Step scaling in X-space: running of the quark mass

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Presentation outlook

1. X-space renormalization scheme
2. Step scaling in X-space
3. Setup and matching of lattices
4. Step scaling – results
5. Conclusions and prospects

Our previous work:

Manuscript in the final stage of preparation:
[K.C., K. Jansen, P. Korcyl, Non-perturbative running of renormalization constants from correlators in coordinate space using step scaling, on arXiv in August]
X-space scheme


- Consider the correlation functions of flavor non-singlet bilinear quark operators of the form \( \langle O_\Gamma(x)O_\Gamma(0) \rangle \), where:
  \[
  O_\Gamma(x) = \overline{\psi}(x)\Gamma\psi(x), \quad \Gamma = \{1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5\}.
  \]

- **Renormalization condition**: for every lattice spacing \( a \) and for each correlator type, impose the following in the chiral limit:
  \[
  (Z_\Gamma^X(x_0, a))^2 \langle O_\Gamma(x_0)O_\Gamma(0) \rangle_{\text{lat}} = \langle O_\Gamma(x_0)O_\Gamma(0) \rangle_{\text{freecont}},
  \]
  from which the renormalization constant \( Z_\Gamma^X(x_0, a) \) can be calculated.

- \( x_0 \) is the renormalization point, which must satisfy:
  \[
  a \ll x_0 \ll \Lambda_{\text{QCD}}^{-1}
  \]
  to keep the discretization & non-perturbative effects under control.

- 4-loop continuum PT formulae exist to convert to the \( \overline{\text{MS}} \) scheme:
  [K.G. Chetyrkin, A. Maier, Massless correlators of vector, scalar and tensor currents in position space at orders \( \alpha_s^3 \) and \( \alpha_s^4 \): Explicit analytical results, Nucl.Phys. B844, 266].
Idea of step scaling

We want to compute the running of RCs non-perturbatively on the lattice, using the step-scaling function:

$$\Sigma_G(\mu, 2\mu) = \lim_{a \to 0} \frac{Z_G(2\mu, a)}{Z_G(\mu, a)}.$$

For example:

$$\frac{Z_G(\sqrt{X^2} = 0.02 \text{ fm})}{Z_G(\sqrt{X^2} = 0.04 \text{ fm})} = \left. \frac{Z_G(\mu = 12 \text{ GeV})}{Z_G(\mu = 6 \text{ GeV})} \right|_{\beta=7.90, \beta=8.62, \beta=9.00, \beta=9.50}.$$

This corresponds to the following ratios:

$$\frac{Z_G(X=(1,1,1,1))}{Z_G(X=(2,2,2,2))} \bigg|_{\beta=7.90, L=8, 16} \quad \frac{Z_G(X=(2,2,2,2))}{Z_G(X=(4,4,4,4))} \bigg|_{\beta=8.62, L=16, 32}$$

$$\frac{Z_G(X=(3,3,3,3))}{Z_G(X=(6,6,6,6))} \bigg|_{\beta=9.00, L=24, 48} \quad \frac{Z_G(X=(4,4,4,4))}{Z_G(X=(8,8,8,8))} \bigg|_{\beta=9.50, L=32, 64}$$
We consider 3 types of points:

- \((\frac{x}{a}, \frac{x}{a}, \frac{x}{a}, \frac{x}{a})\) – call it type IV,
- \((\frac{x}{a}, \frac{x}{a}, \frac{x}{a}, 0)\) (4 permutations) – type III,
- \((\frac{x}{a}, \frac{x}{a}, 0, 0)\) (6 permutations) – type II.

Allow to obtain 3 scale changes for a given step scaling step, e.g.:

- 8.360 to 16.719 GeV (type II),
- 6.826 to 13.651 GeV (type III),
- 5.911 to 11.822 GeV (type IV)

for the first step.

We always take \(x/L = 1/8\) (maximal to have FVE under control).
Matching of lattices

To compute the continuum limit of the step-scaling function, we need to properly match lattices. For this, we use the Creutz ratio:

\[ R_C(l, t) \equiv \frac{W(l, t) W(l + a, t - a)}{W(l, t + a) W(l + a, t)}, \]

where \( W(l, t) \) is a Wilson-loop with spatial size \( l \) and temporal size \( t \).

