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Beyond complex Langevin equations:

from simple examples to positive representation

of Feynman path integrals directly in Minkowski time

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In general quantum averages result from complex "densities"

- **A harmonic oscillator**

$$\langle x^2(T)\rangle \;=\; \frac{\int dx x^2 K(x,x;T)}{\int dx K(x,x;T)}.$$

- **QED/QCD with external charges**

$$Z = \int DA \exp\left\{\frac{i}{\hbar}S[A]\right\}\exp\left\{\frac{i}{\hbar}\int d^4x A_\mu J^\mu_{ext}\right\} \leftrightarrow \int DU \exp\left\{S_{latt}[U]\right\}W[U] \qquad (1)$$

- **QCD with finite chemical potential**

$$Z = \int DU \exp\left\{S_{latt}[U]\right\}Det[A_0 \to A_0 + i\mu] \qquad (2)$$

P. Hasenfratz, F. Karsch, 1983

- **"Complex Langevin" approach**
- **Avoiding the trouble - one degree of freedom**
- **Two examples**
- **Path integrals – a free particle**
- **Path integrals – an harmonic oscillator**
- **Classic quantum applications**

  free particle

  harmonic oscillator

  particle in a constant magnetic field
- **Possible interpretation**
- **Outlook**

I. The Langevin method - real and complex cases

Real case

$$S(x) \quad \xrightarrow{\dot{x}(\tau)=-\partial_x S+\eta(\tau)} \quad x(\tau) \quad \longrightarrow \quad P(x,\tau) \quad \xrightarrow{\tau\to\infty} \quad P(x) = e^{-S(x)}$$

$$\int f(x)e^{-S(x)}dx \,\Big/\!\int e^{-S(x)}dx = \int f(x)P(x)dx \,\Big/\!\int P(x)dx \,.$$

Complex case, e.g. $\rho(x) \equiv e^{-S(x)} = e^{-\sigma x^2/2}, \sigma \in \mathcal{C}$,

G. Parisi, 1983
J.Klauder, 1983

$$S(x) \quad \xrightarrow{\dot{z}(\tau)=-\partial_z S+\eta(\tau)} \quad z(\tau) \quad \longrightarrow \quad P(x,y,\tau)$$

But $P(x,y,\tau)$ does not $\xrightarrow{\tau\to\infty}$ e^{-S}

Nevertheless in some cases $P(x,y,\infty)$ exists, and indeed

$$\frac{\int f(x)e^{-S(x)}dx}{\int e^{-S(x)}dx} = \frac{\int\int f(x+iy)P(x,y,\infty)dxdy}{\int\int P(x,y,\infty)dxdy}. \tag{3}$$

Example (Ambjorn and Yang, 1985)

$$\rho(x) = \exp\left(-S_g(x)\right) = \exp\left(-\frac{\sigma}{2}x^2\right), \quad \sigma = \sigma_R + i\sigma_I, \quad \sigma_R > 0$$

$$P(x,y) = \exp\left(-\sigma_R(x^2 + 2rxy + (1+2r^2)y^2)\right), \qquad r = \frac{\sigma_R}{\sigma_I},$$

indeed

$$\langle x^n \rangle_{\rho(x)} = \langle (x+iy)^n \rangle_{P(x,y)}$$

or for the generating function

$$G(t) = \frac{\int_{-\infty}^{\infty} e^{tx} e^{-S_g(x)} dx}{\int_{-\infty}^{\infty} e^{-S_g(x)} dx} = \exp\left(\frac{t^2}{2\sigma}\right) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx dy e^{t(x+iy)} P(x,y)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx dy P(x,y)}.$$

In general, however

- $P(x, y, \tau)$ **not known**
- **not connected with** S

Moreover

- $z(\tau) \to \infty$, **i.e. diverges**
- $z(\tau) \to$ **wrong answer**

G. Aarts et al. (2 - 6), 2005 - present

II. Avoiding the trouble

- Construct ρ, P without any reference to the stochastic process.

