Lattice operators for scattering of particles with spin

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in collaboration with

Ursa Skerbis (Ljubljana) & Christian B. Lang (Graz)

based on S. Prelovsek, U. Skerbis, C.B. Lang: arXiv:1607:06738

Motivation

- Mainly PP scattering was simulated on lattice up to now → scattering phase shift extracted (P has no spin)

- $H^{(1)}\,H^{(2)}$: where one or both H carry spin was explored mostly only for L=0

  many interesting channels still unexplored, particularly for L>0

I will consider construction of $H^{(1)}\,H^{(2)}$ interpolators

where H is one of P,V,N hadrons, which is (almost) stable with respect to strong decay:

P=psuedoscalar ($J^P=0^{-}$) = π , K, D, B, η_c , ...

V=vector ($J^P=1^{-}$) = D*, B*, J/ψ, Υ_b , B_c^*,... (but not directly applicable to ρ as is unstable...)

N=nucleon ($J^P=1/2^{+}$) = p, n, Λ, Λ_c, Σ, ... (but not directly applicable to $N^-(1535)$ as is unstable...)

I will consider interpolators for channels :

PV: meson resonances and $\underline{Q}$Q-like exotics (e.g. π J/ψ, D $\underline{D}$* ..)

PN: baryon resonances (e.g. π N, K N ...) and pentaquarks

NV: baryon resonances and pentaquarks

NN: nucleon-nucleon and deuterium, baryon-baryon

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$$\langle O_i(t)|O_j^\dagger(0)\rangle \;\rightarrow\; E_n \;\rightarrow\; \delta(E)$$

- O=HH needed to create/annihilate HH system
- E_n related to phase shifts for HH scattering
  - two spinless particles Luscher (1991):
  - two particles with arbitrary spin
    Briceno, PRD89, 074507 (2014)
    (other authors: some specific cases)

Some previous related work on lattice HH operators for hadrons with spin and L≠0

Partial-wave method for HH:
Berkowitz, Kurth, Nicolson, Joo, Rinaldi, Strother, Walker-Loud, 1508.00886
Wallace, Phys. Rev. D92, 034520 (2015), [arXiv:1506.05492]

Projection method for HH:
M. Göckeler et al., Phys.Rev. D86, 094513 (2012), [arXiv:1206.4141].

Helicity operators for single-H:
Thomas, Edwards and Dudek, Phys. Rev. D85, 014507 (2012), [arXiv:1107.1930]

Some aspects of helicity operators for HH:
Wallace, Phys. Rev. D92, 034520 (2015), [arXiv:1506.05492].
Dudek, Edwards and Thomas, Phys. Rev. D86, 034031 (2012), [arXiv:1203.6041].

Which CG of H_1 and H_2 to H_1H_2 irreps are nonzero; values of CG not published:
Moore and Fleming, Phys. Rev. D 74, 054504 (2006), [arXiv:hep- lat/0607004].

etc ...

However: for a lattice practitioner who was interested in a certain channel, for example
(PV scattering in L=2 or VN scattering with λ_V=1 and λ_N=1/2)
there were still lots of puzzles to beat before constructing a reliable interpolator ..

Outline

I will present

- three different methods to construct operators
- illuminate the proofs (given in the paper)
- verify they lead to consistent operators (that gives confidence in each one of them)
- they lead to complementary physics info
- present explicit ops for PV, PN, VN, NN for lowest two momentum shells.

We restrict to total momentum zero

$H^{(1)}(p)\, H^{(2)}(-p)$, $P_{tot}=0$

Advantage of $P_{tot}=0$:

- parity P is a good number
- channels with even and odd L do not mix in the same irrep

not true for $P_{tot}\neq 0$

Building blocks H: required transformation properties of H

to prove correct transformation properties of HH

rotations R | inversion I | $|p,s,m_s\rangle \equiv H^\dagger_{m_s}(p)|0\rangle$

$$R|p,s,m_s\rangle = \sum_{m'_s} \overbrace{D^s_{m'_s m_s}}^{\text{Wigner D matrix}}(R)|Rp,s,m'_s\rangle \ , \qquad I|p,s,m_s\rangle = (-1)^P|-p,s,m_s\rangle$$

state

$$RH^\dagger_{m_s}(p)R^{-1} = \sum_{m'_s} D^s_{m'_s m_s}(R) H^\dagger_{m'_s}(Rp) \ , \qquad IH^\dagger_{m_s}(p)I = (-1)^P H^\dagger_{m_s}(-p) \, .$$

creation field

note:

D → D*

$$RH_{m_s}(p)R^{-1} = \sum_{m'_s} \underbrace{D^s_{m'_s m_s}(R)^*}_{D^s_{m_s m'_s}(R^{-1})} H_{m'_s}(Rp) \ , \qquad IH_{m_s}(p)I = (-1)^P H_{m_s}(-p)$$

annihilation field

m_s is a good quantum number at p=0: $S_z H_{m_s}(0) S_z^{-1} = m_s H_{m_s}(0)$

m_s is not good quantum number in general for p≠0: in this case it denotes m_s of corresponding $H_{ms}(p=0)$

Non-practical choice of H: canonical fields $H^{(c)}$

with correct transformation properties under R and I

$$H^{(c)}_{m_s}(p) \equiv L(p) H_{m_s}(0)$$

L(p) is boost from 0 to p; drawback: $H^{(c)}(p)$ depend on m, E,..

$$V_{m_s=1}(0) = \tfrac{1}{\sqrt{2}}[-V_x(0) + iV_y(0)] \;\rightarrow\; V^{(c)}_{m_s=1}(p_x) = \tfrac{1}{\sqrt{2}}[-\gamma V_x(p_x) + iV_y(p_x)]$$

$$\begin{pmatrix}-1\\ i\\ 0\end{pmatrix} \xrightarrow{\Lambda^1(p_x)} \begin{pmatrix}\gamma & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-1\\ i\\ 0\end{pmatrix} = \begin{pmatrix}-\gamma\\ i\\ 0\end{pmatrix}$$

$$N_{m_s=1/2}(0) = \mathcal{N}_1(0) \;\rightarrow\; N^{(c)}_{m_s=1/2}(p_x) \propto \mathcal{N}_1(p_x) + \tfrac{p_x}{E+m}\mathcal{N}_4(p_x)$$

$$\begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix} \xrightarrow{\Lambda^{1/2}(p_x)} \begin{pmatrix}1\\ 0\\ 0\\ \frac{p_x}{E+m}\end{pmatrix}$$

