Lefschetz-thimble approach to the Silver Blaze problem of one-site fermion model

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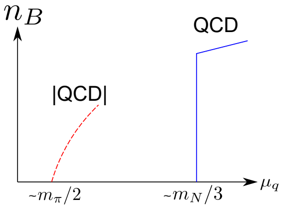
Sign problem of finite-density QCD

QCD partition function:

$$Z_{\mathrm{QCD}}(T,\mu)=\int \mathcal{D}A\ \underbrace{\mathrm{Det}(\not{D}(A,\mu_q)+m)}_{\text{quark}}\underbrace{\exp\left(-S_{\mathrm{YM}}(A)\right)}_{\text{gluon}}.$$

Sign problem: $\mathrm{Det}(\not{D}(A,\mu_q)+m)\not\geq 0$ at $\mu_q\neq 0$.

At $T=0$, the sign problem becomes severe at $\mu_q\geq m_\pi/2$ (e.g., Barbour et. al. (PRD **56** (1998) 7063))



Baryon Silver Blaze problem: Explain why $n_B=0$ for $\mu_q\lesssim m_N/3$ via path integral. (Cohen, PRL **91** (2003) 222001)

Sign problem of path integrals

Consider the path integral:

$$Z = \int \mathcal{D}x \exp(-S[x]).$$

- $S[x]$ is real $\Rightarrow$ No sign problem. Monte Carlo works.
- $S[x]$ is complex $\Rightarrow$ Sign problem appears!

If $S[x] \in \mathbb{C}$, eom $S'[x] = 0$ may have no real solutions $x(t) \in \mathbb{R}$.

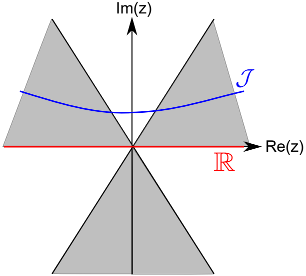
Idea: Complexify $x(t) \in \mathbb{C}$!

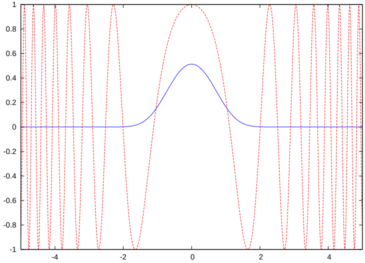
Lefschetz thimble for Airy integral

Airy integral is given as

$$\mathrm{Ai}(a) = \int_{\mathbb{R}} \frac{\mathrm{d}x}{2\pi} \exp \mathrm{i} \left(\frac{x^3}{3} + ax \right)$$

Complexify the integration variable: $z = x + \mathrm{i}y$.





Integrand on $\mathbb{R}$, and on $\mathcal{J}_1$ $(a = 1)$

One-site Fermi Hubbard model

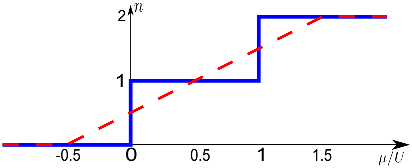
One-site Hubbard model:

$$\hat{H} = U\hat{n}_{\uparrow}\hat{n}_{\downarrow} - \mu(\hat{n}_{\uparrow} + \hat{n}_{\downarrow}).$$

Fock state gives the number density immediately:

$$\langle \hat{n} \rangle = \frac{1}{\beta}\frac{\partial}{\partial \mu} \ln Z = \frac{2(\mathrm{e}^{\beta\mu} + \mathrm{e}^{\beta(2\mu - U)})}{1 + 2\mathrm{e}^{\beta\mu} + \mathrm{e}^{\beta(2\mu-U)}}.$$

In the zero-temperature limit,



(YT, Hidaka, Hayata, NJP 18 (2016) 033002)(cf. Monte Carlo with 1-thimble ansatz: Fujii, Kamata, Kikukawa,1509.08176, 1509.09141; Alexandru, Basar, Bedaque,1510.03258.)

Path integral for one-site model

Effective Lagrangian of the one-site Hubbard model:

$$\mathcal{L} = \frac{\varphi^2}{2U} + \psi^* \left[\partial_\tau - (U/2 + \mathrm{i}\varphi + \mu)\right] \psi.$$

The path-integral expression is

$$Z = \sqrt{\frac{\beta}{2\pi U}} \int_{\mathbb{R}} \mathrm{d}\varphi \underbrace{\left(1 + \mathrm{e}^{\beta(\mathrm{i}\varphi + \mu + U/2)}\right)^2}_{\text{Fermion Det}} \mathrm{e}^{-\beta\varphi^2/2U}.$$

Integrand has complex phases causing the sign problem.

φ is an auxiliary field for the number density:

$$\langle \hat{n} \rangle = \mathrm{Im}\langle \varphi \rangle / U.$$

Silver Blaze problem for $\mu < -U/2$, $\mu < m_\pi/2$

One-site Hubbard model: As $\beta U \gg 1$ and $-U/2-\mu > 0$,

$$\mathrm{Det}\left[\partial_\tau - \left(\mu + \frac{U}{2} + \mathrm{i}\varphi\right)\right] = \left(1+\mathrm{e}^{-\beta(-U/2-\mu)}\mathrm{e}^{\mathrm{i}\beta\varphi}\right)^2 \simeq 1.$$

The sign problem almost disappears, so that $\mathcal{J}_* \simeq \mathbb{R}$.