The only requirement

$$\int_R f(x)\rho(x)dx \Big/ \int_R \rho(x)dx = \int\int f(x+iy)P(x,y)dxdy \Big/ \int\int P(x,y)dxdy \,.$$

- • Introduce two, independent complex variables

$$z = x + iy, \quad \bar{z} = x - iy$$

then

$$\int_{\Gamma_z} f(z)\rho(z)dz \Big/ \int_{\Gamma_z} \rho(z)dz = \int_{\Gamma_z}\int_{\Gamma_{\bar{z}}} f(z)P(z,\bar{z})dzd\bar{z} \Big/ \int_{\Gamma_z}\int_{\Gamma_{\bar{z}}} P(z,\bar{z})dzd\bar{z}$$

sufficient condition

$$\rho(z) = \int_{\Gamma_{\bar{z}}} P(z,\bar{z})d\bar{z}$$

additionally require

$$P(z,\bar{z})|_{x+iy,x-iy} \quad \text{positive and normalizable}$$

The construction

1. Find $P(z,\bar{z})$ which satisfies

$$\rho(z) = \int_{\Gamma_{\bar{z}}} P(z,\bar{z}) d\bar{z}$$

and

2. Is positive and normalizable at

$$P(z,\bar{z})|_{x+iy,x-iy} = P(x,y)$$

- **Then**

$$\int f(z)\rho(z)dz \Big/ \int \rho(z)dz = \int\int f(x+iy)P(x,y)dxdy \Big/ \int\int P(x,y)dxdy$$

Example 1 - generalized gaussian model

$$S(z,\bar z) = a^* z^2 + 2bz\bar z + a\bar z^2 \ |_{x+iy,x-iy} = \ 2(b+\alpha)x^2 + 4\beta xy + 2(b-\alpha)y^2,$$

$$a = \alpha + i\beta, \quad b = b^*, \qquad \lambda_\pm = 2(b \pm |a|)$$

$$P(x,y) = e^{-S(x,y)} \qquad \text{is positive and normalizable for} \quad b > |a|$$

At the same time

$$\rho(z) = \int_{\Gamma_{\bar z}} P(z,\bar z) d\bar z = \frac{1}{2}\sqrt{\frac{\pi}{-a}} \exp\left(-sz^2\right), \quad s = \frac{|a|^2 - b^2}{a}.$$

This reduces to $\exp\left(-\frac{\sigma}{2}z^2\right)$ if

$$b = \frac{\sigma_R}{2}(1+r^2), \quad \alpha = -\frac{\sigma_R}{2}r^2, \quad \beta = \frac{\sigma_R}{2}r, \quad r = \frac{\sigma_R}{\sigma_I}, \quad \sigma_R > 0$$

- But is more general.
- •• Provides positive representation for any complex a.

$$\langle x^n \rangle_{\rho(x)} \longleftarrow \langle z^n \rangle_{\rho(z)} = \langle (x+iy)^n \rangle_{P(x,y)}$$

$\langle$ real obs. over complex weight $\rangle$ = $\langle$ complex obs. over real weight $\rangle$

• • • For example:

$\alpha = 0, \beta \neq 0$ – $e^{i|s|x^2}$ – "Minkowski" integrals

$\alpha > 0, \beta = 0$ – $e^{+|s|x^2}$ – "a striking example:

complex stochastic process has the correct fixed point also for $\sigma_R < 0$"

$\int f(x+iy) P(x,y) dx dy$ = analytic continuation of $\int \rho(x) f(x) dx$

H. Okamoto et al., 1988
L.L. Salcedo, 1997
D. Weingarten, 2002

Example 2 - quartic action

$$S_4(z,\bar{z}) = (a^* z^2 + 2bz\bar{z} + a\bar{z}^2)(c^* z^2 + 2dz\bar{z} + c\bar{z}^2) \xrightarrow{c=ad/b} \frac{d}{b}(a^* z^2 + 2bz\bar{z} + a\bar{z}^2)^2,$$

$$\rho_4(z) \;=\; \frac{i}{2}\int_{\Gamma_{\bar{z}}} d\bar{z} e^{-S_4(z,\bar{z})} = \frac{i}{2}\left(\frac{b}{2da^2}\right)^{\frac{1}{4}} \exp\left(-\sigma z^4\right)\left(\sigma z^4\right)^{\frac{1}{4}} K_{\frac{1}{4}}\left(\sigma z^4\right),$$

with an arbitrary complex

$$\sigma = \frac{d(b^2 - |a|^2)^2}{2ba^2}.$$

$$\rho(z) \sim e^{\left(-2\sigma z^4\right)}, \quad z \longrightarrow \infty$$

again

$$\langle x^n \rangle_{\rho(x)} \longleftarrow \langle z^n \rangle_{\rho(z)} = \langle (x+iy)^n \rangle_{P(x,y)}$$

