\gamma_5}, \gamma_0, \mathbf{1}, i\gamma_0\gamma_5\}$ (for $\gamma_0, \mathbf{1}$ as parity partners)

$$\mathbf{1}, i\gamma_0\gamma_5 \in \{\widehat{\Gamma}^{\eta E}\}, \quad \widehat{\gamma_5} \in \{\widehat{\Gamma}^{\eta O}\}, \quad \gamma_0 \in \{\widehat{\Gamma}^{\omega O}\}$$

Other spin-0 channels:

- γ_0- & $\mathbf{1}$-channels: pseudoscalar **parity partners**, **opposite sign of m_3^2** in connected part, **no quark-disconnected contribution** $\Rightarrow$ **taste-broken charged pion states**
- $i\gamma_0\gamma_5$-channel: **quark-disconnected** part due to $\widehat{S}^{\eta E}$ persists in chiral & continuum limits, **same sign of m_3^2** in connected part as $\widehat{\gamma_5}$-channel $\Rightarrow$ flavor singlet $\leftrightarrow$ **eta meson**
- NB: **scalars** are found in $\mathbf{1}$- and $i\gamma_0\gamma_5$-channels, the latter as **taste-broken parity partner**

Summary

- Constructed a **flavored** **Minimally Doubled Fermion action**
- Broken symmetries and **su(2) algebra** are manifest at finite cutoff

- **MDF quark determinant** is **ignorant of broken discrete symmetries** for **arbitrary gauge fields** and **arbitrary bare parameters**
- QCD vacuum is **invariant under charge conjugation and time reflection** and receives **no $\mathcal{O}(a)$ corrections**

- Quark propagators are **even-odd decomposable wrt. symmetries**; decompositions for **various discrete symmetries** can be nested
- **Flavored meson propagators** distinguished unambiguously by dependence on **flavor non-singlet quark mass m_3** and in terms of (non-)existence and **nature of quark-disconnected contributions**
- **Taste breaking** avertible by simple means, **no fine tuning**

A brief history of Minimally Doubled Fermions so far

- Minimally Doubled Fermions suggested in early 80s [L.H. Karsten, 1981]
- Anisotropic patterns for Karsten-Wilczek Fermions [F. Wilczek, 1987]
- Spatial MDF, mirror fermion symmetry [M. Pernici, 1995]

- MDF Revival: Boriçi-Creutz Fermions [M. Creutz, A. Boriçi, 2007/08]
- Symmetries of MDF [P. Bedaque, M. Buchoff, B. Tiburzi, A. Walker-Loud, 2008]
- Renormalization of MDF [S. Capitani, H. Wittig, JHW, 2009/10]
- Flavor interpretation, axial anomaly with KWF [B. Tiburzi, 2010]
- Index Theorem with MDF [T. Kimura, M. Creutz, T. Misumi 2010]

- Numerical Studies of MDF (quenched) [S. Capitani, H. Wittig, JHW, 2014]
- Correlation functions with KWF (quenched):
  oscillations, parity partners, taste breaking, continuum limit [JHW, 2015]

Pernici's mirror fermion symmetry

- **Degenerate** MDF have **mirror fermion symmetry** [M. Pernici, 1995]

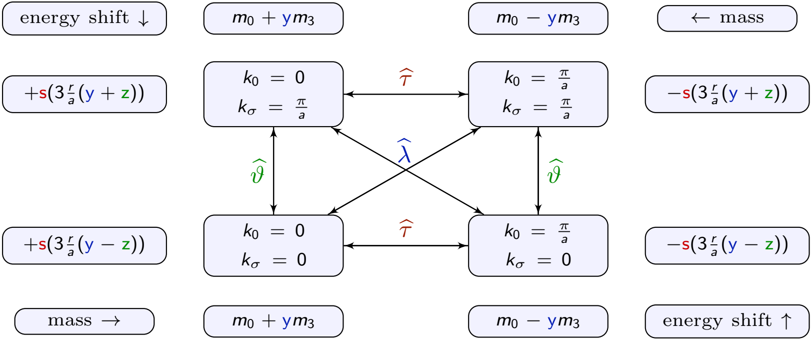
$$\widehat{D}^W \xrightarrow{\widehat{\tau}\,\widehat{\tau}} \widehat{D}^W, \quad \widehat{D}^3 \xrightarrow{\widehat{\tau}\,\widehat{\tau}} \widehat{D}^3$$

  Form-invariance of action: **combined reflection & shift transform**
- **Shift transform** can be defined with **shift operator** $\widehat{\tau}$

$$\bar{\psi} \to \bar{\psi}\widehat{\tau}, \quad \psi \to \widehat{\tau}\psi, \quad \widehat{\tau} \equiv \tau_{m,n} = \tau_{n_0}\delta_{m,n}, \quad \tau_{n_0} = i\gamma_0\gamma_5(-1)^{n_0}$$

- **Flavored** MDF: mirror fermion symmetry is broken by $m_3\widehat{M} \to -m_3\widehat{M}$
- $\bar{\psi}m_3\widehat{M}\psi$ has a continuum limit – treat as **flavor non-singlet mass term**

