Step scaling in X-space: running of the quark mass

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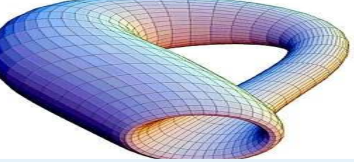


Presentation outlook

1. X-space renormalization scheme
2. Step scaling in X-space
3. Setup and matching of lattices
4. Step scaling – results
5. Conclusions and prospects

Our previous work:

[K.C., K. Jansen, P. Korcyl, *Non-perturbative renormalization in coordinate space for Nf=2 maximally twisted mass fermions with tree-level Symanzik improved gauge action*, Nucl.Phys. B865, 268]

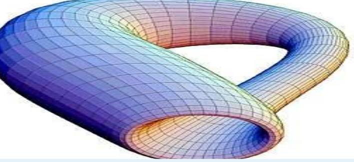
Manuscript in the final stage of preparation:

[K.C., K. Jansen, P. Korcyl, *Non-perturbative running of renormalization constants from correlators in coordinate space using step scaling*, on arXiv in August]

X-space scheme

[V. Gimenez, L. Giusti, S. Guerriero, V. Lubicz, G. Martinelli, S. Petrarca, J. Reyes, B. Taglienti *et al.*, *Non-perturbative renormalization of lattice operators in coordinate space*, Phys.Lett. **B598** (2004) 227-236, hep-lat/0406019]

- Consider the correlation functions of flavor non-singlet bilinear quark operators of the form $\langle \mathcal{O}_\Gamma(x)\mathcal{O}_\Gamma(0)\rangle$, where:
$$\mathcal{O}_\Gamma(x) = \bar{\psi}(x)\Gamma\psi(x), \qquad \Gamma = \{1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5\}.$$
- **Renormalization condition**: for every lattice spacing a and for each correlator type, impose the following in the chiral limit:
$$\left(Z_\Gamma^X(x_0,a)\right)^2 \langle \mathcal{O}_\Gamma(x_0)\mathcal{O}_\Gamma(0)\rangle_{\rm lat} = \langle \mathcal{O}_\Gamma(x_0)\mathcal{O}_\Gamma(0)\rangle_{\rm cont}^{\rm free},$$
from which the renormalization constant $Z_\Gamma^X(x_0,a)$ can be calculated.
- x_0 is the renormalization point, which must satisfy:
$$a \ll x_0 \ll \Lambda_{\rm QCD}^{-1}$$
to keep the discretization & non-perturbative effects under control.
- 4-loop continuum PT formulae exist to convert to the $\overline{\rm MS}$ scheme:
[K.G. Chetyrkin, A. Maier, *Massless correlators of vector, scalar and tensor currents in position space at orders α_s^3 and α_s^4: Explicit analytical results*, Nucl.Phys. B844, 266].



Idea of step scaling

We want to compute the running of RCs non-perturbatively on the lattice, using the step-scaling function:

$$\Sigma_\Gamma(\mu, 2\mu) = \lim_{a\to 0} \frac{Z_\Gamma(2\mu, a)}{Z_\Gamma(\mu, a)}.$$

For example:

$$\frac{Z_\Gamma(\sqrt{X^2} = 0.02\,\text{fm})}{Z_\Gamma(\sqrt{X^2} = 0.04\,\text{fm})} = \frac{Z_\Gamma(\mu = 12\,\text{GeV})}{Z_\Gamma(\mu = 6\,\text{GeV})}\Bigg|_{\beta=7.90}, \Bigg|_{\beta=8.62}, \Bigg|_{\beta=9.00}, \Bigg|_{\beta=9.50}.$$

This corresponds to the following ratios:

$$\frac{Z_\Gamma(X{=}(1,1,1,1))}{Z_\Gamma(X{=}(2,2,2,2))}\Bigg|_{\beta=7.90,\, L=8,\, 16} \qquad \frac{Z_\Gamma(X{=}(2,2,2,2))}{Z_\Gamma(X{=}(4,4,4,4))}\Bigg|_{\beta=8.62,\, L=16,\, 32}$$

$$\frac{Z_\Gamma(X{=}(3,3,3,3))}{Z_\Gamma(X{=}(6,6,6,6))}\Bigg|_{\beta=9.00,\, L=24,\, 48} \qquad \frac{Z_\Gamma(X{=}(4,4,4,4))}{Z_\Gamma(X{=}(8,8,8,8))}\Bigg|_{\beta=9.50,\, L=32,\, 64}$$

Types of points in X-space

We consider 3 types of points:

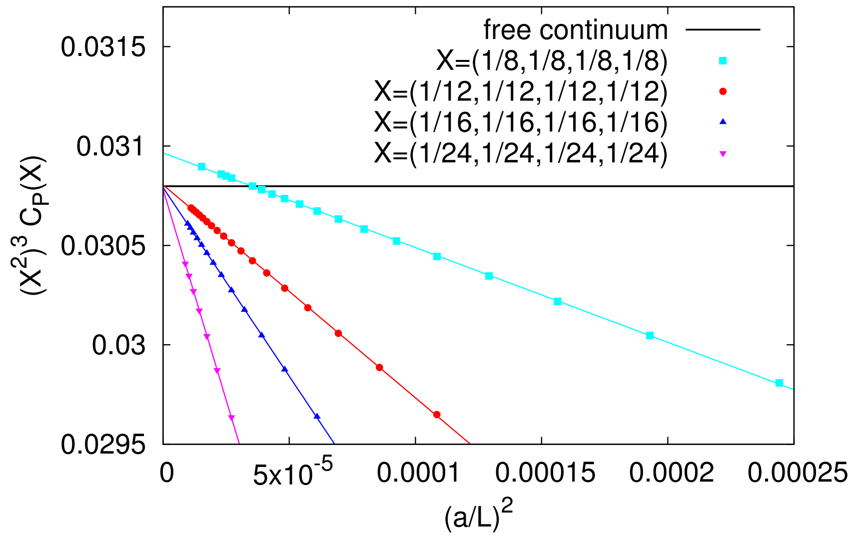
- $(\frac{x}{a}, \frac{x}{a}, \frac{x}{a}, \frac{x}{a})$ – call it type IV,
- $(\frac{x}{a}, \frac{x}{a}, \frac{x}{a}, 0)$ (4 permutations) – type III,
- $(\frac{x}{a}, \frac{x}{a}, 0, 0)$ (6 permutations) – type II.