III. Path integrals in Minkowski time – a free particle

2N variables $z_i, \bar{z}_i \; (\equiv z, \bar{z})$ with periodic boundary conditions

$$S_N(z,\bar{z}) = \sum_{i=1}^{N} a\bar{z}_i^2 + 2b\bar{z}_i z_i + 2c\bar{z}_i z_{i+1} + 2c^* z_i \bar{z}_{i+1} + a^* z_i^2, \quad a, c \in C, b \in R.$$

$$\rho(z) \sim \int \prod_{i=1}^{N} d\bar{z}_i \exp\left(-S_N(z,\bar{z})\right) \sim \exp\left(\mathcal{A} \sum_{i=1}^{N} \left(z_{i+1} - z_i\right)^2 - r\left(z_{i+1} - z_{i-1}\right)^2\right)$$

$$2c \equiv 2\gamma = -b + |a|, \qquad \mathcal{A} = \frac{b(b - |a|)}{a}, \qquad r = \frac{b - |a|}{4b}.$$

to be compared with $$S_N^{free} = \frac{im}{2\hbar\epsilon} \sum_{i=1}^{N} (z_{i+1} - z_i)^2$$

$$S_N(x,y) = 2\sum_{i=1}^{N} (b+\alpha)x_i^2 + 2\beta x_i y_i + (b-\alpha)y_i^2 + 2\gamma(x_i x_{i+1} + y_i y_{i+1}).$$

$P_N(x_i, y_i)$ is real and positive

$P_N(x_i, y_i)$ normalizable up to one zero mode $\lambda_i > 0, i > 0$

$$\lambda_0 = 2(b - |a| + 2\gamma),$$

The continuum limit of the propagator

$$K_N(z_N, z_1) = e^{-\mathcal{A}(z_N - z_1)^2} \int dz_2 ... dz_{N-1} e^{-S_N^{\rho}(z_1, ..., z_N)},$$

Transfer matrix $\rightarrow$ recursion

$$K_N(z_N, z_1) \sim \exp\left(\mathcal{A}\sigma_N(r)(z_N - z_1)^2\right), \qquad \sigma_N(r) = \frac{P_N(r)}{Q_N(r)}$$

$$\sigma_N(r) = v_0(r) + \frac{v_1(r)}{N} + \frac{v_2(r)}{N^2} + ..., N \longrightarrow \infty$$

$r = 0 \Longrightarrow v_1 = 1, \;\; v_i = 0, i \neq 1$ – Feynman case

$r \neq 0 \Longrightarrow v_i \neq 0$ – no continuum limit since $\mathcal{A} \sim 1/\epsilon$

• • • define the new limit (lim_1)

$$|a|, b \rightarrow \infty, b - |a| = \frac{m}{2\hbar\epsilon} = const., \;\; a = -i|a|.$$

• • • than physics $= lim_{N\rightarrow\infty} \; lim_1 \; \langle O \rangle_{P_N(x,y)}$

VI. Path integrals in Minkowski time – an harmonic oscillator

$$\rho(z) = \left(\frac{i}{2}\right)^N \int \prod_{i=1}^{N} d\bar{z}_i P(z, \bar{z}) \sim \exp\left(-S_N^\rho(z)\right)$$

gaussian integrations

$$-S_N^\rho(z) = \sum_{i=1}^{N} \frac{b^2 + (c+c^*)^2 - |a|^2}{2a}\left(z_i^2 + 2\frac{2b(c+c^*)}{b^2+(c+c^*)^2-|a|^2} z_i z_{i+1} + z_{i+1}^2\right) + O\left(\frac{cc^*}{a}\right)$$

if $2c = 2\gamma = -b + |a|$, then a free case obtains.