Quark modes for flavored Minimally Doubled Fermions (tree level)



- Exactly **two non-degenerate quark modes** in the continuum limit
- **su(2) generators** of $(\widehat{\lambda}, \widehat{\tau}, \widehat{\vartheta})$ **swap between pairs** of fermion modes
- **Charge conjugation/time reflection invert the sign** of the energy shift

Broken discrete symmetries for flavored Minimally Doubled Fermions

$$\widehat{D}_{\kappa_{s,x,y,z}}[U] = \widehat{X}[U] + xm_0 + szr\widehat{Z}[U] + sy\rho\widehat{Y} + xym_3\widehat{M}[U]$$

Transform	Effect	$\widehat{X}$	$\widehat{Y}$	$\widehat{Z}$	$\widehat{1}$	$\widehat{M}$	U^0	U^σ
$\widehat{\chi}$	$\bar{\psi}\gamma_5\widehat{A}\gamma_5\psi$	$-\widehat{X}$	$-\widehat{Y}$	$-\widehat{Z}$	$+\widehat{1}$	$+\widehat{M}$	$+U^0$	$+U^\sigma$
$\widehat{\tau}$	$\bar{\psi}\widehat{\tau}\widehat{A}\widehat{\tau}\psi$	$+\widehat{X}$	$-\widehat{Y}$	$-\widehat{Z}$	$+\widehat{1}$	$-\widehat{M}$	$+U^0$	$+U^\sigma$
$\widehat{\lambda}$	$\bar{\psi}\widehat{\lambda}\widehat{A}\widehat{\lambda}\psi$	$+\widehat{X}$	$-\widehat{Y}$	$+\widehat{Z}$	$+\widehat{1}$	$-\widehat{M}$	$+U^0$	$+U^\sigma$
$\widehat{\vartheta}$	$\bar{\psi}\widehat{\vartheta}\widehat{A}\widehat{\vartheta}\psi$	$+\widehat{X}$	$+\widehat{Y}$	$-\widehat{Z}$	$+\widehat{1}$	$+\widehat{M}$	$+U^0$	$+U^\sigma$
$\widehat{\mathcal{C}}$	$\bar{\psi}\widehat{\mathcal{C}}\widehat{A}\widehat{\mathcal{C}}\psi$	$\widehat{X}^T$	$-\widehat{Y}^T$	$-\widehat{Z}^T$	$+\widehat{1}^T$	$+\widehat{M}^T$	$+U^{0*}$	$+U^{\sigma *}$
$\widehat{\mathcal{T}}$	$\bar{\psi}\widehat{\mathcal{T}}\widehat{A}\widehat{\mathcal{T}}\psi$	$+\widehat{X}$	$-\widehat{Y}$	$-\widehat{Z}$	$+\widehat{1}$	$+\widehat{M}$	$-U^{0\dagger}$	$+U^\sigma$
$\widehat{\mathcal{P}}$	$\bar{\psi}\widehat{\mathcal{P}}\widehat{A}\widehat{\mathcal{P}}\psi$	$+\widehat{X}$	$+\widehat{Y}$	$+\widehat{Z}$	$+\widehat{1}$	$+\widehat{M}$	$+U^0$	$-U^{\sigma\dagger}$

- **Four out of five independent** symmetry transforms are broken for MDF
- Operator mixing due to broken symmetries **restricted in loop functions**!

Quark determinant in the free theory

- **Momentum modes decouple**, $\det D_{\kappa_{s,x,y,z}}$ is explicitly calculable
- Factor out imaginary factors and $GL(4, \mathbb{C})$ matrices

$$\widehat{X} = i \sum_{\mu} X_{\mu}\gamma_{\mu}, \quad \widehat{Z} = iZ\gamma_0, \quad \widehat{Y} = iY\gamma_0, \quad \widehat{M} = M$$

$$X_0 = \frac{\xi}{a}\sin(\xi a k_0), \quad X_j = \frac{1}{a}\sin(ak_j), \quad M = \cos(\xi a k_0),$$

$$Y = \frac{1}{a}, \quad Z = \frac{1}{a}\sum_{j=1}^{3}\cos(ak_j).$$

- The integrand is a **real, scalar function** for each momentum mode

$$\Delta_{\kappa_{s,x,y,z}} = \Delta_{\kappa_{s,y,z}} = \left[\left(\{X_0 + s(y\rho Y + zrZ)\}^2 + \boldsymbol{X}^2\right) + (m_0 + y m_3 M)^2\right]^2$$

- Infinite volume, absorb lattice spacing $k_j \to k_j' = ak_j$, $k_0 \to k_0' = \xi a k_0$

$$\det \widehat{D}_{\kappa_{s,x,y,z}} = \int_{-\pi}^{+\pi} \frac{d^4 k}{(2\pi)^4}\, \Delta_{\kappa_{s,y,z}}(k),$$