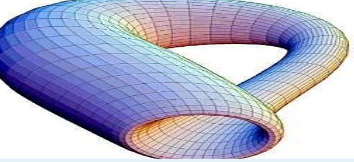
Allow to obtain 3 scale changes for a given step scaling step, e.g.:

- 8.360 to 16.719 GeV (type II),
- 6.826 to 13.651 GeV (type III),
- 5.911 to 11.822 GeV (type IV)

for the first step.

We always take $x/L = 1/8$
(maximal to have FVE under control).





Matching of lattices

To compute the continuum limit of the step-scaling function, we need to properly match lattices. For this, we use the Creutz ratio:

$$R_C(l,t) \equiv \frac{W(l,t)W(l+a,t-a)}{W(l,t+a)W(l+a,t)},$$

where $W(l,t)$ is a Wilson-loop with spatial size l and temporal size t.
The Creutz ratio can be used to define the renormalized coupling. Plugging $W(l,t) \propto e^{-\alpha t/l}$ (for $t \gg 1$), we get:

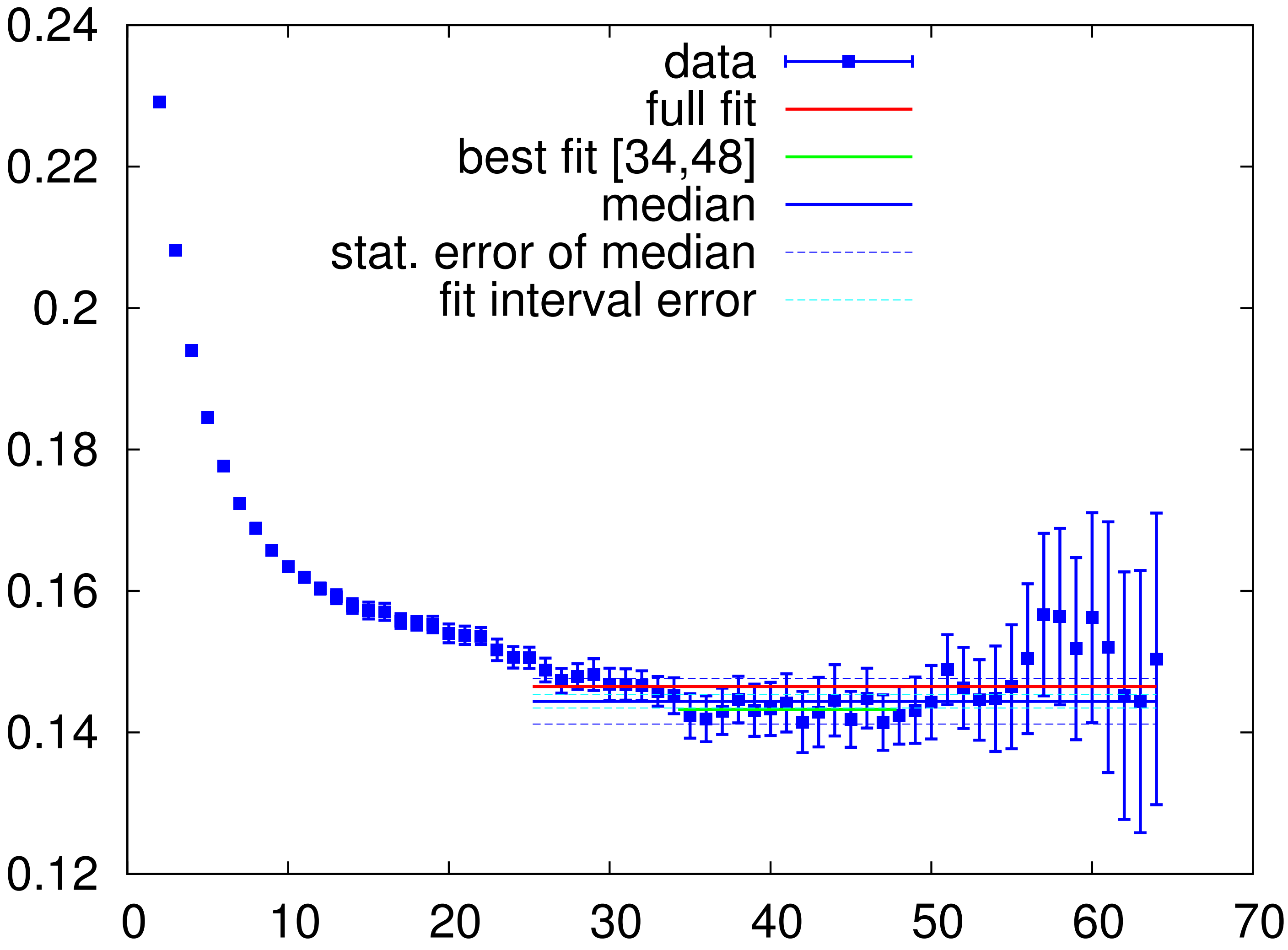
$$R_{(}l,t) = e^{-\alpha \frac{a}{l} \frac{a}{l+a}}.$$

Hence, the effective coupling can be defined as:

$$\alpha_{\text{eff}} = -\frac{l}{a}\frac{l+a}{a} \ln R_C(l,t).$$

For l, use always $l = L/4$.