$\mathcal{N}_{\mu=1,..,4}$ are Dirac components in Dirac basis

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$$\begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix} \xrightarrow{\Lambda^{1/2}(p_x)} \begin{pmatrix}1\\ 0\\ 0\\ \frac{p_x}{E+m}\end{pmatrix}$$

$\mathcal{N}_{\mu=1,..,4}$ are Dirac components in Dirac basis

Practical choice of H

with correct transformation properties under R and I

$$V_{m_s=\pm 1}(p) = \tfrac{1}{\sqrt{2}}[\mp V_x(p) + iV_y(p)], \quad V_{m_s=0}(p) = V_z(p)$$

$$N_{m_s=1/2}(p) = \mathcal{N}_{\mu=1}(p)\;, \quad N_{m_s=-1/2}(p) = \mathcal{N}_{\mu=2}(p)$$

These H are employed as building block in our HH operators

$$P(p) = \sum_x \bar{q}(x)\gamma_5 q(x) e^{ipx}$$

$$V_i(p) = \sum_x \bar{q}(x)\gamma_i q(x) e^{ipx}, \quad i = x, y, z$$

$$\mathcal{N}_\mu(p) = \sum_x \epsilon_{abc}[q^{aT}(x) C\gamma_5 q^b(x)]\; q^c_\mu(x)\; e^{ipx}, \quad \mu = 1, .., 4$$

simple examples

Required transformation properties of O=HH

$$R O^{J,m_J}(P_{tot}\!=\!0) R^{-1} = \sum_{m'_J} D^J_{m_J m'_J}(R^{-1}) O^{J,m'_J}(0) \qquad I O^{J,m_J}(0) I = (-1)^P O^{J,m_J}(0)$$

continuum R

good parity since P_{tot}=0 !

relevant rotations: $R \in O^{(2)}$ O with 24 el. for J=integer ; O^2 with 48 elements for J=half-integer

The group including inversion I: O_h with 48 el. for J=integer ; O^2_h with 96 elements for J=half-integer

The representation O^J reducible under $O^{(2)}$. Irreducible representations (irreps) are denoted by Γ and rows r

$$R|\Gamma,r\rangle = \sum_{r'} T^{\Gamma}_{r',r}(R)|\Gamma,r'\rangle \quad R \in O^{(2)}, \qquad I|\Gamma,r\rangle = (-1)^P|\Gamma,r\rangle \ ,$$

$$R O_{\Gamma,r} R^{-1} = \sum_{r'} \underbrace{T^{\Gamma}_{r,r'}(R^{-1})} O_{\Gamma,r'} \quad R \in O^{(2)}, \qquad I O_{\Gamma,r} I = (-1)^P O_{\Gamma,r}$$

discrete R

T(R) given for all irreps in
Bernard, Lage, Meißner, Rusetsky, JHEP 2008, 0806.4495
We use same conventions for rows.

J	Γ (dim$_\Gamma$)
0	$A_1(1)$
$\frac{1}{2}$	$G_1(2)$
1	$T_1(3)$
$\frac{3}{2}$	$H(4)$
2	$E(2) \oplus T_2(3)$
$\frac{5}{2}$	$H(4) \oplus G_2(2)$
3	$A_2(1) \oplus T_1(3) \oplus T_2(3)$

Method I: Projection operators

$$\boxed{O_{|p|,\Gamma,r,n} = \sum_{\tilde{R}\in O_h^{(2)}} \underbrace{T^{\Gamma}_{r,r}(\tilde{R})}_{} \ \tilde{R}\ \underbrace{H^{(1),a}(p)\ H^{(2),a}(-p)}_{}\ \tilde{R}^{-1}\ ,} \qquad n = 1,..,n_{max}$$

T(R) given for all irreps in
Bernard, Lage, Meißner, Rusetsky, JHEP 2008, 0806.4495

"seed": Each H^a can have any polarization m_s and direction p with given |p|. Different choices lead to different linearly independent O_n

Some examples for |p|=1:

PV in T_1^+ , n_{max}=2:

$$O_{T_1^+,r=3,n=1} = \mathrm{P}(e_z)V_z(-e_z) + \mathrm{P}(-e_z)V_z(e_z)$$
$$O_{T_1^+,r=3,n=2} = \mathrm{P}(e_x)V_z(-e_x) + \mathrm{P}(-e_x)V_z(e_x) + \mathrm{P}(e_y)V_z(-e_y) + \mathrm{P}(-e_y)V_z(e_y)$$

PN in H^+ , n_{max}=1:

$$O_{H^+,r=1} = -iN_{\frac{1}{2}}(-e_x)\mathrm{P}(e_x) + iN_{\frac{1}{2}}(e_x)\mathrm{P}(-e_x) - N_{\frac{1}{2}}(-e_y)\mathrm{P}(e_y) + N_{\frac{1}{2}}(e_y)\mathrm{P}(-e_y)$$

VN in H^- , n_{max}=3:

$$O_{H^-,r=1,n=1} = iN_{\frac{1}{2}}(e_x)V_x(-e_x) + iN_{\frac{1}{2}}(-e_x)V_x(e_x) + N_{\frac{1}{2}}(e_y)V_y(-e_y) + N_{\frac{1}{2}}(-e_y)V_y(e_y)$$
$$O_{H^-,r=1,n=2} = \ldots$$
$$O_{H^-,r=1,n=3} = \ldots$$

<u>Disadvantage:</u>

not informative which continuum numbers (partial wave L or helicity) each O_n corresponds

This is remedied in next two method.s

Method II: Partial-wave operators

building blocks H mentioned on slide 6 below

Starting annihilation operator
(before sobduction to irreps)