On the other hand

$$Dz_i^2 + 2Ez_i z_{i+1} + Dz_{i+1}^2, \quad D = \frac{b^2 + 4\gamma^2 - |a|^2}{2a}, \quad E = \frac{2b\gamma}{a}$$

rewrite

$$-E\left((z_{i+1} - z_i)^2 - \left(\frac{D}{E} + 1\right)(z_i^2 + z_{i+1}^2)\right)$$

compare with an harmonic oscillator

$$\frac{im}{2\hbar\epsilon}\left((x_1 - x_2)^2 - \frac{\omega^2\epsilon^2}{2}(x_1^2 + x_2^2)\right)$$

hence

$$-\frac{2b\gamma}{a} = \frac{im}{2\hbar\epsilon}, \quad \frac{b^2 + 4\gamma^2 - |a|^2}{4b\gamma} + 1 = \frac{\omega^2\epsilon^2}{2}.$$

The continuum limit for HO

$$\rho = \frac{\omega^2 T^2}{2(N-1)^2}, \quad \mu = \frac{m(N-1)}{2\hbar T},$$

$$a = -i|a|, \quad b = \frac{\mu}{\nu}, \quad |a| = \frac{\mu}{\nu}\zeta(\nu,\rho), \quad 2\gamma = -\mu\zeta(\nu,\rho)$$

where

$$\zeta(\nu,\rho) = \frac{\sqrt{1-2\nu^2\rho+\nu^2\rho^2} - \nu(1-\rho)}{1-\nu^2}$$

$$\langle O \rangle = \overbrace{\lim_{N\to\infty}}^{lim2} \quad \overbrace{\lim_{\nu\to 0}}^{lim1} \quad \langle O \rangle_{P_N(x,y)}$$

IV. Applications – a free particle

The propagator from the new (x, y) representation

$$K_N(z_N, z_1) = e^{\left(-\mathcal{A}(z_1 - z_N)^2\right)} \int d\bar{z}_1 \prod_{j=2}^{N-1} dx_j dy_j d\bar{z}_N \exp\left(-X^T M X\right)$$

$$X^T = (z_1, \bar{z}_1, x_2, y_2, x_3 ..., y_{N-1}, z_N, \bar{z}_N)$$

$$K_N(z_N, z_1) \sim \exp\left(\sigma_N(a, b)(z_N - z_1)^2\right),$$

N	$\sigma_N(a, b)$	$lim\ 1$
5	$\frac{id(16b^2+28bd-19d^2)}{8(8b-3d)(b-d)}$	$i\frac{d}{4}$
8	$\frac{id(16b^4+40b^3d-70b^2d^2+23bd^3-d^4)}{(b-d)(112b^3-120b^2d+30bd^2-d^3)}$	$i\frac{d}{7}$
11	$\frac{id(1024b^5+3328b^4d-9472b^3d^2+6832b^2d^3-1700bd^4+109d^5)}{(b-d)(1280b^4-2304b^3d+1344b^2d^2-280bd^3+15d^4)}$	$i\frac{d}{10}$

$$lim1: \qquad |a|, b \to \infty, b - |a| = \frac{m}{2\hbar\epsilon} = const. \equiv d, \quad a = -i|a|.$$

$$\lim_{N\to\infty}\ lim1\ K_N \sim \exp\left(\frac{im}{2\hbar}\frac{(z_N - z_1)^2}{T}\right) \longrightarrow \exp\left(\frac{im}{2\hbar}\frac{(x_N - x_1)^2}{T}\right) \equiv K$$

An average

Classics

$$\langle x^2(t)\rangle = \int K(0,x;T-t)x^2K(x,0;t)dx \,/K(0,0;T) = \frac{i\hbar}{m}\frac{t(T-t)}{T}$$