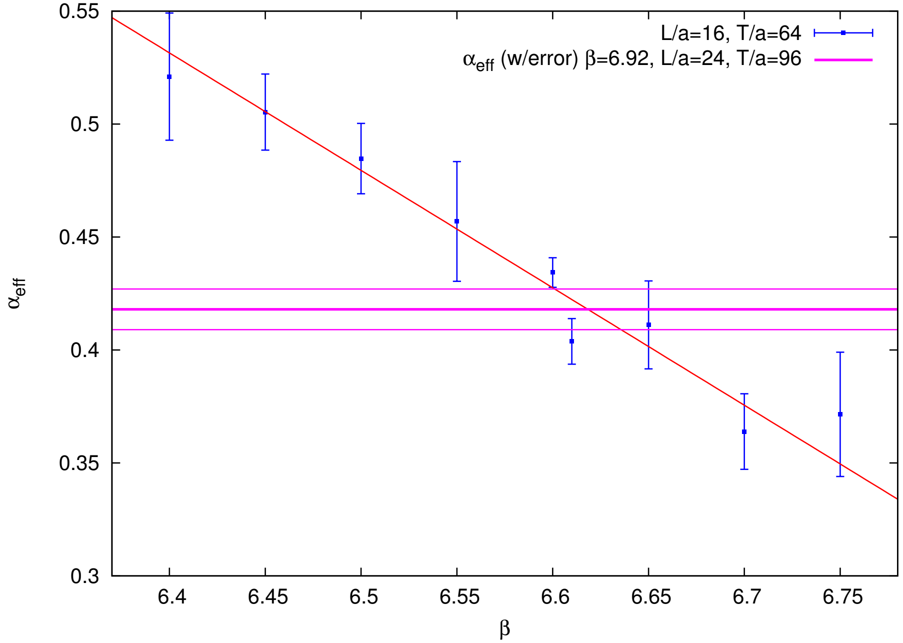
[Acknowledgment: code to compute Creutz ratios from Pan Kessel.]



Result: $\alpha_{\text{eff}} = 0.1444(32)_{\text{stat}}(9)_{\text{sys}}$.

Consider the ensemble: $\beta = 6.92$, $L/a = 24$, $T/a = 96$, with effective coupling $\alpha_{\mathrm{eff}} = 0.418(8)(3)$.

First, match α_{eff} on $16^3 \times 64$ lattices:



From this, we know that $\beta = 6.618(17)$ is matched (has the same α_{eff}) to our $\beta = 6.92$ ensemble.

Result from formula by [S. Necco, R. Sommer, Nucl.Phys. B622, 328] that gives r_0/a vs. β for the range $\beta \in [5.7, 6.92]$: $\beta = 6.613 \Rightarrow$ **CONSISTENT!**

Results of matching

Finally, we get the following set of matched ensembles:

step	32/64	a [fm]	24/48	a [fm]	16/32	a [fm]	8/16	a [fm]
1	9.50(7)	0.0024	9.00	0.0031	8.62(7)	0.0047	7.90(13)	0.0094
2	8.62(7)	0.0047	8.24(6)	0.0063	7.90(13)	0.0094	7.18(2)	0.0188
3	7.90(13)	0.0094	7.56(11)	0.0125	7.18(2)	0.0188	6.61(2)	0.0376

Details:

- quenched ensembles generated with Chroma (Wilson plaquette gauge action, heatbath with overrelaxation),
- valence quarks – $N_f = 2$ twisted mass tuned to maximal twist,
- inversions done using tmLQCD inverter,
- always 3 valence quark masses, extrapolate to the chiral limit,
- 200 configurations (1000 for $L/a = 8$) with a step of 40 updates.

Systematic effects

For a reliable continuum result, we need to take into account systematic effects:

1. chiral extrapolation (linear in $a\mu$, 3 masses),
2. finite volume effects (next slide),
3. choice of the continuum extrapolation fitting ansatz:

$$\text{(A)} \quad \Sigma_\Gamma(\mu, 2\mu, a)_{\text{corrected}} = \Sigma_\Gamma(\mu, 2\mu)_{\text{cont,A}} + c_{\text{A}} a^2,$$

$$\text{(B)} \quad \Sigma_\Gamma(\mu, 2\mu, a)_{\text{corrected}} = \Sigma_\Gamma(\mu, 2\mu)_{\text{cont,B}} + c_{\text{B}} a^2 + d_{\text{B}} a^4,$$

$$\text{(C1)} \quad \Sigma_\Gamma(\mu, 2\mu, a)_{\text{corrected}} = \Sigma_\Gamma(\mu, 2\mu)_{\text{cont,C}} + c_{\text{C1}} a^2,$$
$$\text{(C2)} \quad \Sigma_\Gamma(\mu, 2\mu, a)_{\text{non-corrected}} = \Sigma_\Gamma(\mu, 2\mu)_{\text{cont,C}} + c_{\text{C2}} a^2,$$

$$\text{(D1)} \quad \Sigma_\Gamma(\mu, 2\mu, a)_{\text{corrected}} = \Sigma_\Gamma(\mu, 2\mu)_{\text{cont,D}} + c_{\text{D1}} a^2 + d_{\text{D1}} a^4,$$
$$\text{(D2)} \quad \Sigma_\Gamma(\mu, 2\mu, a)_{\text{non-corrected}} = \Sigma_\Gamma(\mu, 2\mu)_{\text{cont,D}} + c_{\text{D2}} a^2 + d_{\text{D2}} a^4,$$

4. effects from non-ideal matching – estimate derivatives like:

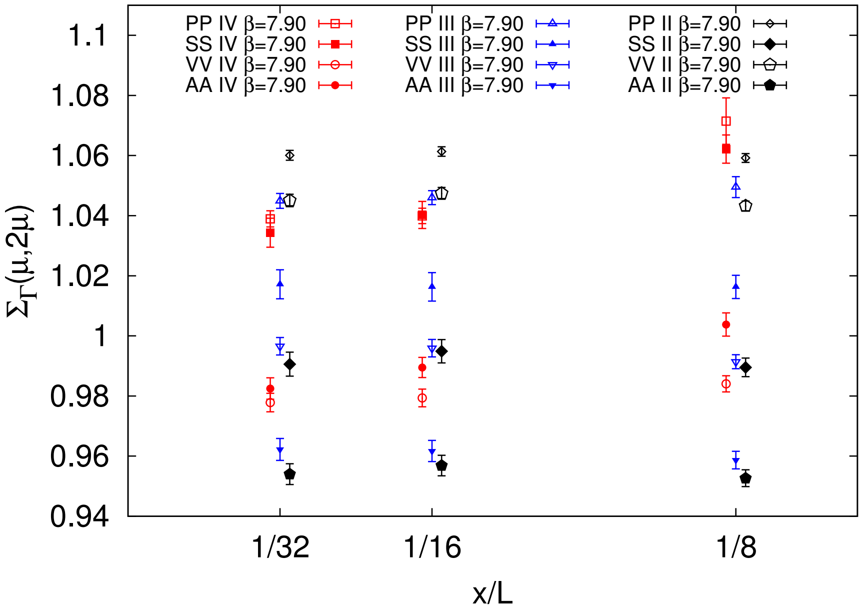
$$\left.\frac{\partial \Sigma_\Gamma(\mu, 2\mu, a(\beta))}{\partial \beta}\right|_{\beta,\beta_-} \approx \frac{\Sigma_\Gamma(\mu, 2\mu, a(\beta)) - \Sigma_\Gamma(\mu, 2\mu, a(\beta_-))}{\beta - \beta_-},$$

   Our estimate of the matching uncertainty is then obtained as the product of $\left.\frac{\partial \Sigma_\Gamma(\mu,2\mu,a)}{\partial \beta}\right|_{\beta,\beta_-}$ or $\left.\frac{\partial \Sigma_\Gamma(\mu,2\mu,a)}{\partial \beta}\right|_{\beta_+,\beta}$ and $\Delta\beta$.

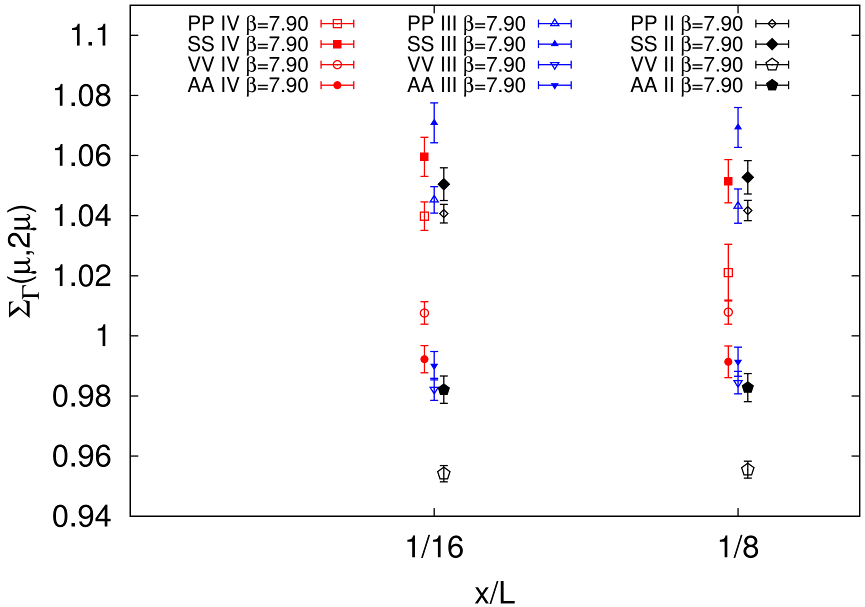
5. choice of the value of $\Lambda_{\overline{\text{MS}}} = 238(19)$ MeV, $r_0 = 0.48(2)$ fm.

Finite volume effects

Work at fixed lattice spacing $\beta = 7.90$, which has 4 volumes available:
$L/a = 8,\ 16,\ 32,\ 64$. Change pairs of lattice sizes for a **fixed** X-space point.



$x/a = 1$ and $x/a = 2$
on $(8, 16)$, $(16, 32)$, $32, 64$ lattices
thus x/L changes from $1/8$ to $1/16$ to $1/32$



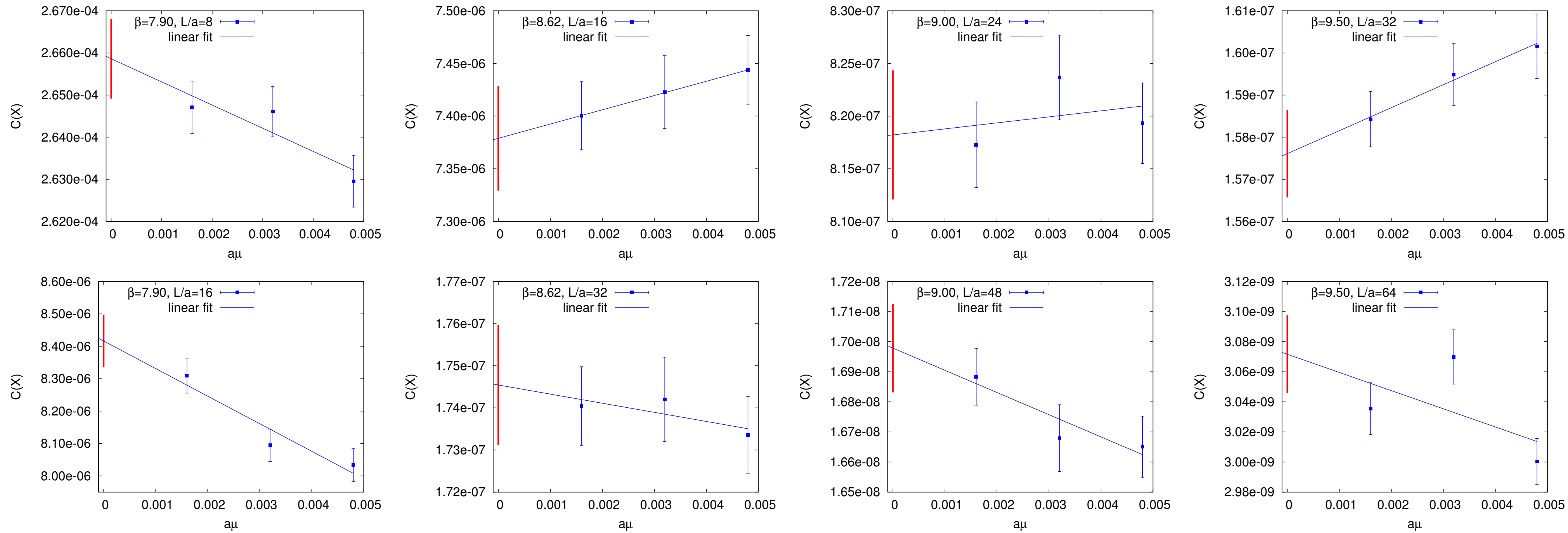
$x/a = 2$ and $x/a = 4$
on $(16, 32)$, $32, 64$ lattices
thus x/L changes from $1/8$ to $1/16$