Clebsch-Gordans

Spherical Harmonics

$$\boxed{O^{|p|,J,m_J,S,L} = \sum_{m_L,m_S,m_{s1},m_{s2}} \underbrace{C^{Jm_J}_{Lm_L,Sm_S}\, C^{Sm_S}_{s_1 m_{s1},s_2 m_{s2}}}_{\text{Clebsch-Gordans}} \sum_{R\in O} \underbrace{Y^{*}_{Lm_L}(\widehat{Rp})}_{\text{Spherical Harmonics}}\ \underbrace{H^{(1)}_{m_{s1}}(Rp)\ H^{(2)}_{m_{s2}}(-Rp)}_{\text{building blocks H}}}$$

Proposed for NN in [Berkowitz, Kurth, Nicolson, Joo, Rinaldi, Strother, Walker-Loud, CALLAT, 1508.00886] There $Y_{lm}{}^{*}$ appears where we have Y_{lm}

Proof (in our paper and backup slides): the correct transformation properties

$$R_a O^{J,m_J,S,L} R_a^{-1} = \sum_{m'_J} D^J_{m_J m'_J}(R_a^{-1}) O^{J,m'_J,S,L}$$

follow from transformations of H (slide 4) and properties of C, Y_{lm} and D.

Example of PV operators

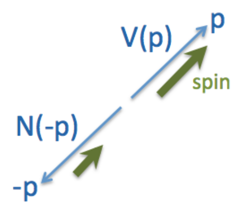
$$O^{|p|=1,J=1,m_J=0,L=0,S=1} = \sum_{p=\pm e_x,\pm e_y,\pm e_z} \mathrm{P}(p) V_z(-p)\ ,$$

$$O^{|p|=1,J=1,m_J=0,L=2,S=1} = \sum_{p=\pm e_x,\pm e_y} \mathrm{P}(p) V_z(-p) - 2\sum_{p=\pm e_z} \mathrm{P}(p) V_z(-p)$$

Subduction to irreps discussed later on.

Method III: helicity operators

[HH in continuum: Jacob, Wick (1959)]
[for single H on lattice: HSC, Thomas et al. (2012)]
[not widely used for HH on lattice yet]



- building blocks in partial-wave operators are $H_{m_s}(p)$ and m_s is not good for $p \neq 0$:

- Helicity λ is projection of S to p. It is good also for particles in flight $$h \equiv S \cdot p \;/\; |p|$$

- Definition of single-hadron helicity operator
  denoted by superscript h
  rotation from p_z to p
  $$H^h_\lambda(p) \equiv R_0^p \; H_{m_s=\lambda}(p_z) \; (R_0^p)^{-1}$$
  good m_s

- Helicity is not modified under R (p and S transform the same way) $$R H^h_\lambda(p) R^{-1} = e^{i\varphi(R)} H^h_\lambda(Rp)$$

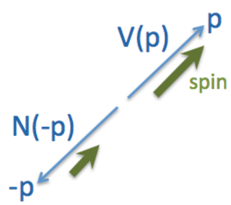
- Two-hadron O: $$\boxed{O^{|p|,J,m_J,\lambda_1,\lambda_2,\lambda} = \sum_{R\in O^{(2)}} D^J_{m_J,\lambda}(R)\; R\; H^{(1),h}_{\lambda_1}(p)\; H^{(2),h}_{\lambda_2}(-p)\; R^{-1}}$$
  p is arbitrary momentum in given shell |p|; R does not modify $\lambda_{1,2}$, so $H_{1,2}$ have chosen $\lambda_{1,2}$ in all terms

- Proof:
$$\begin{aligned} R_a O^{J,m_J,\lambda_1,\lambda_2} R_a^{-1} &= \sum_{R\in O^{(2)}} D^J_{m_J,\lambda}(R)\; R_a R\; H^h_{\lambda_1}(p) H^h_{\lambda_2}(-p)\; R^{-1} R_a^{-1} \\ &= \sum_{R\in O^{(2)}} D^J_{m_J,\lambda}(R_a^{-1} R')\; R'\; H^h_{\lambda_1}(p) H^h_{\lambda_2}(-p)\; R'^{-1} \\ &= \sum_{R'\in O^{(2)}} \sum_{m'_J} D^J_{m_J,m'_J}(R_a^{-1}) D^J_{m'_J,\lambda}(R')\;\; R'\; H^h_{\lambda_1}(p) H^h_{\lambda_2}(-p)\; R'^{-1} \\ &= \sum_{m'_J} D^J_{m_J,m'_J}(R_a^{-1})\; O^{J,m'_J,\lambda_1,\lambda_2}\,, \end{aligned}$$

$R' = R_a R$

$$D(R_1 R_2) = D(R_1) D(R_2)$$

Method III: helicity operators (continued)



Using definitions of $H^h_\lambda(p) \equiv R^p_0\ H_{m_s=\lambda}(p_z)\ (R^p_0)^{-1}$ and parity projection $\frac{1}{2}(\mathcal{O} + PI\mathcal{O}I)$

$$O^{|p|,J,m_J,P,\lambda_1,\lambda_2,\lambda} = \frac{1}{2}\sum_{R\in O^{(2)}} D^J_{m_J,\lambda}(R)\ RR^p_0\ [H^{(1)}_{m_{s_1}=\lambda_1}(p_z)H^{(2)}_{m_{s_2}=-\lambda_2}(-p_z)$$

$$+ P\ I\ H^{(1)}_{m_{s_1}=\lambda_1}(p_z)H^{(2)}_{m_{s_2}=-\lambda_2}(-p_z)\ I]\ (R^p_0)^{-1}R^{-1}$$

$$\lambda = \lambda_1 - \lambda_2$$

- H are building blocks from slide 6 below: actions of R and I on $H_{ms}(p)$ are given in slide 4
- There are several choices of $R_0{}^p$ which rotate from p_z to p:
  - these lead to different phases in definition of $H_\lambda{}^h$: inconvenience
  - but they lead to the same O above (modulo irrelevant overall factor): so no problem for such construction
- Simple choice for momentum shell |p|=1 : $p=p_z$ and $R_0{}^p$=Identity
- paper provides details how to use functions from Mathematica for construction, also since Mathematica uses non-conventional defnition of D

$$D^j_{m,m'}[R^\omega_{\alpha\beta\gamma}] = F \cdot \texttt{WignerD}[\{j,m,m'\},-\alpha,-\beta,-\gamma], \qquad F = \begin{cases} 1\ :\ j = \text{integer} \\ \pm 1\ :\ j = \text{halfinteger},\ \ \mathrm{F}(\omega+2\pi) = -\mathrm{F}(\omega) \end{cases}$$, choice of sign in our paper

$$\{\alpha,\beta,\gamma\} = \underbrace{\texttt{EulerAngles}}_{\text{MATHEMATICA}}[T] \quad T = \exp(-i\vec{n}\vec{J}\omega) \text{ and } (J_k)_{ij} = -i\epsilon_{ijk}$$

one last step before reaching the results ...