Origin of the symmetry of the quark determinant

- The **integrand breaks the symmetries explicitly**, depends on $\{s, y, z\}$:

$$\Delta_{\kappa_{s,y,z}} = \left[X^2 + \mu_+ + \epsilon_+\right]^2 + 4X_0^2\epsilon_+ + 4\mu_-^2 + 4\epsilon_-^2$$
$$+ 4\left[X^2 + \mu_+ + \epsilon_+\right]\left[sX_0\sqrt{\epsilon_+ + 2yz\epsilon_-} + y\left(z\epsilon_- + \mu_-\right)\right]$$
$$+ 8\left[syX_0\sqrt{\epsilon_+ + 2yz\epsilon_-}\left(z\epsilon_- + \mu_-\right) + z\epsilon_-\left(yX_0^2 + \mu_-\right)\right] .$$

where signs $\{s, y, z\}$ have been factored out from

$$\mu_+ = m_0^2 + (m_3 M)^2 \geq 0, \qquad \mu_- = m_0 m_3 M,$$
$$\epsilon_+ = \rho^2 Y^2 + r^2 Z^2 \geq 0, \qquad \epsilon_- = \rho r Y Z$$

- Any terms with signs $\{s, y, z\}$ attached contain **odd powers** of $\cos k_\mu$ or $\sin k_\mu$ for some direction $\Rightarrow$ **symmetries restored upon integration**.

$$\boxed{\det \widehat{D}_{\kappa_{s,x,y,z}} = d(m_0^2, m_3^2, r^2)}$$

Origin of the symmetry of the quark determinant

- The **integrand breaks the symmetries explicitly**, depends on $\{s, y, z\}$:

$$\Delta_{\kappa_{s,y,z}} = \left[X^2 + \mu_+ + \epsilon_+\right]^2 + 4X_0^2\epsilon_+ + 4\mu_-^2 + 4\epsilon_-^2$$
$$+ 4\left[X^2 + \mu_+ + \epsilon_+\right]\left[sX_0\sqrt{\epsilon_+ + 2yz\epsilon_-} + y\left(z\epsilon_- + \mu_-\right)\right]$$
$$+ 8\left[syX_0\sqrt{\epsilon_+ + 2yz\epsilon_-}\left(z\epsilon_- + \mu_-\right) + z\epsilon_-\left(yX_0^2 + \mu_-\right)\right)\right].$$

where signs $\{s, y, z\}$ have been factored out from

$$\mu_+ = m_0^2 + (m_3 M)^2 \geq 0, \qquad \mu_- = m_0 m_3 M,$$
$$\epsilon_+ = \rho^2 Y^2 + r^2 Z^2 \geq 0, \qquad \epsilon_- = \rho r Y Z$$

- Any terms with signs $\{s, y, z\}$ attached contain **odd powers** of $\cos k_\mu$ or $\sin k_\mu$ for some direction $\Rightarrow$ **symmetries restored upon integration**.

Symmetry breaking terms do not vanish by themselves, symmetry violations actually cancel between different quark modes.

Eigenvalue equations

- **Eigenvalue equations** of the MDF Dirac operator

$$\bar{\phi}_\omega^{\kappa_{s,x,y,z}} \widehat{D}_{\kappa_{s,x,y,z}} \phi_\omega^{\kappa_{s,x,y,z}} = \omega_{\kappa_{s,x,y,z}}$$

Eigenvectors $\phi_\omega^{\kappa_{s,x,y,z}}$ and eigenvalues $\omega_{\kappa_{s,x,y,z}}$ depend on all parameters

- Prototype: γ_5 **hermiticity** $\Rightarrow$ **reality & positivity** of determinant

$$\omega_{\kappa_{s,x,y,z}} = \underbrace{\bar{\phi}_\omega^{\kappa_{s,x,y,z}} \gamma_5}_{=-(\psi_\omega^{\kappa_{s,x,y,z}})^\dagger} \underbrace{\gamma_5 \widehat{D}_{\kappa_{s,x,y,z}} \gamma_5}_{\widehat{D}^\dagger_{\kappa_{s,x,y,z}}} \underbrace{\gamma_5 \phi_\omega^{\kappa_{s,x,y,z}}}_{=(\bar{\psi}_\omega^{\kappa_{s,x,y,z}})^\dagger} = \left[\bar{\psi}_\omega^{\kappa_{s,x,y,z}} \widehat{D}_{\kappa_{s,x,y,z}} \psi_\omega^{\kappa_{s,x,y,z}}\right]^\dagger$$

$\Rightarrow$ Eigenvalues only in complex conjugate pairs: $\omega_{\kappa_{s,x,y,z}} \Leftrightarrow \omega^*_{\kappa_{s,x,y,z}}$

$$\det \widehat{D}_{\kappa_{s,x,y,z}} = \prod_\omega \omega_{\kappa_{s,x,y,z}} = \prod_{0 \leq \arg(\omega_{\kappa_{s,x,y,z}}) < \pi} |\omega_{\kappa_{s,x,y,z}}|^2 \geq 0$$

using that **MDF have imaginary eigenvalues** $\omega_{\kappa_{s,x,y,z}}$ in chiral limit and demanding that **quark masses are positive** $(m_0^{\mathrm{ren}} \geq |m_3^{\mathrm{ren}}| \geq 0)$

