Diagrammatic Monte-Carlo for non-Abelian field theories and resurgence

Pavel Buividovich (Regensburg University)

Lattice 2016, Southampton, 25-30 July 2016
Diagrammatic Monte-Carlo for dense $\textbf{QCD}$ and sign problem

So far lattice strong-coupling expansion: (leading order or few lowest orders) 
[de Forcrand, Philipsen, Unger, Gattringer, ...]

- Worldlines of quarks/mesons/baryons
- Confining strings

Very good approximation!

Physical degrees of freedom!

✓ Phase diagram, tri-critical

X Hadron spectrum, potentials
Lattice strong-coupling expansion

- Confinement
- Dynamical mass gap generation ARE NATURAL, BUT...

Continuum physics is at weak-coupling!

DiagMC @ Weak-coupling?

Non-perturbative physics via RESURGENCE

DiagMC algorithms from Schwinger-Dyson
Perturbative DiagMC

it should be first-principle and automatic

• Take $N \to \infty$ to reduce diagram space
• Small fluctuations of SU(N) fields
• Map SU(N) to Hermitian matrices

Cayley map

\[
g = \frac{1 + i\alpha \phi}{1 - i\alpha \phi}
\]

\[
\int_{SU(N)} dg \Rightarrow \int_{\mathbb{H}_{N \times N}} d\phi \ \text{det} \ (1 + \alpha^2 \phi^2)^{-N} = \\
= \int_{\mathbb{H}_{N \times N}} d\phi \ \exp \left( -N \alpha^2 \text{Tr} \ \phi^2 + O (\alpha^4 \phi^4) \right)
\]
SU(N) principal chiral model

\[ Z = \int_{U(N)} \, dg_x \exp \left( -\frac{N}{\lambda} \sum_{<x,y>} \text{Tr} \left( g_x^\dagger g_y \right) \right) \]

\[ \alpha^2 = \frac{\lambda}{8} \]

Expand action and Jacobian in \( \phi \)

Infinitely many interaction vertices

\[ S[\phi_x] = \frac{1}{2} \sum_{x,y} \left( D_{xy} + \frac{\lambda}{4} \delta_{xy} \right) \text{Tr} \left( \phi_x \phi_y \right) + \]

\[ + \sum_{n=2}^{+\infty} \left( -\frac{\lambda}{8} \right)^{n-1} \left( \frac{\lambda}{8n} \sum_x \text{Tr} \phi_x^{2n} + \right) \]

\[ \frac{1}{2} \sum_{l=1}^{2n-1} (-1)^{l-1} \sum_{x,y} D_{xy} \text{Tr} \left( \phi_x^{2n-l} \phi_y^l \right) \]
SU(N) principal chiral model

Power series in t’Hooft $\lambda$?

Factorial growth even at large $N$ due to IR renormalons ... [Bali, Pineda]

Can be sampled, but resummation difficult

...Bare mass term $\sim \lambda$ from Jacobian???

[a-la Fujikawa for axial anomaly]

✓ Massive planar field theory

✓ Suitable for DiagMC

? How to expand in $\lambda$?

Count vertices !??
Minimal working example:
2D O(N) sigma model @ large N

\[
\int d\vec{n}_x \exp \left( -\frac{1}{\alpha^2} \sum_{<x,y>} \vec{n}_x \cdot \vec{n}_y \right) \\
\sim \exp \left( -m^2 |x - y| \right)
\]

Non-perturbative mass gap

Jacobian reads

Again, bare mass term from the Jacobian...

\[
m^2 = 32 \exp \left( -\frac{4\pi}{\alpha^2} \right)
\]

\[
\mathcal{D}n_x = \mathcal{D}\phi_x \left( 1 + \frac{\lambda}{4} \phi_x^2 \right)^{-N}
\]

Cayley map

[PB, 1510.06568]
O(N) sigma model @ large N

Full action in new coordinates

\[
S[\phi_x] = \frac{1}{2} \sum_{x,y} \left( D_{xy} + \frac{\lambda}{2} \delta_{xy} \right) \phi_x \cdot \phi_y +
\]

\[
+ \sum_{k=2}^{+\infty} \frac{(-1)^{k-1} \lambda^k}{4^k k} \sum_x \left( \phi_x^2 \right)^k +
\]

\[
+ \sum_{k,l=0 \atop k+l \neq 0}^{+\infty} \frac{(-1)^{k+l} \lambda^{k+l}}{2 \cdot 4^{k+l}} \sum_{x,y} D_{xy} \left( \phi_x^2 \right)^k \left( \phi_y^2 \right)^l (\phi_x \cdot \phi_y)
\]

We blindly do perturbation theory ...

Only cactus diagrams @ large N
Trans-series and Resurgence

From our perturbative expansion we get

\[ m^2 = \sum_{p,q=0}^{+\infty} c_{p,q} \lambda^p (\log \lambda)^q \]

Same for PCM!!!

Resurgent trans-series \([\text{Écalle}, 81]\)

\[ f(z) = \sum_{p,q,r} c_{p,q,r} z^p (\log z)^q \left(e^{-\frac{s}{z}}\right)^r \]

PT Zero modes Classical solutions

\([\text{Argyres, Dunne, Unsal, ...}, 2011\text{-present}]\)

\[ \exp \left(-\frac{1}{\lambda}\right) = \exp \left(-e^{-\log(\lambda)}\right) = \sum_{k} c_k (\log \lambda)^{k} \]
O(N) sigma model @ large N

Good convergence in practice
(But no proof of convergence!!!(3)
Now we need DiagMC, all planar diagrams
Stochastically solve Schwinger-Dyson equations

Recursive structure for diagrams:
V vertices, L legs -> V-v vertices, L+l legs
Results from DiagMC

(several hours on laptop)

Mean link vs $1$/max order

$\lambda = 4.0 \ (l_c \sim 1)$

$\lambda = 3.23 \ (l_c \sim 5)$

Mean Link vs [Vicari, Rossi, 9307014]
Results from DiagMC

Restoration of SU(N) x SU(N)
Errors due to sign problem at high orders

$\langle \text{tr}(g_x)/N\rangle$ vs $1/\text{max order}$

$\lambda = 4.0 \ (l_c \sim 1)$

$\lambda = 3.23 \ (l_c \sim 5)$
Discussion

DiagMC in the large-N limit:
• # of diagrams grows exponentially
• All contributions are finite
• Good for DiagMC!!!
• No IR renormalons at the expense of generating transseries
• Numerical evidence of convergence

BUT: sign problem at high orders
Inevitable for any (naive) approach close to continuum!
No signs-no continuum limit
Discussion

• With lattice perturbation theory, we have lost the beauty of strong coupling (Strings for gauge theories!)

• Another series expansion is possible, basing on Schwinger-Dyson equations with original field variables, so far convergence not so good???

Thank you for your attention!