Subduction of O^J to irreducible representations

subduction

continum R

discrete R in discrete group $O^{(2)}$

Partial-wave operators $O^{J,\, mJ,\, L,\, S}$

Helicity operators $O^{J,\, mJ,\, \lambda 1,\, \lambda 2}$

$$O^{[J,S,L]}_{|p|,\Gamma,r} = \sum_{m_J} \mathcal{S}^{J,m_J}_{\Gamma,r} O^{|p|,J,m_J,S,L}$$

$$O^{[J,P,\lambda_1,\lambda_2,\lambda]}_{|p|,\Gamma,r} = \sum_{m_J} \underbrace{\mathcal{S}^{J,m_J}_{\Gamma,r}}_{} O^{|p|,J,m_J,P,\lambda_1,\lambda_2,\lambda}$$

The representation O^J is irreducible under continuum R.

But it is reducible under R in discrete group lattice $O^{(2)}$.

Operators that transform according to irrep Γ and row r obtained via subduction.

Subduction matrices S

[Dudek et al., PRD82, 034508 (2010)

Edwards et al, PRD84, 074508 (2011)]

Single-hadron operators H: experience by Hadron Spectrum collaboration Phys. Rev. D 82, 034508 (2010)

- subduced operators $O^{[J]}{}_{\Gamma}$ carry memory of continuum spin and dominantly couple to states with this J

Expectation for partial-wave and helicity operators HH obtained by subduction :

- $O^{[J,S,L]}_{|p|,\Gamma,r}$ would dominantly couple to eigen-states with continuum (J,L,S)
- $O^{[J,P,\lambda_1,\lambda_2,\lambda]}_{|p|,\Gamma,r}$ would dominantly couple to eigen-states with continuum (J,λ1,λ2)

valuable for simulations

give physics intuition on quant. num.

Explicit expressions all for $H^{(1)}(p)H^{(2)}(-p)$

PV, PN, VN, NN

in three methods

all irreps, $|p|=0,1$

given in

[S. Prelovsek, U. Skerbis, C.B. Lang, arXiv:1607. 1607:06738]

Example:
P(p)V(-p) operators

J	Γ (dim$_\Gamma$)
0	$A_1(1)$
$\frac{1}{2}$	$G_1(2)$
1	$T_1(3)$
$\frac{3}{2}$	$H(4)$
2	$E(2) \oplus T_2(3)$
$\frac{5}{2}$	$H(4) \oplus G_2(2)$
3	$A_2(1) \oplus T_1(3) \oplus T_2(3)$

row=1 provided

Conventions for row
Bernard et al. , 0806.4495

rows of T1: (x,y,z)

rows of T2: (yz,xz,xy)

$|p| = 0$

T_1^+ :

$$O_{T_1^+,r=1} = \mathrm{P}(0)V_x(0)$$

$$O^{[J=1,L=0,S=1]}_{T_1^+,r=1} = O_{T_1^+,r=1}$$

other irreps: O=0

$|p| = 1$

A_1^- :

$$O_{A_1^-,r=1} = \mathrm{P}(e_x)V_x(-e_x) - \mathrm{P}(-e_x)V_x(e_x) + \mathrm{P}(e_y)V_y(-e_y) - \mathrm{P}(-e_y)V_y(e_y) + \mathrm{P}(e_z)V_z(-e_z) - \mathrm{P}(-e_z)V_z(e_z)$$

$$O^{[J=0,m_J=0,P=-,\lambda_V=0,\lambda_P=0]}_{A_1^-,r=1} = O^{[J=0,m_J=0,L=1,S=1]}_{A_1^-,r=1} = O_{A_1^-,r=1}$$

T_1^+ :

$$O_{T_1^+,r=1,n=1} = \mathrm{P}(e_x)V_x(-e_x) + \mathrm{P}(-e_x)V_x(e_x)$$
$$O_{T_1^+,r=1,n=2} = \mathrm{P}(e_y)V_x(-e_y) + \mathrm{P}(-e_y)V_x(e_y) + \mathrm{P}(e_z)V_x(-e_z) + \mathrm{P}(-e_z)V_x(e_z)$$

$$O^{[J=1,P=+,\lambda_V=\pm1,\lambda_P=0]}_{T_1^+,r=1} = O_{T_1^+,r=1,n=2}$$
$$O^{[J=1,P=+,\lambda_V=0,\lambda_P=0]}_{T_1^+,r=1} = O_{T_1^+,r=1,n=1}$$

$$O^{[J=1,L=0,S=1]}_{T_1^+,r=1} = O_{T_1^+,r=1,n=1} + O_{T_1^+,r=1,n=2}$$
$$O^{[J=1,L=2,S=1]}_{T_1^+,r=1} = -2\ O_{T_1^+,r=1,n=1} + O_{T_1^+,r=1,n=2}$$

T_1^- :

$$O_{T_1^-,r=1} = -\mathrm{P}(e_y)V_z(-e_y) + \mathrm{P}(-e_y)V_z(e_y) + \mathrm{P}(e_z)V_y(-e_z) - \mathrm{P}(-e_z)V_y(e_z)$$

$$O^{[J=1,P=-,\lambda_V=\pm1,\lambda_P=0]}_{T_1^-,r=1} = O^{[J=1,L=1,S=1]}_{T_1^-,r=1} = O_{T_1^-,r=1}$$