FVE small for $x/L = 1/8$ apart from pair of volumes $(8,\ 16)$ – hence prefer fitting ansätze **A** or **C** to 3 finest lattice spacings (4th one to estimate systematic effects – fits **B** or **D**).

Summary of our procedure

1. Compute the relevant correlation functions in X-space, at three values of the valence quark mass.
2. Extrapolate to the chiral limit.
3. Apply the tree-level correction.
4. Use the chirally extrapolated and tree-level corrected values of correlators to compute the step scaling function $\Sigma_\Gamma(\mu, 2\mu, a(\beta))$.
5. Extrapolate to the continuum, using fit **C** or **A** (depending whether non-corrected data points have cut-off effects under control).
6. Convert from the X-space to the $\overline{\mathrm{MS}}$ renormalization scheme.
7. Calculate systematic uncertainties:
   a. from the fitting ansatz (by comparing fits **C**/**D** or **A**/**B**),
   b. non-ideal matching,
   c. uncertainties of $\Lambda^{(0)}_{\overline{\mathrm{MS}}}$ and r_0.

Example – SS correlator, scale change from 5.911 to 11.822 GeV



$\beta = 7.90$	$\beta = 8.62$	$\beta = 9.00$	$\beta = 9.50$
$L/a = 8,\ 16$	$L/a = 16,\ 32$	$L/a = 24,\ 48$	$L/a = 32,\ 64$
$(1,1,1,1)$	$(2,2,2,2)$	$(3,3,3,3)$	$(4,4,4,4)$
$(2,2,2,2)$	$(4,4,4,4)$	$(6,6,6,6)$	$(8,8,8,8)$

Continuum extrapolation – Σ_S

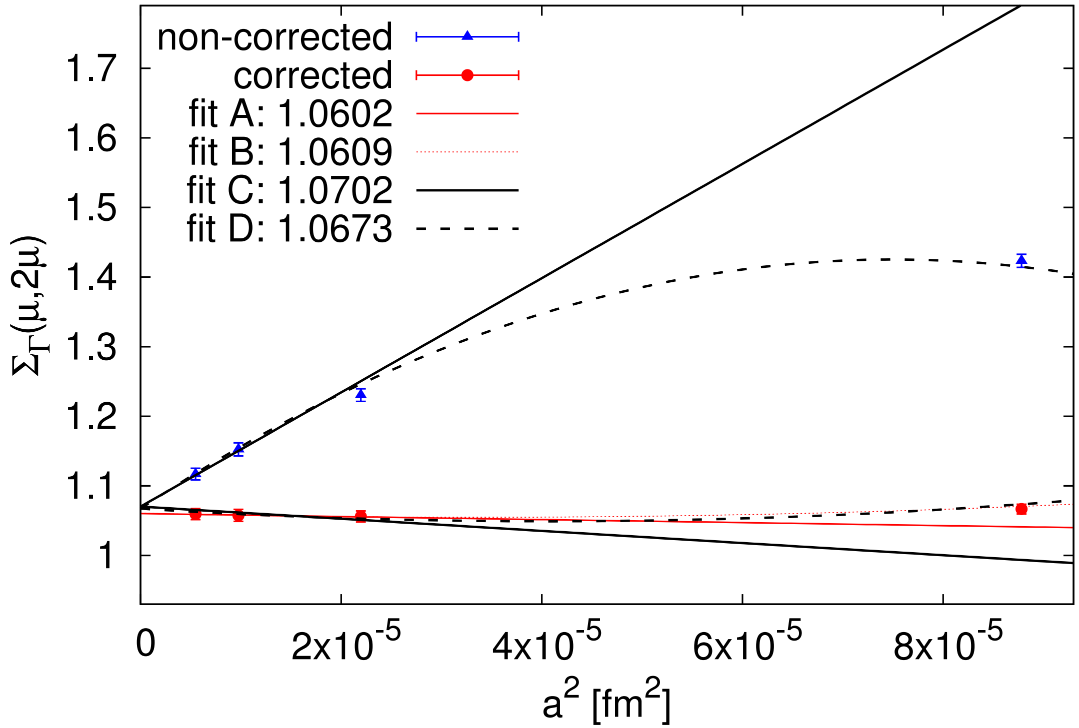
SS correlator

scale change:

from 5.911 GeV

to 11.822 GeV

i.e. $\frac{Z_S(\mu=11.822\,\text{GeV})}{Z_S(\mu=5.911\,\text{GeV})}$



Hence, our final estimate of the step scaling function (fit **C**) in the $\overline{\text{MS}}$ scheme is:

$$\Sigma_S(5.911\,\text{GeV},\,11.822\text{GeV}) = 1.0728(71)_{\text{stat}}(30)_{\text{fit}}(3)_{\text{match.}}(2)_{\Lambda^{(0)}_{\overline{\text{MS}}}}(9)_{r_0},$$

The obtained value can be compared to the prediction of PT:

$$\Sigma_S(5.911\,\text{GeV},\,11.822\text{GeV})_{\text{PT}} = 1.0713(18),$$

where the uncertainty comes from the uncertainty of $\Lambda^{(0)}_{\overline{\text{MS}}}$.