New representation

$$\langle z_k^2\rangle\Big|_{z_1=z_N=0} = \int d\bar{z}_1 dx_2 dy_2 ... dy_{N-1} d\bar{z}_N (x_k+iy_k)^2 \exp\left(-X^T M X\right)/Z$$
$$= \frac{1}{2}\left(M^{-1}_{2k-2,2k-2} + i(M^{-1}_{2k-2,2k-1} + M^{-1}_{2k-1,2k-2}) - M^{-1}_{2k-1,2k-1}\right),$$

$$\langle z_k^2\rangle \quad \xrightarrow{lim1} \quad \frac{i}{2d}\frac{(k-1)(N-k)}{N-1} \quad \xrightarrow{lim2} \quad \frac{i\hbar}{m}\frac{t(T-t)}{T}$$

zero mode $\Longrightarrow$ fixed end points $\Longrightarrow$ two integrals ($d\bar{z}_1, d\bar{z}_N$) not positive

V. Applications – an harmonic oscillator

$$\langle x^2(T)\rangle = \langle x^2(0)\rangle \;=\; \frac{\int dx x^2 K(x,x;T)}{\int dx K(x,x;T)}.$$

Classics

$$K(x_b,x_a;T) \;\sim\; \exp\left\{\frac{i}{\hbar}\frac{m\omega}{2\sin\omega T}\Big((x_a^2+x_b^2)\cos\omega T - 2x_a x_b\Big)\right\},$$

gives

$$\langle x^2(T)\rangle \;=\; -\frac{i\hbar T}{4m}\frac{\cot\frac{\omega T}{2}}{\frac{\omega T}{2}}.$$