Quark mass dependence

- Use positivity, expand determinant as square root

$$\det \widehat{D}_{\kappa_{s,x,y,z}} = \sqrt{\det \widehat{D}_{\kappa_{s,x,y,z}} \det \widehat{D}_{\kappa_{s,x,y,z}}} = \sqrt{\det \widehat{D}_{\kappa_{s,x,y,z}} \det \widehat{D}^{\dagger}_{\kappa_{s,x,y,z}}}$$

- Sort in x, i.e. **sort even and odd terms** under action of $\widehat{\chi}$

$$\widehat{D}_{\kappa_{s,x,y,z}} = x\widehat{E} + \widehat{O}, \qquad \widehat{E} = m_0 + y m_3 \widehat{M}, \qquad \widehat{O} = \widehat{X} + sy\rho\widehat{Y} + szr\widehat{Z}$$

- Expand determinant in $\widehat{E}$ and $\widehat{O}$

$$\det \widehat{D}_{ksxyz} = \sqrt{\det(x\widehat{E} + \widehat{O}) \det(x\widehat{E} - \widehat{O})} = \sqrt{\det \widehat{E}^2 \det(1 - [\widehat{E}^{-1}\widehat{O}]^2)}$$

assuming that $\widehat{E}^{-1}$ exists (both quarks are massive) $\Rightarrow$ x cancels

$$\Rightarrow \qquad \det \widehat{D}_{\kappa_{s,x,y,z}} = d(m_0^2, m_3^2, y m_3 m_0; sy\rho, szr, \xi; \beta, \xi_\beta)$$

su(2) transforms

- Transform with $\widehat{\lambda}$: cancel sign y ($\Rightarrow$ average combinations of m_3 and s)

$$\omega_{\kappa_{s,x,y,z}} = \underbrace{\underbrace{\bar{\phi}_\omega^{\kappa_{s,x,y,z}}\widehat{\lambda}}\,\underbrace{\widehat{\lambda}\widehat{D}_{\kappa_{s,x,y,z}}\widehat{\lambda}}\,\underbrace{\widehat{\lambda}\phi_\omega^{\kappa_{s,x,y,z}}}}_{=\widehat{D}_{\kappa_{s,x,-y,z}}} = \omega_{\kappa_{s,x,-y,z}}$$

$$\Rightarrow \quad \omega_{\kappa_{s,x,+y,z}} = \omega_{\kappa_{s,x,-y,z}} \text{ (though different eigenvectors)}$$

- Transform with $\widehat{\vartheta}$: cancel sign z ($\Rightarrow$ averaging with the $k_\sigma = \frac{\pi}{a}$ modes)

$$\omega_{\kappa_{s,x,y,z}} = \underbrace{\underbrace{\bar{\phi}_\omega^{\kappa_{s,x,y,z}}\widehat{\vartheta}}\,\underbrace{\widehat{\vartheta}\widehat{D}_{\kappa_{s,x,y,z}}\widehat{\vartheta}}\,\underbrace{\widehat{\vartheta}\phi_\omega^{\kappa_{s,x,y,z}}}}_{=\widehat{D}_{\kappa_{s,x,y,-z}}} = \omega_{\kappa_{s,x,y,-z}}$$

$$\Rightarrow \quad \omega_{\kappa_{s,x,y,+z}} = \omega_{\kappa_{s,x,y,-z}} \text{ (though different eigenvectors)}$$

$$\Rightarrow \qquad \det \widehat{D}_{\kappa_{s,x,y,z}} = d(m_0^2, m_3^2, r^2, \rho^2, s\rho m_3 m_0; \xi; \beta, \xi_\beta)$$

Invariance of the vacuum under all discrete symmetries is demonstrated for $m_3 = 0$ or $m_0 = 0$. **Arbitrary quark masses** require one further step. . .

Chiral even-odd structure of quark propagators

- Introduce **chiral projectors** $\widehat{P}_\chi^\pm = \frac{1}{2}(\mathbf{1} \pm \gamma_5)$

$\rightarrow$ Sort quark matrix and propagator by $\widehat{P}_\chi^\pm$ **even-odd structures**

$$\widehat{D}_{\kappa_{s,x,y,z}} = x\widehat{E} + \widehat{O}, \qquad \widehat{E} = m_0 + y m_3 \widehat{M}, \qquad \widehat{O} = \widehat{X} + sy\rho\widehat{Y} + szr\widehat{Z}$$

$$\widehat{S}_{\kappa_{s,x,y,z}} = x\widehat{S}^E + \widehat{S}^O, \quad \widehat{S}^E = \left(\widehat{E} - \widehat{O}\widehat{E}^{-1}\widehat{O}\right)^{-1}, \quad \widehat{S}^O = -\widehat{S}^E\widehat{O}\widehat{E}^{-1}$$