The Creutz ratio can be used to define the renormalized coupling. Plugging \( W(l, t) \propto e^{-\alpha t/l} \) (for \( t \gg 1 \)), we get:

\[ R(l, t) = e^{-\alpha \frac{t}{l+a}}. \]

Hence, the effective coupling can be defined as:

\[ \alpha_{\text{eff}} = -\frac{l}{a} \frac{l + a}{a} \ln R_C(l, t). \]

For \( l \), use always \( l = L/4 \).

[Acknowledgment: code to compute Creutz ratios from Pan Kessel.]
Example of $\alpha_{\text{eff}}$ extraction

Result: $\alpha_{\text{eff}} = 0.1444(32)_{\text{stat}}(9)_{\text{sys}}$. 
Matching

Consider the ensemble: $\beta = 6.92, L/a = 24, T/a = 96$, with effective coupling $\alpha_{\text{eff}} = 0.418(8)(3)$.

First, match $\alpha_{\text{eff}}$ on $16^3 \times 64$ lattices:

From this, we know that $\beta = 6.618(17)$ is matched (has the same $\alpha_{\text{eff}}$) to our $\beta = 6.92$ ensemble.

Result from formula by [S. Necco, R. Sommer, Nucl.Phys. B622, 328] that gives $r_0/a$ vs. $\beta$ for the range $\beta \in [5.7, 6.92]: \beta = 6.613 \Rightarrow \text{CONSISTENT!}$
Finally, we get the following set of matched ensembles:

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.50(7)</td>
<td>0.0024</td>
<td>9.00</td>
<td>0.0031</td>
<td>8.62(7)</td>
<td>0.0047</td>
<td>7.90(13)</td>
<td>0.0094</td>
</tr>
<tr>
<td>2</td>
<td>8.62(7)</td>
<td>0.0047</td>
<td>8.24(6)</td>
<td>0.0063</td>
<td>7.90(13)</td>
<td>0.0094</td>
<td>7.18(2)</td>
<td>0.0188</td>
</tr>
<tr>
<td>3</td>
<td>7.90(13)</td>
<td>0.0094</td>
<td>7.56(11)</td>
<td>0.0125</td>
<td>7.18(2)</td>
<td>0.0188</td>
<td>6.61(2)</td>
<td>0.0376</td>
</tr>
</tbody>
</table>

Details:

- quenched ensembles generated with Chroma (Wilson plaquette gauge action, heatbath with overrelaxation),
- valence quarks – $N_f = 2$ twisted mass tuned to maximal twist,
- inversions done using tmLQCD inverter,
- always 3 valence quark masses, extrapolate to the chiral limit,
- 200 configurations (1000 for $L/a = 8$) with a step of 40 updates.
Systematic effects

For a reliable continuum result, we need to take into account systematic effects:

1. chiral extrapolation (linear in $a\mu$, 3 masses),
2. finite volume effects (next slide),
3. choice of the continuum extrapolation fitting ansatz:

   \[
   \begin{align*}
   (A) \quad & \Sigma \Gamma(\mu, 2\mu, a)_{\text{corrected}} = \Sigma \Gamma(\mu, 2\mu)_{\text{cont}, A} + c_A a^2, \\
   (B) \quad & \Sigma \Gamma(\mu, 2\mu, a)_{\text{corrected}} = \Sigma \Gamma(\mu, 2\mu)_{\text{cont}, B} + c_B a^2 + d_B a^4, \\
   (C1) \quad & \Sigma \Gamma(\mu, 2\mu, a)_{\text{corrected}} = \Sigma \Gamma(\mu, 2\mu)_{\text{cont}, C} + c_{C1} a^2, \\
   (C2) \quad & \Sigma \Gamma(\mu, 2\mu, a)_{\text{non-corrected}} = \Sigma \Gamma(\mu, 2\mu)_{\text{cont}, C} + c_{C2} a^2, \\
   (D1) \quad & \Sigma \Gamma(\mu, 2\mu, a)_{\text{corrected}} = \Sigma \Gamma(\mu, 2\mu)_{\text{cont}, D} + c_{D1} a^2 + d_{D1} a^4, \\
   (D2) \quad & \Sigma \Gamma(\mu, 2\mu, a)_{\text{non-corrected}} = \Sigma \Gamma(\mu, 2\mu)_{\text{cont}, D} + c_{D2} a^2 + d_{D2} a^4, \\
   \end{align*}
   \]