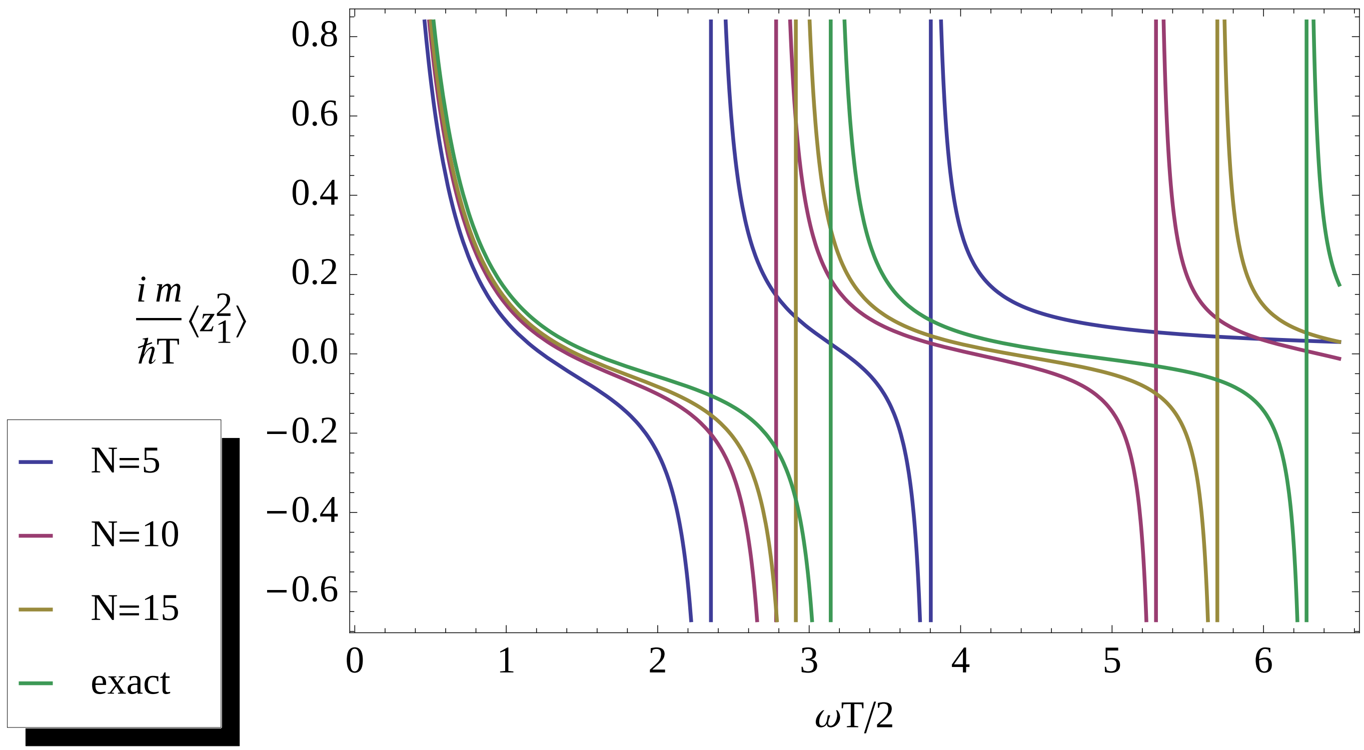
The new way

$$\langle z_1^2\rangle = \frac{1}{Z}\int\prod_{j=1}^{N} dx_j dy_j (x_1+iy_1)^2 \exp\left\{-X^T M X\right\}.$$

$$lim_1\;\langle z_1^2\rangle = \lim_{\nu\to 0}\langle z_1^2\rangle = -\frac{i\hbar T}{m}\frac{P_N(\omega T/2)}{Q_N(\omega T/2)}.$$

TABLE II

N	$P_N(x)/Q_N(x)$
5	$\frac{(x^2-2x-4)(x^2+2x-4)}{x^2(x^4-20x+80)}$
8	$\frac{7(128x^8-12544x^6+384160x^4-3764768x^2+5764801)}{32x^2(x^2-49)(2x^2-49)(8x^4-392x^2+2401)}$
11	$\frac{5(x^5-5x^4-100x^3+375x^2+1875x-3125)(x^5+5x^4-100x^3-375x^2+1875x+3125)}{2x^2(x^{10}-275x^8+27500x^6-1203125x^4+21484375x^2-107421875)}$



0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
$\frac{i\,m}{\hbar T}\langle z_1^2\rangle$
0
1
2
3
4
5
6
ωT/2
N=5
N=10
N=15
exact

Figure 1:

- the average is over positive probability

- lim1 of our average reproduces the standard discretization

- one negative mode

$$\boldsymbol{S_N(x,y) = 2\sum_{i=1}^{N}(b+\alpha)x_i^2 + 2\beta x_i y_i + (b-\alpha)y_i^2 + 2\gamma(x_i x_{i+1} + y_i y_{i+1}).}$$

The lowest eigenvalue

$$\boldsymbol{\lambda_0 = 2(b - |a| + 2\gamma) = \begin{cases} 0 & \text{a free particle} \\ \xrightarrow{lim1} -\frac{m\omega^2 T}{4\hbar(N-1)}, & \text{an harmonic oscillator} \end{cases}} \quad (4)$$

One can:
fix it
regularize it – M^{-1} **exists**
$\longrightarrow$ **use our trick again**
relax periodic boundary conditions

$\exp\{-S_N\}$ **provides a positive representation for the path integral of HO**

VI. Applications – a charged particle in a constant magnetic field

A cute observable

One can see classical orbits looking at the time dependence of quantum averages with fixed-end-points propagators as weights.

Define

$$\langle \vec{x} \rangle = \int d^2x K(\vec{x}_b, \vec{x}; T - t)\vec{x} K(\vec{x}, \vec{x}_a; t) / K(\vec{x}_b, \vec{x}_a; T) = x^{cl}_{x_a, x_b, T}(t)$$

- is gauge invariant
- can be reduced to HO averages

$$\langle x(t) \rangle_B = \langle x(t) \rangle_{O=O_x}$$
$$\langle y(t) \rangle_B = \langle y(t) \rangle_{O=O_y}$$

Two gauges

$$\vec{A}^{FG} = \frac{B}{2}\left\{\begin{array}{c} -y \\ x \\ 0 \end{array}\right\} \qquad \vec{A}^{LG} = B\left\{\begin{array}{c} 0 \\ x \\ 0 \end{array}\right\} \tag{5}$$