- Proof is straightforward

$$\widehat{S}\widehat{D} = \left(x\widehat{S}^E + \widehat{S}^O\right)\left(x\widehat{E} + \widehat{O}\right) = \frac{\left(x - \widehat{O}\widehat{E}^{-1}\right)\left(x\widehat{E} + \widehat{O}\right)}{\widehat{E} - \widehat{O}\widehat{E}^{-1}\widehat{O}} = \frac{\widehat{E} - \widehat{O}\widehat{E}^{-1}\widehat{O}}{\widehat{E} - \widehat{O}\widehat{E}^{-1}\widehat{O}} = \delta$$

- $\widehat{S}^E$ and $\widehat{S}^O$ connect different sets of chiral projectors $\widehat{P}_\chi^\pm = \frac{1}{2}(\mathbf{1} \pm \gamma_5)$

$$\widehat{P}_\chi^\pm \widehat{S}_{\kappa_{s,x,y,z}} \widehat{P}_\chi^\pm = \widehat{P}_\chi^\pm \widehat{S}^E \widehat{P}_\chi^\pm, \quad \widehat{P}_\chi^\pm \widehat{S}_{\kappa_{s,x,y,z}} \widehat{P}_\chi^\mp = \widehat{P}_\chi^\pm \widehat{S}^O \widehat{P}_\chi^\mp$$

Projection operator formalism

- Meson two-point function has **quark-disconnected & -connected** parts

$$C_{\widehat{\Gamma}_a,\widehat{\Gamma}_b}(n_0) = \sum_{\boldsymbol{n}} \left[\left\langle \left(\bar{\psi}_0 \widehat{\Gamma}_a \psi_0 \right) \left(\bar{\psi}_n \widehat{\Gamma}_b \psi_n \right) \right\rangle \right]_U$$
$$= \sum_{\boldsymbol{n}} \left[\left\langle \widehat{S}_{0,0} \widehat{\Gamma}_a \right\rangle \left\langle \widehat{S}_{n,n} \widehat{\Gamma}_b \right\rangle \right]_U - \left[\left\langle \widehat{S}_{n,0} \widehat{\Gamma}_a \widehat{S}_{0,n} \widehat{\Gamma}_b \right\rangle \right]_U$$

- Insert $\mathbf{1} = \widehat{P}^+_{\mathcal{C}} + \widehat{P}^-_{\mathcal{C}}$ into the traces, use cyclic property, e.g.

$$\left\langle \widehat{S} \widehat{\Gamma}_a \right\rangle = \sum_{\sigma=\pm1} \left\langle \widehat{P}^\sigma_{\mathcal{C}} \widehat{S}^\eta \widehat{P}^\sigma_{\mathcal{C}} \widehat{\Gamma}_a + \mathsf{s} \widehat{P}^\sigma_{\mathcal{C}} \widehat{S}^\omega \widehat{P}^{-\sigma}_{\mathcal{C}} \widehat{\Gamma}_a \right\rangle$$
$$= \sum_{\sigma=\pm1} \left\langle \widehat{S}^\eta \underbrace{\widehat{P}^\sigma_{\mathcal{C}} \widehat{\Gamma}_a \widehat{P}^\sigma_{\mathcal{C}}} + \mathsf{s} \widehat{S}^\omega \underbrace{\widehat{P}^{-\sigma}_{\mathcal{C}} \widehat{\Gamma}_a \widehat{P}^\sigma_{\mathcal{C}}} \right\rangle$$

- Sort kernels $\widehat{\Gamma}_{a,b}$ of interpolating operators by symmetries, e.g.

$$\{\widehat{\Gamma}^\eta\} = \{\mathbf{1}, \gamma_5, i\gamma_\mu\gamma_5\}, \qquad \{\widehat{\Gamma}^\omega\} = \{\gamma_\mu, i\gamma_\mu\gamma_\nu\}$$

- **Quark propagators linked to interpolators through even-odd structure**

$$\left\langle \widehat{S} \widehat{\Gamma}_a \right\rangle = \sum_{\sigma=\pm1} \left\langle \widehat{S}^\eta \widehat{\Gamma}^\eta_a \widehat{P}^\sigma_{\mathcal{C}} + \mathsf{s} \widehat{S}^\omega \widehat{\Gamma}^\omega_a \widehat{P}^\sigma_{\mathcal{C}} \right\rangle = \left\langle \widehat{S}^\eta \widehat{\Gamma}^\eta_a \right\rangle + \mathsf{s} \left\langle \widehat{S}^\omega \widehat{\Gamma}^\omega_a \right\rangle$$

$\widehat{P}_{\mathcal{C}}^{\pm}$ projections

- Quark-disconnected contribution:

$$C^{\mathrm{disc}}_{\widehat{\Gamma}_a,\widehat{\Gamma}_b}(n_0) = \sum_{\boldsymbol{n}} \left[\left\langle \widehat{S}^{\eta}_{0,0}\widehat{\Gamma}^{\eta}_a\right\rangle \left\langle \widehat{S}^{\eta}_{n,n}\widehat{\Gamma}^{\eta}_b\right\rangle + \left\langle \widehat{S}^{\omega}_{0,0}\widehat{\Gamma}^{\omega}_a\right\rangle \left\langle \widehat{S}^{\omega}_{n,n}\widehat{\Gamma}^{\omega}_b\right\rangle\right]_U + \mathsf{s}\left[\left\langle \widehat{S}^{\eta}_{0,0}\widehat{\Gamma}^{\eta}_a\right\rangle \left\langle \widehat{S}^{\omega}_{n,n}\widehat{\Gamma}^{\omega}_b\right\rangle + \left\langle \widehat{S}^{\omega}_{0,0}\widehat{\Gamma}^{\omega}_a\right\rangle \left\langle \widehat{S}^{\eta}_{n,n}\widehat{\Gamma}^{\eta}_b\right\rangle\right]_U$$