Full results – PP/SS

Results for the step scaling function $\Sigma_{P/S}(\mu, 2\mu)$, determined from the PP and SS correlators.

μ [GeV]	2μ [GeV]	point type	4-loop PT ($N_f = 0$)	lattice PP	lattice SS
1.478	2.956	IV	1.1318(68)	1.0995(104)(66)(33)(13)(37)	1.1134(121)(56)(37)(13)(37)
1.706	3.413	III	1.1206(56)	1.1027(91)(19)(36)(10)(29)	1.1210(115)(6)(41)(11)(29)
2.090	4.180	II	1.1080(44)	1.1012(101)(33)(49)(8)(23)	1.1337(140)(13)(52)(8)(23)
2.956	5.911	IV	1.0919(31)	1.0787(81)(31)(21)(4)(16)	1.0856(90)(27)(21)(5)(16)
3.413	6.826	III	1.0866(27)	1.0743(72)(18)(19)(4)(14)	1.0846(90)(14)(18)(4)(14)
4.180	8.360	II	1.0802(23)	1.0691(78)(22)(23)(3)(12)	1.0961(109)(1)(22)(3)(12)
5.911	11.822	IV	1.0713(18)	1.0721(57)(22)(3)(2)(9)	1.0728(71)(30)(3)(2)(9)
6.826	13.651	III	1.0682(16)	1.0736(57)(35)(4)(2)(9)	1.0802(73)(8)(5)(2)(9)
8.360	16.719	II	1.0643(15)	1.0571(65)(24)(5)(2)(8)	1.0755(91)(1)(7)(1)(8)

Uncertainties:

1. statistical,
2. from the fitting ansatz,
3. from non-ideal matching,
4. from $\Lambda_{\overline{\rm MS}}^{(0)}$,
5. from r_0.

Full results – AA/VV

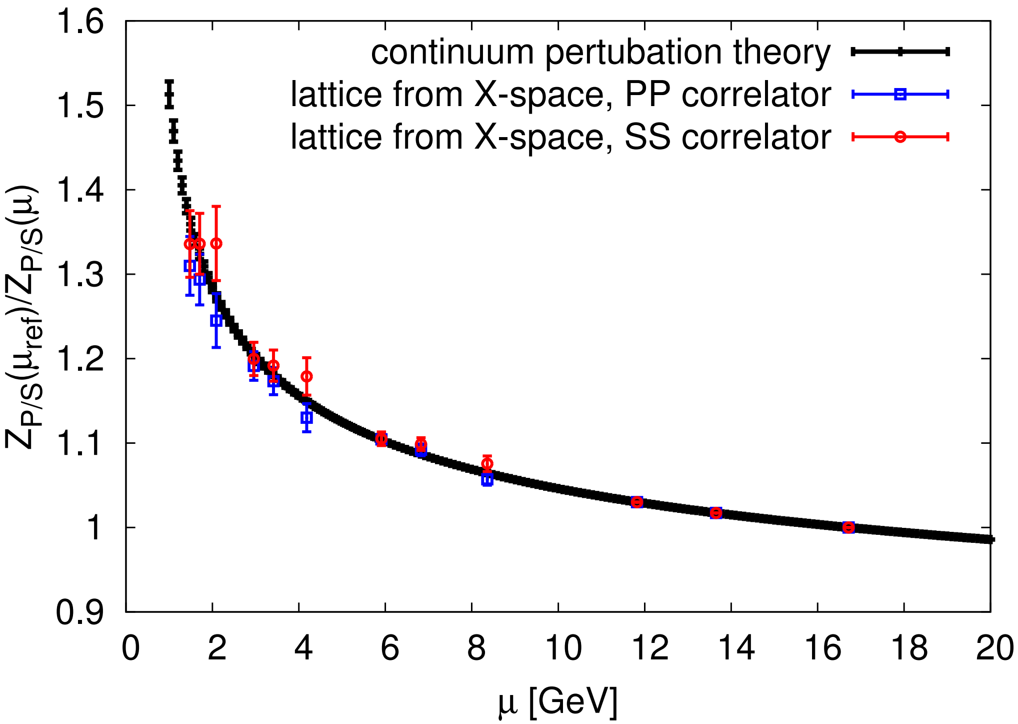
Results for the step scaling function $\Sigma_{A/V}(\mu, 2\mu)$, determined from the VV and AA correlators.