T_2^+ :

$$O_{T_2^+,r=1} = \mathrm{P}(e_y)V_x(-e_y) + \mathrm{P}(-e_y)V_x(e_y) - \mathrm{P}(e_z)V_x(-e_z) - \mathrm{P}(-e_z)V_x(e_z)$$

$$O^{[J=2,P=+,\lambda_V=\pm1,\lambda_P=0]}_{T_2^+,r=1} = O^{[J=2,L=2,S=1]}_{T_2^+,r=1} = O_{T_2^+,r=1}$$

T_2^- :

$$O_{T_2^-,r=1} = \mathrm{P}(e_y)V_z(-e_y) - \mathrm{P}(-e_y)V_z(e_y) + \mathrm{P}(e_z)V_y(-e_z) - \mathrm{P}(-e_z)V_y(e_z)$$

$$O^{[J=2,P=-,\lambda_V=\pm1,\lambda_P=0]}_{T_2^-,r=1} = O^{[J=2,L=1,S=1]}_{T_2^-,r=1} = O^{[J=2,L=3,S=1]}_{T_2^-,r=1} = O_{T_2^-,r=1}$$

E^- :

$$O_{E^-,r=1} = \mathrm{P}(e_x)V_x(-e_x) - \mathrm{P}(-e_x)V_x(e_x) + \mathrm{P}(e_y)V_y(-e_y) - \mathrm{P}(-e_y)V_y(e_y) - 2\mathrm{P}(e_z)V_z(-e_z) + 2\mathrm{P}(-e_z)V_z(e_z)$$

$$O^{[J=2,P=-,\lambda_V=0,\lambda_P=0]}_{E^-,r=1} = O^{[J=2,L=1,S=1]}_{E^-,r=1} = O^{[J=2,L=3,S=1]}_{E^-,r=1} = O_{E^-,r=1}$$

$$O_{A_1^+} = O_{A_2^+} = O_{A_2^-} = O_{E^+} = 0\,.$$

Example: P(p)V(-p) operators

$|p|=1$

A_1^- :

$$O_{A_1^-,r=1} = \mathrm{P}(e_x)V_x(-e_x) - \mathrm{P}(-e_x)V_x(e_x) + \mathrm{P}(e_y)V_y(-e_y) - \mathrm{P}(-e_y)V_y(e_y) + \mathrm{P}(e_z)V_z(-e_z) - \mathrm{P}(-e_z)V_z(e_z)$$

$$O^{[J=0,m_J=0,P=-,\lambda_V=0,\lambda_P=0]}_{A_1^-,r=1} = O^{[J=0,m_J=0,L=1,S=1]}_{A_1^-,r=1} = O_{A_1^-,r=1}$$

T_1^+ :

projection operators

$$O_{T_1^+,r=1,n=1} = \mathrm{P}(e_x)V_x(-e_x) + \mathrm{P}(-e_x)V_x(e_x)$$

$$O_{T_1^+,r=1,n=2} = \mathrm{P}(e_y)V_x(-e_y) + \mathrm{P}(-e_y)V_x(e_y) + \mathrm{P}(e_z)V_x(-e_z) + \mathrm{P}(-e_z)V_x(e_z)$$

provides lin. combination of projection operators O_n that enhances the coupling to state with continuum (J^P, λ_V)

helicity operators

$$O^{[J=1,P=+,\lambda_V=\pm1,\lambda_P=0]}_{T_1^+,r=1} = O_{T_1^+,r=1,n=2}$$ $J^P=1^+$, $\lambda_V=0$

$$O^{[J=1,P=+,\lambda_V=0,\lambda_P=0]}_{T_1^+,r=1} = O_{T_1^+,r=1,n=1}$$ $J^P=1^+$, $\lambda_V=1$

provides lin. combination of projection operators O_n that enhances the coupling to state with continuum (J^P, S,L)

partial-wave operators

$$O^{[J=1,L=0,S=1]}_{T_1^+,r=1} = O_{T_1^+,r=1,n=1} + O_{T_1^+,r=1,n=2}$$ $J^P=1^+$, S=1,L=0

$$O^{[J=1,L=2,S=1]}_{T_1^+,r=1} = -2\, O_{T_1^+,r=1,n=1} + O_{T_1^+,r=1,n=2}$$ $J^P=1^+$, S=1,L=2

T_1^- :

$$O_{T_1^-,r=1} = -\mathrm{P}(e_y)V_z(-e_y) + \mathrm{P}(-e_y)V_z(e_y) + \mathrm{P}(e_z)V_y(-e_z) - \mathrm{P}(-e_z)V_y(e_z)$$

$$O^{[J=1,P=-,\lambda_V=\pm1,\lambda_P=0]}_{T_1^-,r=1} = O^{[J=1,L=1,S=1]}_{T_1^-,r=1} = O_{T_1^-,r=1}$$

T_2^+ :

$$O_{T_2^+,r=1} = \mathrm{P}(e_y)V_x(-e_y) + \mathrm{P}(-e_y)V_x(e_y) - \mathrm{P}(e_z)V_x(-e_z) - \mathrm{P}(-e_z)V_x(e_z)$$

$$O^{[J=2,P=+,\lambda_V=\pm1,\lambda_P=0]}_{T_2^+,r=1} = O^{[J=2,L=2,S=1]}_{T_2^+,r=1} = O_{T_2^+,r=1}$$

T_2^- :

$$O_{T_2^-,r=1} = \mathrm{P}(e_y)V_z(-e_y) - \mathrm{P}(-e_y)V_z(e_y) + \mathrm{P}(e_z)V_y(-e_z) - \mathrm{P}(-e_z)V_y(e_z)$$

$$O^{[J=2,P=-,\lambda_V=\pm1,\lambda_P=0]}_{T_2^-,r=1} = O^{[J=2,L=1,S=1]}_{T_2^-,r=1} = O^{[J=2,L=3,S=1]}_{T_2^-,r=1} = O_{T_2^-,r=1}$$