- Quark-connected contribution:

$$\begin{aligned} C^{\mathrm{con}}_{\widehat{\Gamma}_a,\widehat{\Gamma}_b}(n_0) = \sum_{\boldsymbol{n}} \; & - \left[\left\langle \widehat{S}^{\eta}_{n,0}\widehat{\Gamma}^{\eta}_a\widehat{S}^{\eta}_{0,n}\widehat{\Gamma}^{\eta}_b\right\rangle + \left\langle \widehat{S}^{\omega}_{n,0}\widehat{\Gamma}^{\eta}_a\widehat{S}^{\omega}_{0,n}\widehat{\Gamma}^{\eta}_b\right\rangle\right]_U \\ & - \left[\left\langle \widehat{S}^{\eta}_{n,0}\widehat{\Gamma}^{\omega}_a\widehat{S}^{\eta}_{0,n}\widehat{\Gamma}^{\omega}_b\right\rangle + \left\langle \widehat{S}^{\omega}_{n,0}\widehat{\Gamma}^{\omega}_a\widehat{S}^{\omega}_{0,n}\widehat{\Gamma}^{\omega}_b\right\rangle\right]_U \\ & - \mathsf{s}\left[\left\langle \widehat{S}^{\eta}_{n,0}\widehat{\Gamma}^{\eta}_a\widehat{S}^{\omega}_{0,n}\widehat{\Gamma}^{\omega}_b\right\rangle + \left\langle \widehat{S}^{\omega}_{n,0}\widehat{\Gamma}^{\eta}_a\widehat{S}^{\eta}_{0,n}\widehat{\Gamma}^{\omega}_b\right\rangle\right]_U \\ & - \mathsf{s}\left[\left\langle \widehat{S}^{\omega}_{n,0}\widehat{\Gamma}^{\omega}_a\widehat{S}^{\eta}_{0,n}\widehat{\Gamma}^{\eta}_b\right\rangle + \left\langle \widehat{S}^{\eta}_{n,0}\widehat{\Gamma}^{\omega}_a\widehat{S}^{\omega}_{0,n}\widehat{\Gamma}^{\eta}_b\right\rangle\right]_U \end{aligned}$$

- Terms $\propto \mathsf{s}$ are $\sim \mathcal{O}(a)$, vanish for symmetric $\widehat{\Gamma}_{a,b} \Rightarrow$ **no $\widehat{\mathcal{C}}$ or $\widehat{\mathcal{T}}$ violation**
- **Vacuum is ignorant** of $\mathsf{s} = \pm 1 \Rightarrow$ **averaging correlators for** $\mathsf{s} = +1$ **and** $\mathsf{s} = -1$ on same configurations legitimate, no change of continuum limit

Flavored pseudoscalar mesons

Pseudoscalars with $\widehat{\Gamma}_{a,b} = \{\widehat{\gamma}_5, \gamma_0, \mathbf{1}, i\gamma_0\gamma_5\}$ (for $\gamma_0, \mathbf{1}$ as parity partners)

$$\mathbf{1}, i\gamma_0\gamma_5 \in \{\Gamma^{\eta E}\}, \quad \widehat{\gamma}_5 \in \{\Gamma^{\eta O}\}, \quad \gamma_0 \in \{\Gamma^{\omega O}\}$$

$\widehat{\gamma}_5$-channel:

$$C_{\widehat{\gamma}_5,\widehat{\gamma}_5}(n_0) = C^{\text{con}}_{\widehat{\gamma}_5,\widehat{\gamma}_5}(n_0) + C^{\text{disc}}_{\widehat{\gamma}_5,\widehat{\gamma}_5}(n_0)$$
$$= -\frac{1}{2}\sum_{\boldsymbol{n}} \left[\left\langle \widehat{S}^{\eta E}_{n,0}\widehat{\gamma}_5\widehat{S}^{\eta E}_{0,n}\widehat{\gamma}_5 \right\rangle + \left\langle \widehat{S}^{\omega O}_{n,0}\widehat{\gamma}_5\widehat{S}^{\omega O}_{0,n}\widehat{\gamma}_5 \right\rangle \right]_U$$
$$+ m_3^2 \left[\left\langle \widehat{S}^{\eta O}_{n,0}\widehat{\gamma}_5\widehat{S}^{\eta O}_{0,n}\widehat{\gamma}_5 \right\rangle + \left\langle \widehat{S}^{\omega E}_{n,0}\widehat{\gamma}_5\widehat{S}^{\omega E}_{0,n}\widehat{\gamma}_5 \right\rangle \right]_U$$
$$+ \sum_{\boldsymbol{n}} m_3^2 \left[\left\langle \widehat{S}^{\eta O}_{0,0}\widehat{\gamma}_5 \right\rangle \left\langle \widehat{S}^{\eta O}_{n,n}\widehat{\gamma}_5 \right\rangle \right]_U$$