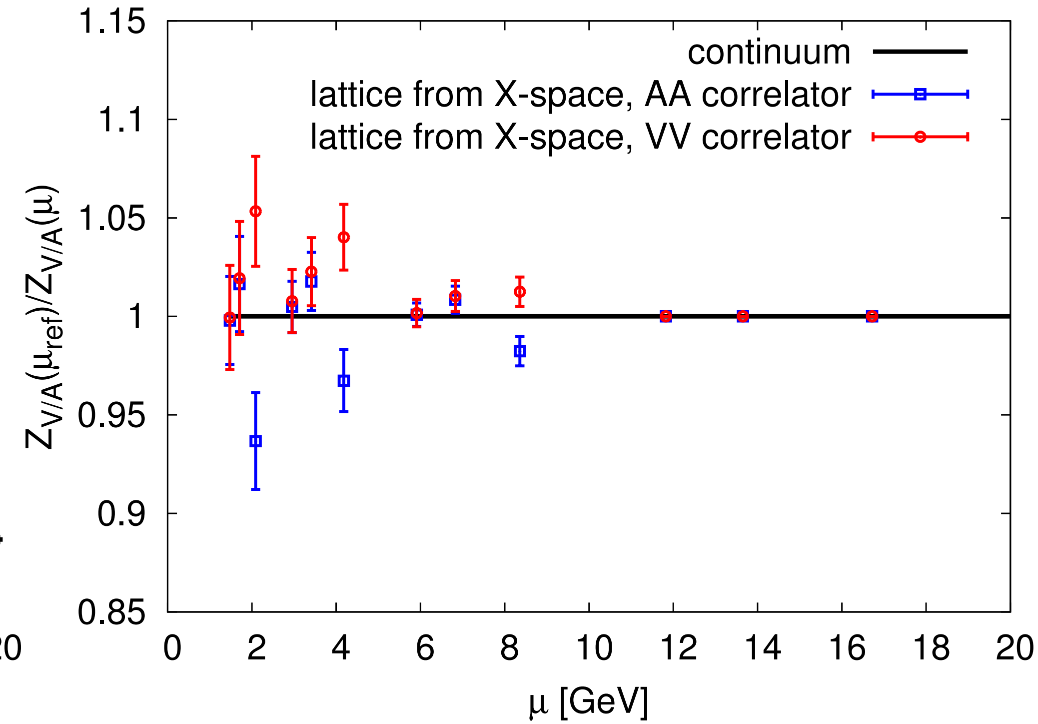
μ [GeV]	2μ [GeV]	point type	continuum result	lattice VV	lattice AA
1.478	2.956	IV	1.0	0.9918(103)(1)(15)(2)(1)	0.9931(86)(30)(9)(2)(1)
1.706	3.413	III	1.0	0.9968(108)(6)(22)(2)(1)	0.9987(87)(20)(16)(2)(1)
2.090	4.180	II	1.0	1.0127(99)(15)(24)(1)(1)	0.9683(69)(61)(23)(1)(1)
2.956	5.911	IV	1.0	1.0061(87)(10)(6)(1)(1)	1.0039(70)(12)(6)(1)(1)
3.413	6.826	III	1.0	1.0122(92)(2)(10)(1)(0)	1.0092(74)(19)(8)(1)(0)
4.180	8.360	II	1.0	1.0273(85)(22)(12)(1)(0)	0.9848(59)(60)(8)(1)(0)
5.911	11.822	IV	1.0	1.0017(69)(12)(0)(1)(0)	1.0009(58)(6)(2)(1)(0)
6.826	13.651	III	1.0	1.0103(77)(12)(2)(1)(0)	1.0085(63)(27)(3)(1)(0)
8.360	16.719	II	1.0	1.0125(73)(14)(8)(1)(0)	0.9823(52)(53)(1)(1)(0)

Uncertainties:

1. statistical,
2. from the fitting ansatz,
3. from non-ideal matching,
4. from $\Lambda^{(0)}_{\overline{\rm MS}}$,
5. from r_0.

Full running from X-space





Summary and prospects

- We have computed the running of RCs in the X-space scheme for quenched ensembles.
- The range of scales between approx. 1.5 and 17 GeV.

- X-space step scaling feasible, but not ideally suited for step scaling.
- Satisfactory agreement with perturbation theory.
- Nevertheless, to make the approach more precise and reliable, one needs to understand better hypercubic artefacts.

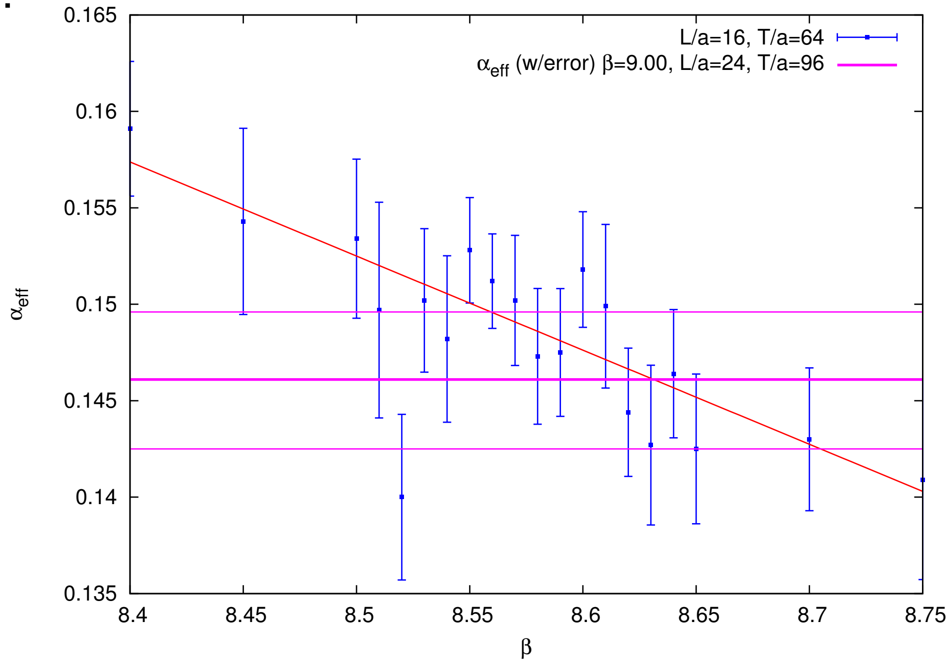
Thank you for attention!