Landau: particle in a magnetic field ↔ 1 d harmonic oscillator

$$H = \frac{1}{2m}\hat{P}_x^2 + \frac{1}{2m}\left(\hat{P}_y - \frac{eB}{c}x\right)^2 = \frac{1}{2m}\hat{P}_x^2 + \frac{m\omega^2}{2}(x - O_x)^2$$

$$\Psi(x,y) = \exp\left(\frac{i}{\hbar}p_y y\right)\psi(x), \quad \omega = \frac{eB}{mc}, \quad O_x = \frac{c}{Be}p_y,$$

for propagators

$$K_{FG} = \exp\left\{\frac{im}{2\hbar}\left(\frac{\omega}{2}\cot\frac{\omega T}{2}\left(\Delta x^2 + \Delta y^2\right) + \omega\left(x_a y_b - y_a x_b\right)\right)\right\}$$

in the Landau gauge

$$K_{LG} = \exp\left\{\frac{im}{2\hbar}\left(\frac{\omega}{2}\cot\frac{\omega T}{2}\left(\Delta x^2 + \Delta y^2\right) + \omega(x_a + x_b)\Delta y\right)\right\}$$

$\longrightarrow$ split the LG propagator as follows

$$K_{LG} = \underbrace{\exp\left\{\frac{i}{\hbar}\frac{m}{2}\left(\omega(x_a + x_b)\Delta y + \omega ct\Delta y^2\right)\right\}}_{\exp\left(\frac{i}{\hbar}p_y(y_b - y_a)\right) = \exp\left(\frac{i}{\hbar}m\omega O_x(y_b - y_a)\right)} \underbrace{\exp\left\{\frac{i}{\hbar}\frac{m}{2}\left(\frac{\omega}{2}\cot\frac{\omega T}{2}\left(\Delta x^2 - \Delta y^2\right)\right)\right\}}_{K^{HO}_{O_x}} \quad (6)$$

with the classical expression for the center of oscillations

$$O_x = \frac{1}{2}(x_a + x_b) + \frac{1}{2}\cot\frac{\omega T}{2}(y_b - y_a)$$

more precisely

$$K^B(\vec{x}_b, \vec{x}_a; T) = \int dO \exp\left\{\frac{i}{\hbar}m\omega O(y_b - y_a)\right\} K^{HO}_O(x_b, x_a; T)|_{saddlepoint} = (6)$$