- **Quark-disconnected part** at leading order $\mathcal{O}(m_3^2)$, **zero in isospin limit**
- Compare with e.g. SU(2) ChPT @ NLO [J. Gasser, H. Leutwyler, 1984]

$$l_7 \left\langle \chi_- \right\rangle^2 = l_7 \left[2B_0(m_d - m_u) \right]^2 \left\langle \tau^3 \mathrm{Im}\, U \right\rangle^2$$

- **Characteristic quark-disconnected term** $\Rightarrow$ ground state is **neutral pion**

Flavored pseudoscalar mesons

Pseudoscalars with $\widehat{\Gamma}_{a,b} = \{\widehat{\gamma}_5, \gamma_0, \mathbf{1}, i\gamma_0\gamma_5\}$ (for $\gamma_0, \mathbf{1}$ as parity partners)

$$\mathbf{1}, i\gamma_0\gamma_5 \in \{\Gamma^{\eta E}\}, \quad \widehat{\gamma}_5 \in \{\Gamma^{\eta O}\}, \quad \gamma_0 \in \{\Gamma^{\omega O}\}$$

γ_0-**channel**:

$$\begin{aligned} C_{\gamma_0,\gamma_0}(n_0) &= C^{\mathrm{con}}_{\gamma_0,\gamma_0}(n_0) \\ &= -\frac{1}{2}\sum_{\boldsymbol{n}} \Bigg\{ \left[\left\langle \widehat{S}^{\eta E}_{n,0}\gamma_5 \widehat{S}^{\eta E}_{0,n}\gamma_5 \right\rangle - \left\langle \widehat{S}^{\omega O}_{n,0}\gamma_5 \widehat{S}^{\omega O}_{0,n}\gamma_5 \right\rangle \right]_U \\ &\qquad - m_3^2 \left[\left\langle \widehat{S}^{\eta O}_{n,0}\gamma_5 \widehat{S}^{\eta O}_{0,n}\gamma_5 \right\rangle - \left\langle \widehat{S}^{\omega E}_{n,0}\gamma_5 \widehat{S}^{\omega E}_{0,n}\gamma_5 \right\rangle \right]_U \Bigg\} \times (-1)^{n_0} \end{aligned}$$

- **No quark-disconnected term at all** (**charged pion**)
- Leading power m_3^2 in numerator has **opposite sign** (**different flavors**)

$$\langle \chi_- \chi_- \rangle = [2B_0(m_d - m_u)]^2 \left\langle \tau^3 \mathrm{Im}\, U \tau^3 \mathrm{Im}\, U \right\rangle^2$$

$\rightarrow$ L_8 and H_2 terms of SU(3) ChPT @ NLO [J. Gasser, H. Leutwyler, 1985]

- **Opposite sign** of $\mathcal{O}(\widehat{\omega}^2)$ terms of numerator (**different tastes**)

Flavored pseudoscalar mesons

Pseudoscalars with $\widehat{\Gamma}_{a,b} = \{\widehat{\gamma}_5, \gamma_0, \mathbf{1}, i\gamma_0\gamma_5\}$ (for $\gamma_0, \mathbf{1}$ as parity partners)

$$\mathbf{1}, i\gamma_0\gamma_5 \in \{\Gamma^{\eta E}\}, \quad \widehat{\gamma}_5 \in \{\Gamma^{\eta O}\}, \quad \gamma_0 \in \{\Gamma^{\omega O}\}$$

γ_0-**channel**: flip s $\to$ $-$s for **only one propagator**

$$\begin{aligned} C_{\gamma_0,\gamma_0}(n_0) &= C^{\text{con}}_{\gamma_0,\gamma_0}(n_0) \\ &= -\frac{1}{2}\sum_{\boldsymbol{n}} \Big\{ \Big[\Big\langle \widehat{S}^{\eta E}_{n,0}\gamma_5 \widehat{S}^{\eta E}_{0,n}\gamma_5 \Big\rangle + \Big\langle \widehat{S}^{\omega O}_{n,0}\gamma_5 \widehat{S}^{\omega O}_{0,n}\gamma_5 \Big\rangle \Big]_U \\ &\qquad - m_3^2 \Big[\Big\langle \widehat{S}^{\eta O}_{n,0}\gamma_5 \widehat{S}^{\eta O}_{0,n}\gamma_5 \Big\rangle + \Big\langle \widehat{S}^{\omega E}_{n,0}\gamma_5 \widehat{S}^{\omega E}_{0,n}\gamma_5 \Big\rangle \Big]_U \Big\} \times (-1)^{n_0} \end{aligned}$$

- **No quark-disconnected term at all** (**charged pion**)
- Leading power m_3^2 in numerator has **opposite sign** (**different flavors**)

$$\langle \chi_- \chi_- \rangle = [2B_0(m_d - m_u)]^2 \left\langle \tau^3 \mathrm{Im}\, U \tau^3 \mathrm{Im}\, U \right\rangle^2$$