E^- :

$$O_{E^-,r=1} = \mathrm{P}(e_x)V_x(-e_x) - \mathrm{P}(-e_x)V_x(e_x) + \mathrm{P}(e_y)V_y(-e_y) - \mathrm{P}(-e_y)V_y(e_y) - 2\mathrm{P}(e_z)V_z(-e_z) + 2\mathrm{P}(-e_z)V_z(e_z)$$

$$O^{[J=2,P=-,\lambda_V=0,\lambda_P=0]}_{E^-,r=1} = O^{[J=2,L=1,S=1]}_{E^-,r=1} = O^{[J=2,L=3,S=1]}_{E^-,r=1} = O_{E^-,r=1}$$

$$O_{A_1^+} = O_{A_2^+} = O_{A_2^-} = O_{E^+} = 0\,.$$

J	Γ ($\dim_\Gamma$)
0	$A_1(1)$
$\frac{1}{2}$	$G_1(2)$
1	$T_1(3)$
$\frac{3}{2}$	$H(4)$
2	$E(2)\oplus T_2(3)$
$\frac{5}{2}$	$H(4)\oplus G_2(2)$
3	$A_2(1)\oplus T_1(3)\oplus T_2(3)$

$|p|=0$

T_1^+ :

$$O_{T_1^+,r=1} = \mathrm{P}(0)V_x(0)$$

$$O^{[J=1,L=0,S=1]}_{T_1^+,r=1} = O_{T_1^+,r=1}$$

other irreps: O=0

row=1 provided

Conventions for row
Bernard et al. , 0806.4495

rows of T1: (x,y,z)

rows of T2: (yz,xz,xy)

P(1)V(-1) operators, T_1^+, row=r=1

T_1^+ :

projection op.

$$O_{T_1^+,r=1,n=1} = \mathrm{P}(e_x)V_x(-e_x) + \mathrm{P}(-e_x)V_x(e_x)$$
$$O_{T_1^+,r=1,n=2} = \mathrm{P}(e_y)V_x(-e_y) + \mathrm{P}(-e_y)V_x(e_y) + \mathrm{P}(e_z)V_x(-e_z) + \mathrm{P}(-e_z)V_x(e_z)$$

partial-wave op.

$$O^{[J=1,L=0,S=1]}_{T_1^+,r=1} = O_{T_1^+,r=1,n=1} + O_{T_1^+,r=1,n=2}$$

$J^P=1^+$, S=1,L=0

$$O^{[J=1,L=2,S=1]}_{T_1^+,r=1} = -2\ O_{T_1^+,r=1,n=1} + O_{T_1^+,r=1,n=2}$$

$J^P=1^+$, S=1,L=2

helicity op.

$$O^{[J=1,P=+,\lambda_V=0,\lambda_P=0]}_{T_1^+,r=1} = O_{T_1^+,r=1,n=1}$$

$J^P=1^+$, $\lambda_V=0$

$$O^{[J=1,P=+,\lambda_V=\pm1,\lambda_P=0]}_{T_1^+,r=1} = O_{T_1^+,r=1,n=2}$$

$J^P=1^+$, $\lambda_V=1$

Partial-wave and helicity operators expressed in terms of projection operators throughout.

Some other examples of HH operators

J	Γ ($\dim_\Gamma$)
0	$A_1(1)$
$\frac{1}{2}$	$G_1(2)$
1	$T_1(3)$
$\frac{3}{2}$	$H(4)$
2	$E(2) \oplus T_2(3)$
$\frac{5}{2}$	$H(4) \oplus G_2(2)$
3	$A_2(1) \oplus T_1(3) \oplus T_2(3)$

PN, |p|=1, H^+ irrep, J^P=3/2$^+$, 5/2$^+$,.. (n=1)

H^+

$$O_{H^+,r=1} = -iN_{\frac{1}{2}}(-e_x)\mathrm{P}(e_x) + iN_{\frac{1}{2}}(e_x)\mathrm{P}(-e_x) - N_{\frac{1}{2}}(-e_y)\mathrm{P}(e_y) + N_{\frac{1}{2}}(e_y)\mathrm{P}(-e_y)$$

$$O_{H^+,r=1}^{[J=\frac{3}{2},m_J=\frac{3}{2},P=+,\lambda_N=\pm\frac{1}{2},\lambda_P=0]} = O_{H^+,r=1}^{[J=\frac{3}{2},m_J=\frac{3}{2},L=1,S=\frac{1}{2}]} = O_{H^+,r=1}$$

NN' , |p|=1, T_2^+ irrep, J^P=2$^+$,.. (n=1)

T_2^+

$$\begin{aligned} O_{T_2^+,r=1} = &-\mathrm{N}_{\frac{1}{2}}(e_y)\mathrm{N'}_{\frac{1}{2}}(-e_y) - \mathrm{N}_{\frac{1}{2}}(-e_y)\mathrm{N'}_{\frac{1}{2}}(e_y) + \mathrm{N}_{-\frac{1}{2}}(e_y)\mathrm{N'}_{-\frac{1}{2}}(-e_y) + \mathrm{N}_{-\frac{1}{2}}(-e_y)\mathrm{N'}_{-\frac{1}{2}}(e_y) \\ &+\mathrm{N}_{\frac{1}{2}}(e_z)\mathrm{N'}_{\frac{1}{2}}(-e_z) + \mathrm{N}_{\frac{1}{2}}(-e_z)\mathrm{N'}_{\frac{1}{2}}(e_z) - \mathrm{N}_{-\frac{1}{2}}(e_z)\mathrm{N'}_{-\frac{1}{2}}(-e_z) - \mathrm{N}_{-\frac{1}{2}}(-e_z)\mathrm{N'}_{-\frac{1}{2}}(e_z) \end{aligned}$$

$$O_{T_2^+,r=1}^{[J=2,P=+,\lambda_N=\frac{1}{2},\lambda_{N'}=-\frac{1}{2}]} = O_{T_2^+,r=1}^{[J=2,P=+,\lambda_N=-\frac{1}{2},\lambda_{N'}=\frac{1}{2}]} = O_{T_2^+,r=1}^{[J=2,L=2,S=1]} = O_{T_2^+,r=1}$$