→ **Need positive distribution representing shifted harmonic oscillator.**

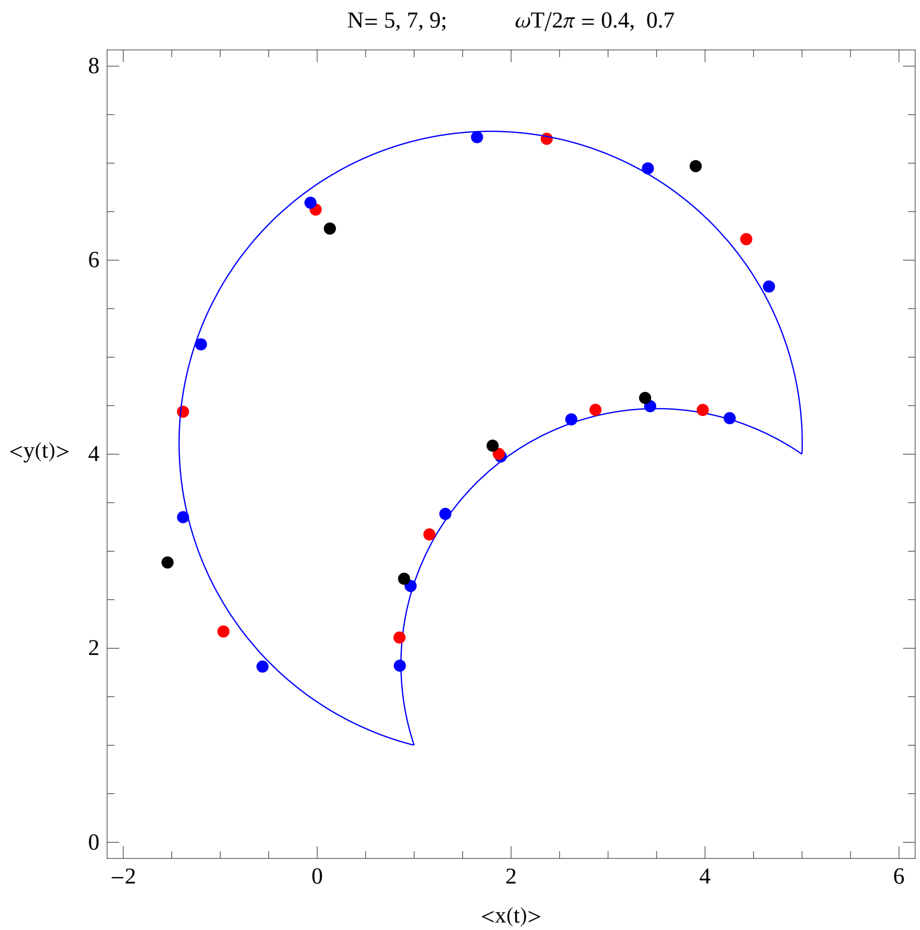
Simply add linear terms

$$S_N(z,\bar{z}) \to S_N(z,\bar{z}) + \sum_i e^* z_i + e\bar{z}_i \leftrightarrow S_N(z - z_c, \bar{z} - z_c^*),$$

then the average

$$\langle z_k \rangle_{P(x,y)} = \langle x_k + i y_k \rangle_{P(x,y)}$$

reproduces, after taking $lim 1$, the magnetic field average.


N= 5, 7, 9; ωT/2π = 0.4, 0.7
<y(t)>
<x(t)>
8
6
4
2
0
−2
0
2
4
6

VII. Summary

Problems of complex Langevin approach were avoided.

Instead of simulating badly converging complex random walks, pairs of corresponding weights were directly constructed.

- **Two complex variables z and $\bar{z}$.**
- **The sum rule $\rho(z) = \int d\bar{z} P(z, \bar{z})$ was derived and used to explicitly obtain P and ρ in few cases of physical interest.**

One degree of freedom:

gaussian model was generalized to arbitrary complex slope

a particular quartic problem was also solved

Quantum mechanical path integrals directly in Minkowski time:

a free particle

an harmonic oscillator

a particle in a constant magnetic field – standard Wick rotation does not give positive representation in this case

VIII. To be done – plenty

- **Explore a freedom in $P(z,\bar{z})$ to derive it from given $\rho(z)$**

- **Mathematics: how to construct positive and normalizable P from ρ ?**
  – deformations (?)

- **Generalizations – 1 DOF: nonabelian models, compact variables, fermions**

- **Generalizations – field theory ...**

IX. An amusing interpretation

New variables ($\bar{z}'s$) $\longrightarrow$

I. **Quantum amplitudes became positive**

II. **$\longrightarrow$ no interference.**

III. **Path integrals have to be evaluated in the limit of $|a|, b \to \infty$**

IV. **$\longrightarrow$ a single trajectory might dominate the path integral (saddle point).**

V. **The variables are integrated over – they are hidden !**

Are these the hidden variables of quantum mechanics?

$\Rightarrow$There are some similarities but there is no revolution .

X. Demystification

I. Quantum amplitudes became positive
– only integrands of $\int dxdy$ did –
full quantum amplitudes are integrals over new variables with fixed end-points $\rightarrow$ they are complex.

II. $\longrightarrow$ No interference
– but true quantum amplitudes add only after integrating new ones at fixed points $\rightarrow$ there is an interference.

III. Path integrals have to be evaluated in the limit of $|a|, b \rightarrow \infty$
– but the difference $b - |a|$ remains fixed and it is $O\,(1/\hbar)$.

IV. $\longrightarrow$ A single trajectory might dominate path integral
$\rightarrow$ finite difference $b - |a|$ drives quantum fluctuations even at infinite $b, |a|$.

V. Nevertheless the variables are integrated over – they are hidden !

• • Therefore quantum mechanics remains intact, Bell inequalities are violated (as they should), etc., etc.

• • Nevertheless some new structure may have shown up.

• • It remains to be seen if these variables tell us more than just help with some numerical calculations.