$\to$ L_8 and H_2 terms of SU(3) ChPT @ NLO [J. Gasser, H. Leutwyler, 1985]

- **Same sign** of $\mathcal{O}(\widehat{\omega}^2)$ terms of numerator (**same tastes**)

Flavored pseudoscalar mesons

Pseudoscalars with $\widehat{\Gamma}_{a,b} = \{\widehat{\gamma}_5, \gamma_0, \mathbf{1}, i\gamma_0\gamma_5\}$ (for $\gamma_0, \mathbf{1}$ as parity partners)

$$\mathbf{1}, i\gamma_0\gamma_5 \in \{\Gamma^{\eta E}\}, \quad \widehat{\gamma}_5 \in \{\Gamma^{\eta O}\}, \quad \gamma_0 \in \{\Gamma^{\omega O}\}$$

$\mathbf{1}$-channel ($i\gamma_0\gamma_5 = \tau_0$):

$$\begin{aligned} C_{\mathbf{1},\mathbf{1}}(n_0) &= C^{\mathrm{con}}_{\mathbf{1},\mathbf{1}}(n_0) \\ &= -\frac{1}{2}\sum_{\boldsymbol{n}} \Bigg\{ \left[\left\langle \widehat{S}^{\eta E}_{n,0}\tau_0 \widehat{S}^{\eta E}_{0,n}\tau_0 \right\rangle - \left\langle \widehat{S}^{\omega O}_{n,0}\tau_0 \widehat{S}^{\omega O}_{0,n}\tau_0 \right\rangle \right]_U \\ &\qquad - m_3^2 \left[\left\langle \widehat{S}^{\eta O}_{n,0}\tau_0 \widehat{S}^{\eta O}_{0,n}\tau_0 \right\rangle - \left\langle \widehat{S}^{\omega E}_{n,0}\tau_0 \widehat{S}^{\omega E}_{0,n}\tau_0 \right\rangle \right]_U \Bigg\} \times (-1)^{n_0} + \ldots \end{aligned}$$

- **No quark-disconnected contribution for pseudoscalar (charged pion)**
- Same pattern of $\mathcal{O}(m_3^2)$ and $\mathcal{O}(\widehat{\omega}^2)$ terms as γ_0-channel
- NB: The **different interpolating operators** (i.e. $\gamma_0, \mathbf{1}$) for charged pions are **needed for decays** with three-point functions (e.g. $\rho_0 \to \pi^+\pi^-, \ldots$)
- NB: **Scalar states** in $\mathbf{1}$-channel receive quark-disconnected contribution

Flavored pseudoscalar mesons

Pseudoscalars with $\widehat{\Gamma}_{a,b} = \{\widehat{\gamma}_5, \gamma_0, \mathbf{1}, i\gamma_0\gamma_5\}$ (for $\gamma_0, \mathbf{1}$ as parity partners)

$$\mathbf{1}, i\gamma_0\gamma_5 \in \{\Gamma^{\eta E}\}, \quad \widehat{\gamma}_5 \in \{\Gamma^{\eta O}\}, \quad \gamma_0 \in \{\Gamma^{\omega O}\}$$

$i\gamma_0\gamma_5$**-channel** ($i\gamma_0\gamma_5 = \tau_0$):

$$\begin{aligned} C_{\tau_0,\tau_0}(n_0) &= C^{\mathrm{con}}_{\tau_0,\tau_0}(n_0) + C^{\mathrm{disc}}_{\tau_0,\tau_0}(n_0) \\ &= -\frac{1}{2}\sum_{\boldsymbol{n}} \left[\left\langle \widehat{S}^{\eta E}_{n,0}\tau_0 \widehat{S}^{\eta E}_{0,n}\tau_0 \right\rangle + \left\langle \widehat{S}^{\omega O}_{n,0}\tau_0 \widehat{S}^{\omega O}_{0,n}\tau_0 \right\rangle \right]_U \\ &\qquad + m_3^2 \left[\left\langle \widehat{S}^{\eta O}_{n,0}\tau_0 \widehat{S}^{\eta O}_{0,n}\tau_0 \right\rangle + \left\langle \widehat{S}^{\omega E}_{n,0}\tau_0 \widehat{S}^{\omega E}_{0,n}\tau_0 \right\rangle \right]_U + \dots \\ &\quad + \sum_{\boldsymbol{n}} \left[\left\langle \widehat{S}^{\eta E}_{0,0}\tau_0 \right\rangle \left\langle \widehat{S}^{\eta E}_{n,n}\tau_0 \right\rangle \right]_U \end{aligned}$$

- **Quark-disconnected part** is $\mathcal{O}(1)$, **nonzero in chiral & continuum limit**

$\Rightarrow$ **peudoscalar flavor singlet**, i.e. ground state is the **eta meson**

- NB: Quark-connected part is indistinguishable from a neutral pion.
- NB: **Scalar states** exist in quark-connected part of the $i\gamma_0\gamma_5$ channel