4. effects from non-ideal matching – estimate derivatives like:

   \[
   \left. \frac{\partial \Sigma \Gamma(\mu, 2\mu, a(\beta))}{\partial \beta} \right|_{\beta, \beta_-} \approx \frac{\Sigma \Gamma(\mu, 2\mu, a(\beta)) - \Sigma \Gamma(\mu, 2\mu, a(\beta_-))}{\beta - \beta_-},
   \]

   Our estimate of the matching uncertainty is then obtained as the product of

   \[
   \left. \frac{\partial \Sigma \Gamma(\mu, 2\mu, a)}{\partial \beta} \right|_{\beta, \beta_-} \quad \text{or} \quad \left. \frac{\partial \Sigma \Gamma(\mu, 2\mu, a)}{\partial \beta} \right|_{\beta, \beta_+} \quad \text{and} \quad \Delta \beta.
   \]

5. choice of the value of $\Lambda_{\overline{\text{MS}}}$ = 238(19) MeV, $r_0$ = 0.48(2) fm.
Finite volume effects

Work at fixed lattice spacing $\beta = 7.90$, which has 4 volumes available: $L/a = 8, 16, 32, 64$. Change pairs of lattice sizes for a fixed X-space point.

FVE small for $x/L = 1/8$ apart from pair of volumes $(8, 16)$ – hence prefer fitting ansätze A or C to 3 finest lattice spacings (4th one to estimate systematic effects – fits B or D).
Summary of our procedure

1. Compute the relevant correlation functions in X-space, at three values of the valence quark mass.
2. Extrapolate to the chiral limit.
3. Apply the tree-level correction.
4. Use the chirally extrapolated and tree-level corrected values of correlators to compute the step scaling function $\Sigma_\Gamma(\mu, 2\mu, a(\beta))$.
5. Extrapolate to the continuum, using fit C or A (depending whether non-corrected data points have cut-off effects under control).
6. Convert from the X-space to the $\overline{\text{MS}}$ renormalization scheme.
7. Calculate systematic uncertainties:
   a. from the fitting ansatz (by comparing fits C/D or A/B),
   b. non-ideal matching,
   c. uncertainties of $\Lambda^{(0)}_{\overline{\text{MS}}}$ and $r_0$. 

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Chiral extrapolations

Example – SS correlator, scale change from 5.911 to 11.822 GeV

\[ \beta = 7.90, \ L/a = 8, \ 16 \]
\[ (1, 1, 1, 1) \]
\[ (2, 2, 2, 2) \]

\[ \beta = 8.62, \ L/a = 16, \ 32 \]
\[ (2, 2, 2, 2) \]
\[ (4, 4, 4, 4) \]

\[ \beta = 9.00, \ L/a = 24, \ 48 \]
\[ (3, 3, 3, 3) \]
\[ (6, 6, 6, 6) \]

\[ \beta = 9.50, \ L/a = 32, \ 64 \]
\[ (4, 4, 4, 4) \]
\[ (8, 8, 8, 8) \]
SS correlator

scale change:

from 5.911 GeV
to 11.822 GeV

i.e. \( \frac{Z_S(\mu=11.822 \text{ GeV})}{Z_S(\mu=5.911 \text{ GeV})} \)

Hence, our final estimate of the step scaling function (fit C) in the \( \overline{\text{MS}} \) scheme is:

\[
\Sigma_S(5.911 \text{ GeV}, 11.822 \text{ GeV}) = 1.0728(71)_{\text{stat}}(30)_{\text{fit}}(3)_{\text{match}}(2)_{\Lambda_{\overline{\text{MS}}}^{(0)}(9) r_0},
\]

The obtained value can be compared to the prediction of PT:

\[
\Sigma_S(5.911 \text{ GeV}, 11.822 \text{ GeV})_{\text{PT}} = 1.0713(18),
\]

where the uncertainty comes from the uncertainty of \( \Lambda_{\overline{\text{MS}}}^{(0)} \).
Full results – PP/SS

Results for the step scaling function $\Sigma_{P/S}(\mu, 2\mu)$, determined from the PP and SS correlators.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$2\mu$</th>
<th>point</th>
<th>4-loop PT ($N_f = 0$)</th>
<th>lattice PP</th>
<th>lattice SS</th>
</tr>
</thead>
<tbody>
<tr>
<td>[GeV]</td>
<td>[GeV]</td>
<td>type</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Uncertainties:
1. statistical,
2. from the fitting ansatz,
3. from non-ideal matching,
4. from $\Lambda_{\overline{MS}}^{(0)}$,
5. from $r_0$. 
Results for the step scaling function $\Sigma_{A/V}(\mu, 2\mu)$, determined from the VV and AA correlators.