VN, |p|=1, H^+ irrep, J^P=3/2$^+$, 5/2$^+$,.. (n=1,2,3)

H^- :

$$O_{H^-,r=1,n=1} = iN_{\frac{1}{2}}(e_x)V_x(-e_x) + iN_{\frac{1}{2}}(-e_x)V_x(e_x) + N_{\frac{1}{2}}(e_y)V_y(-e_y) + N_{\frac{1}{2}}(-e_y)V_y(e_y)$$

$$\begin{aligned} O_{H^-,r=1,n=2} = &-iN_{\frac{1}{2}}(e_x)V_y(-e_x) - iN_{\frac{1}{2}}(-e_x)V_y(e_x) - N_{-\frac{1}{2}}(e_x)V_z(-e_x) - N_{-\frac{1}{2}}(-e_x)V_z(e_x) \\ &+N_{\frac{1}{2}}(e_y)V_x(-e_y) + N_{\frac{1}{2}}(-e_y)V_x(e_y) + N_{-\frac{1}{2}}(e_y)V_z(-e_y) + N_{-\frac{1}{2}}(-e_y)V_z(e_y) \\ &+2N_{\frac{1}{2}}(e_z)\left(V_x(-e_z) - iV_y(-e_z)\right) + 2N_{\frac{1}{2}}(-e_z)\left(V_x(e_z) - iV_y(e_z)\right) \end{aligned}$$

$$\begin{aligned} O_{H^-,r=1,n=3} = &\,iN_{\frac{1}{2}}(e_x)V_y(-e_x) + iN_{\frac{1}{2}}(-e_x)V_y(e_x) - 2N_{-\frac{1}{2}}(e_x)V_z(-e_x) - 2N_{-\frac{1}{2}}(-e_x)V_z(e_x) \\ &-N_{\frac{1}{2}}(e_y)V_x(-e_y) - N_{\frac{1}{2}}(-e_y)V_x(e_y) + 2N_{-\frac{1}{2}}(e_y)V_z(-e_y) + 2N_{-\frac{1}{2}}(-e_y)V_z(e_y) \\ &+N_{\frac{1}{2}}(e_z)\left(V_x(-e_z) - iV_y(-e_z)\right) + N_{\frac{1}{2}}(-e_z)\left(V_x(e_z) - iV_y(e_z)\right) \end{aligned}$$

$$O_{H^-,r=1}^{[J=\frac{3}{2},P=-,\lambda_N=\pm\frac{1}{2},\lambda_V=0]} = O_{H^-,r=1}^{[J=\frac{5}{2},P=-,\lambda_N=\pm\frac{1}{2},\lambda_V=0]} = O_{H^-,r=1,n=1}$$

$$\begin{aligned} O_{H^-,r=1}^{[J=\frac{3}{2},P=-,\lambda_N=\frac{1}{2},\lambda_V=-1]} &= O_{H^-,r=1}^{[J=\frac{3}{2},P=-,\lambda_N=-\frac{1}{2},\lambda_V=1]} = \\ &= O_{H^-,r=1}^{[J=\frac{5}{2},P=-,\lambda_N=\frac{1}{2},\lambda_V=-1]} = O_{H^-,r=1}^{[J=\frac{5}{2},P=-,\lambda_N=-\frac{1}{2},\lambda_V=1]} = O_{H^-,r=1,n=2} \end{aligned}$$

$$\begin{aligned} O_{H^-,r=1}^{[J=\frac{3}{2},P=-,\lambda_N=\frac{1}{2},\lambda_V=1]} &= O_{H^-,r=1}^{[J=\frac{3}{2},P=-,\lambda_N=-\frac{1}{2},\lambda_V=-1]} = \\ &= O_{H^-,r=1}^{[J=\frac{5}{2},P=-,\lambda_N=\frac{1}{2},\lambda_V=1]} = O_{H^-,r=1}^{[J=\frac{5}{2},P=-,\lambda_N=-\frac{1}{2},\lambda_V=-1]} = O_{H^-,r=1,n=2} - 2\ O_{H^-,r=1,n=3} \end{aligned}$$

$$O_{H^-,r=1}^{[J=\frac{3}{2},L=0,S=\frac{3}{2}]} = -3i\ O_{H^-,r=1,n=1} + 2\ O_{H^-,r=1,n=2} - O_{H^-,r=1,n=3}$$

$$O_{H^-,r=1}^{[J=\frac{3}{2},L=2,S=\frac{3}{2}]} = O_{H^-,r=1}^{[J=\frac{5}{2},L=2,S=\frac{3}{2}]} = 3i\ O_{H^-,r=1,n=1} + O_{H^-,r=1,n=2} + O_{H^-,r=1,n=3}$$

$$O_{H^-,r=1}^{[J=\frac{3}{2},L=2,S=\frac{1}{2}]} = O_{H^-,r=1}^{[J=\frac{5}{2},L=2,S=\frac{1}{2}]} = 3i\ O_{H^-,r=1,n=1} + O_{H^-,r=1,n=2} - 2\ O_{H^-,r=1,n=3}$$

$$O_{H^-,r=1}^{[J=\frac{5}{2},L=4,S=\frac{3}{2}]} = 12i\ O_{H^-,r=1,n=1} - 3\ O_{H^-,r=1,n=2} + 4\ O_{H^-,r=1,n=3}$$

For all irreps we verified:

- all three methods give consistent operators
- the number of linearly independent operators agree with Moore & Fleming (2006) [this reference indicates which CG are non-zero but does not provide their values]

H(p)H(-p), |p|>1

- The explicit expressions are not provided in the paper as it would get to lengthy

- One can straightforwardly obtain them using the
  - general relations for three methods and
  - all necessary technical details given in the paper

- |p|=1 expressions can be used as cross-check

Conclusions

- We construct H(p)H(-p) operators for scattering of particles with spin
- General expressions for operators given in three formally independent methods
- Consistent results found in three methods

✧ Projection operators O_n: gives little guidance on underlying quantum numbers
✧ Partial-wave operators: provides linear combinations O_n to enhance coupling to (J, S, L)
✧ Helicity operators: provides linear combinations O_n to enhance coupling to (J, P, λ1, λ2)