Matching

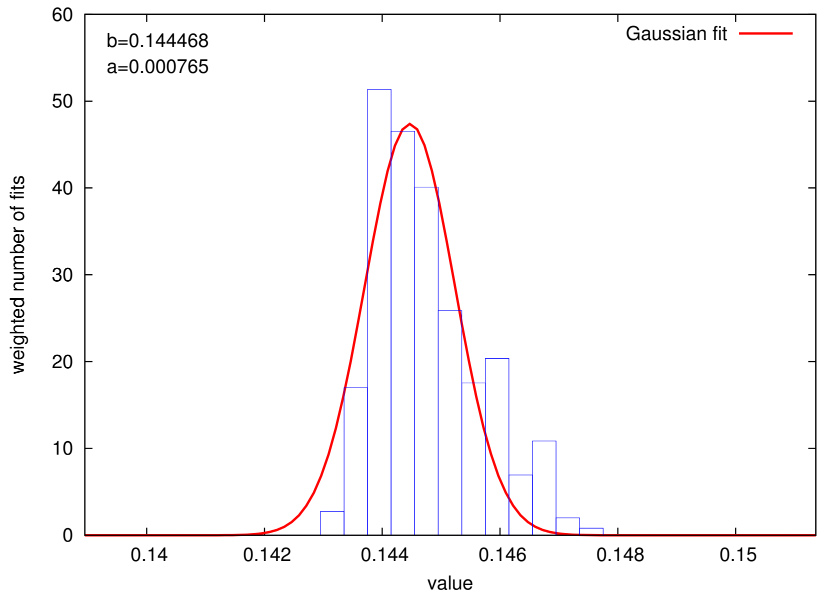
Our starting point – ensemble:

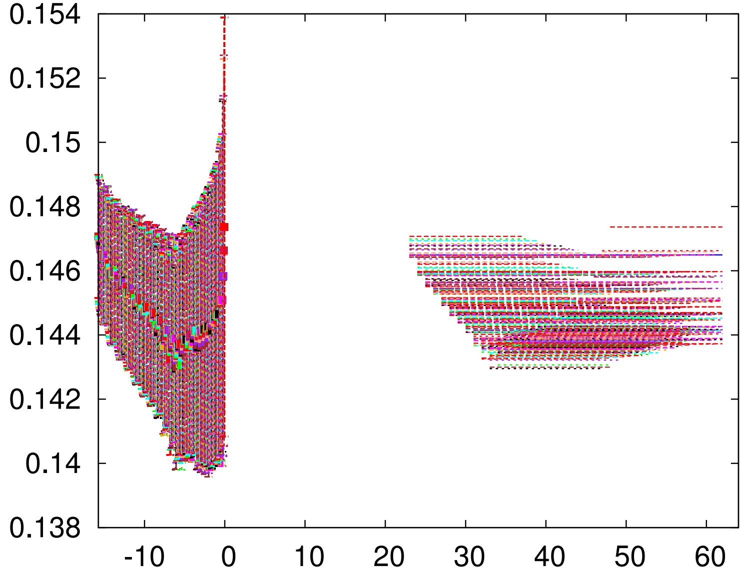
$$\beta = 9.00,\ L/a = 24,\ T/a = 96,$$

with effective coupling $\alpha_{\mathrm{eff}} = 0.1461(32)(15)$. First, match α_{eff} on $16^3 \times 64$ lattices:



From this, we know that $\beta = 8.62(7)$ is matched (has the same α_{eff}) to our starting ensemble.





Result: $\alpha_{\mathrm{eff}} = 0.1444(32)_{\mathrm{stat}}(9)_{\mathrm{sys}}$.

Tuning to maximal twist

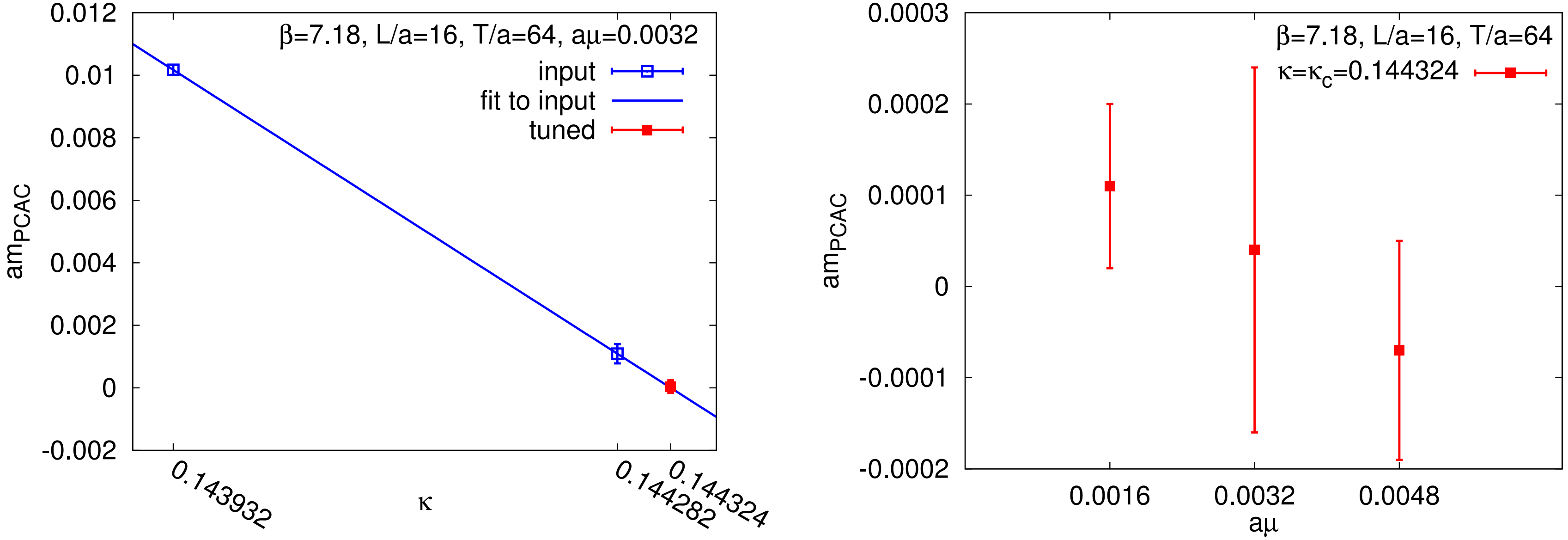


Illustration of tuning to maximal twist for the ensemble:
$\beta = 7.18,\ L/a = 16,\ T/a = 64,\ a\mu = 0.0032$.