<table>
<thead>
<tr>
<th>$\mu$ [GeV]</th>
<th>$2\mu$ [GeV]</th>
<th>point type</th>
<th>continuum result</th>
<th>lattice VV</th>
<th>lattice AA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.478</td>
<td>2.956</td>
<td>IV</td>
<td>1.0</td>
<td>0.9918(103)(1)(15)(2)(1)</td>
<td>0.9931(86)(30)(9)(2)(1)</td>
</tr>
<tr>
<td>1.706</td>
<td>3.413</td>
<td>III</td>
<td>1.0</td>
<td>0.9968(108)(6)(22)(2)(1)</td>
<td>0.9987(87)(20)(16)(2)(1)</td>
</tr>
<tr>
<td>2.090</td>
<td>4.180</td>
<td>II</td>
<td>1.0</td>
<td>1.0127(99)(15)(24)(1)(1)</td>
<td>0.9683(69)(61)(23)(1)(1)</td>
</tr>
<tr>
<td>2.956</td>
<td>5.911</td>
<td>IV</td>
<td>1.0</td>
<td>1.0061(87)(10)(6)(1)(1)</td>
<td>1.0039(70)(12)(6)(1)(1)</td>
</tr>
<tr>
<td>3.413</td>
<td>6.826</td>
<td>III</td>
<td>1.0</td>
<td>1.0122(92)(2)(10)(1)(0)</td>
<td>1.0092(74)(19)(8)(1)(0)</td>
</tr>
<tr>
<td>4.180</td>
<td>8.360</td>
<td>II</td>
<td>1.0</td>
<td>1.0273(85)(22)(12)(1)(0)</td>
<td>0.9848(59)(60)(8)(1)(0)</td>
</tr>
<tr>
<td>5.911</td>
<td>11.822</td>
<td>IV</td>
<td>1.0</td>
<td>1.0017(69)(12)(0)(1)(0)</td>
<td>1.0009(58)(6)(2)(1)(0)</td>
</tr>
<tr>
<td>6.826</td>
<td>13.651</td>
<td>III</td>
<td>1.0</td>
<td>1.0103(77)(12)(2)(1)(0)</td>
<td>1.0085(63)(27)(3)(1)(0)</td>
</tr>
<tr>
<td>8.360</td>
<td>16.719</td>
<td>II</td>
<td>1.0</td>
<td>1.0125(73)(14)(8)(1)(0)</td>
<td>0.9823(52)(53)(1)(1)(0)</td>
</tr>
</tbody>
</table>

Uncertainties:
1. statistical,
2. from the fitting ansatz,
3. from non-ideal matching,
4. from $\Lambda_{\text{MS}}^{(0)}$,
5. from $r_0$. 

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Full running from X-space

![Graph 1](continuum perturbation theory)

- lattice from X-space, PP correlator
- lattice from X-space, SS correlator

![Graph 2](continuum)

- lattice from X-space, AA correlator
- lattice from X-space, VV correlator

Continuum perturbation theory
Summary and prospects

- We have computed the running of RCs in the X-space scheme for quenched ensembles.

- The range of scales between approx. 1.5 and 17 GeV.

- X-space step scaling feasible, but not ideally suited for step scaling.

- Satisfactory agreement with perturbation theory.

- Nevertheless, to make the approach more precise and reliable, one needs to understand better hypercubic artefacts.

Thank you for attention!
Our starting point – ensemble:

$$\beta = 9.00, \ L/a = 24, \ T/a = 96,$$

with effective coupling $$\alpha_{\text{eff}} = 0.1461(32)(15)$$. First, match $$\alpha_{\text{eff}}$$ on $$16^3 \times 64$$ lattices:

From this, we know that $$\beta = 8.62(7)$$ is matched (has the same $$\alpha_{\text{eff}}$$) to our starting ensemble.
Systematic error extraction

Result: \( \alpha_{\text{eff}} = 0.1444(32)_{\text{stat}}(9)_{\text{sys}}. \)
Illustration of tuning to maximal twist for the ensemble: \( \beta = 7.18, \frac{L}{a} = 16, \frac{T}{a} = 64, a\mu = 0.0032 \).