- Proofs of correct transformation for all three methods.
  These demonstrate that simple (non-canonical) H_{ms}(p) can be used as building blocks.
- Explicit expressions for PV, PN, VN, NN for p=0,1
- All necessary technical details for explicit construction

[details in the paper]

- Operators will lead to eigen-energies of HH. These are related to scattering phase shifts
  for H with arbitrary spin in [Briceno, Phys. Rev. D89, 074507 (2014)]

Backup slides

Transformations of the employed H (see also slide 4)

$$RP(p)R^{-1} = P(Rp), \quad IP(p)I = -P(-p)$$

$$RV_i(p)R^{-1} = T^{s=1}_{ji}(R)^* V_j(Rp) = \exp(-i\vec{n}\vec{J}\omega)_{ji} V_j(Rp), \quad IV_i(p)I = -V_i(-p) \quad i,j = x,y,z$$

$$RN_{m_s}(p)R^{-1} = D^s_{m_s' m_s}(R)^* N_{m_s'}(Rp) = [\exp(-\tfrac{i}{2}\vec{n}\vec{\sigma}\omega)]^*_{m_s' m_s} N_{m_s'}(Rp), \quad IN_{m_s}(p)I = N_{m_s}(-p)$$

Properties of Wigner D matrices

$$D(R) = D^\dagger(R^\dagger) \ , \qquad R^\dagger = R^{-1} \ , \qquad D(R_1 R_2) = D(R_1) D(R_2)$$

Proof: partial-wave operators

Proof of correct transformation properties:

$$R_a O^{J,m_J,S,L} R_a^{-1} = \sum_{m_L,m_S,m_{s1},m_{s2}} C^{Jm_J}_{Lm_L,Sm_S} C^{Sm_S}_{s_1m_{s1},s_2m_{s2}} \sum_{R\in O^{(2)}} Y^*_{Lm_L}(\hat{Rp})\ R_a H_{m_{s1}}(Rp) H_{m_{s2}}(-Rp) R_a^{-1}$$

$$= \sum_{m_L,m_S,m_{s1},m_{s2}} C^{Jm_J}_{Lm_L,Sm_S} C^{Sm_S}_{s_1m_{s1},s_2m_{s2}} \sum_{R\in O_h} Y^*_{Lm_L}(\hat{Rp})$$

$$\times \sum_{m'_{s1}} D^{s_1}_{m_{s1}m'_{s1}}(R_a^{-1}) H_{m'_{s1}}(R_aRp) \sum_{m'_{s2}} D^{s_2}_{m_{s2}m'_{s2}}(R_a^{-1}) H_{m'_{s2}}(-R_aRp)\ ,$$

$$Y^*_{Lm_L}(Rp) = Y^*_{Lm_L}(R_a^{-1}(R'p)) = \sum_{m'_L} D^L_{m_Lm'_L}(R_a^{-1}) Y^*_{Lm'_L}(R'p) \qquad R' \equiv R_a R \qquad Y^*_{Lm_L}(R_1p) = \textstyle\sum_{m'_L} D^L_{m_Lm'_L}(R_1) Y^*_{Lm'_L}(p)$$

$$D^{s_1}_{m_{s1}m'_{s1}}(R_a^{-1}) D^{s_2}_{m_{s2}m'_{s2}}(R_a^{-1}) = \sum_{\tilde{S},\tilde{m}_S,m'_S} C^{\tilde{S},\tilde{m}_S}_{s_1m_{s1},s_2m_{s2}} C^{\tilde{S},m'_S}_{s_1m'_{s1},s_2m'_{s2}}\ D^{\tilde{S}}_{\tilde{m}_Sm'_S}(R_a^{-1}) \qquad \sum_{m_{s1},m_{s2}} C^{Sm_S}_{s_1m_{s1},s_2m_{s2}} C^{\tilde{S},\tilde{m}_S}_{s_1m_{s1},s_2m_{s2}} = \delta_{m_S,\tilde{m}_S}\delta_{S,\tilde{S}}$$

$$D^L_{m_Lm'_L}(R_a^{-1}) D^{\tilde{S}}_{\tilde{m}_Sm'_S}(R_a^{-1}) = \sum_{\tilde{J},\tilde{m}_J,m'_J} C^{\tilde{J},\tilde{m}_J}_{Lm_L,\tilde{S}\tilde{m}_S} C^{\tilde{J},m'_J}_{Lm'_L,\tilde{S}m'_S}\ D^{\tilde{J}}_{\tilde{m}_Jm'_J}(R_a^{-1}) \qquad \sum_{m_L,m_S} C^{Jm_J}_{Lm_L,Sm_S} C^{\tilde{J},\tilde{m}_J}_{Lm_L,Sm_S} = \delta_{m_J,\tilde{m}_J}\delta_{J,\tilde{J}}$$

$$R_a O^{J,m_J,S,L} R_a^{-1} =$$

$$= \sum_{m'_J} D^J_{m_Jm'_J}(R_a^{-1}) \sum_{m'_L,m'_S,m'_{s1},m'_{s2}} C^{Jm'_J}_{Lm'_L,Sm'_S} C^{Sm'_S}_{s_1m'_{s1},s_2m'_{s2}} \sum_{R'\in O^{(2)}} Y^*_{Lm'_L}(\hat{R'p}) H_{m'_{s1}}(R'p) H_{m'_{s2}}(-R'p)$$

$$= \sum_{m'_J} D^J_{m_Jm'_J}(R_a^{-1}) O^{J,m'_J,S,L}$$

Vectors

$$V^\dagger|0\rangle = |V\rangle = A_x|V_x\rangle + A_y|V_y\rangle + A_z|V_z\rangle$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}_{m_J=1} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix}, \quad \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}_{m_J=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}_{m_J=-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$(S_k)_{ij} = -i\epsilon_{ijk}$$

$$S_z \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}_{m_J} = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}_{m_J} = m_J \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

The annihilation operators are obtained by hermitian conjugation, so coefficients are complex conjugated.