Finite-density QCD: As $\beta \to \infty$ and $\mu < m_\pi/2$,

$$\frac{\mathrm{Det}\left[\gamma_4(\not{D}_A + m) - \mu\right]}{\mathrm{Det}\left[\gamma_4(\not{D}_A + m)\right]} = \prod_{\mathrm{Re}\lambda_j > 0} \frac{(1+\mathrm{e}^{-\beta(\lambda_j-\mu)})(1+\mathrm{e}^{-\beta(\lambda_j+\mu)})}{(1+\mathrm{e}^{-\beta\lambda_j})(1+\mathrm{e}^{-\beta\lambda_j})} \to 1.$$

(Cohen PRL **91** (2003), Adams, PRD **70** (2004), Nagata et. al. PTEP **2012**)

The sign problem disappears by the reweighting method.

$\Rightarrow$ Lefschetz thimbles $\simeq$ Original integration regions

Flows at $\mu/U < -0.5$ (and $\mu/U > 1/5$)

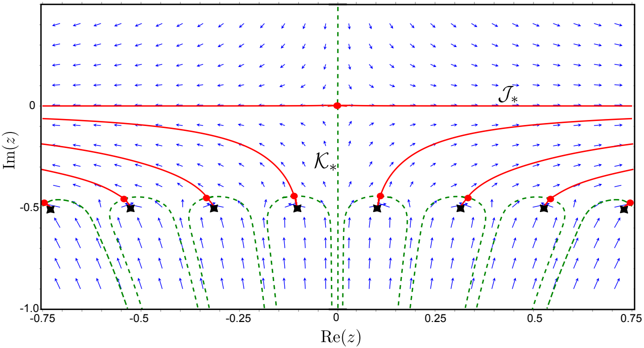


Figure: Flow at $\mu/U = -1$. $\mathcal{J}_* \simeq \mathbb{R}$.

$$Z = \int_{\mathcal{J}_*} \mathrm{d}z \ \mathrm{e}^{-S(z)}.$$

Number density: $n_* = 0$ for $\mu/U < -0.5$, $n_* = 2$ for $\mu/U > 1.5$.

(YT, Hidaka, Hayata, NJP 18 (2016) 033002)

Silver Blaze problem for $\mu > -U/2$, $\mu > m_\pi/2$

One-site Hubbard model: At each real config., the magnitude is exponentially large:

$$\mathrm{Det}\left[\partial_\tau - \left(\mu + \frac{U}{2} + \mathrm{i}\varphi\right)\right] = O\left(\mathrm{e}^{\beta(U+\mu/2)}\right)$$

This large contributions must be canceled exactly in order for $n = 0$.

Finite density QCD: The situation is almost the same, since

$$\frac{\mathrm{Det}(\not{D}(A,\mu_q)+m)}{\mathrm{Det}(\not{D}(A,0)+m)} \simeq \prod_{\mathrm{Re}(\lambda_A)<\mu_q} \exp\beta\left(\mu_q - \lambda_A\right),$$

but $n_B = 0$ for $\mu_q \lesssim m_N/3$.

Flows at $-0.5 < \mu/U < 1.5$

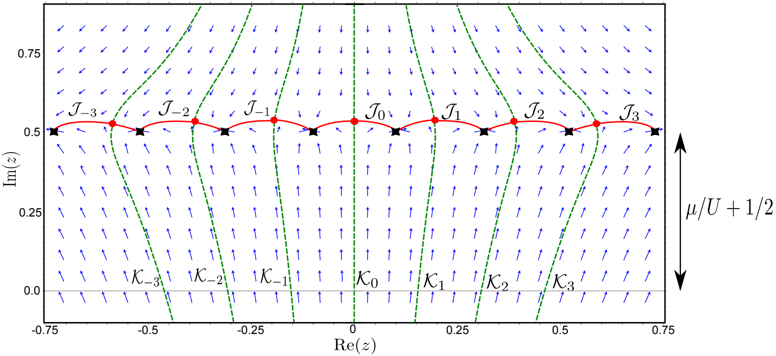


Figure: Flow at $\mu/U = 0$

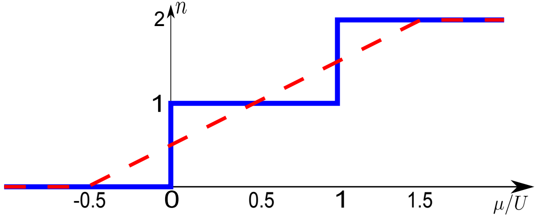
Complex saddle points lie on $\mathrm{Im}(z_m)/U \simeq \mu/U + 1/2$.

This value is far away from $n = \mathrm{Im}\,\langle z\rangle/U$=0, 1, or 2.

(YT, Hidaka, Hayata, NJP 18 (2016) 033002)

Curious incident of n in $-0.5U < \mu < 0$

We have a big difference bet. the exact result and naive expectation:

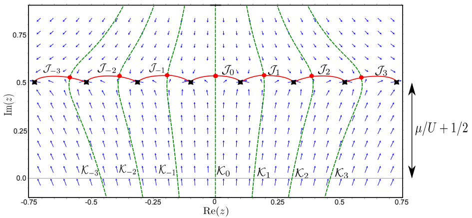


This is similar to what happens for QCD and |QCD|.
$\mu/U = -0.5 \Leftrightarrow \mu_q = m_\pi/2$.

Complex classical solutions

If $\beta U \gg 1$, the classical sol. for $-0.5 < \mu/U < 1.5$ are labeled by $m \in \mathbb{Z}$:

$$z_m \simeq \mathrm{i}\left(\mu + \frac{U}{2}\right) + 2\pi m T.$$

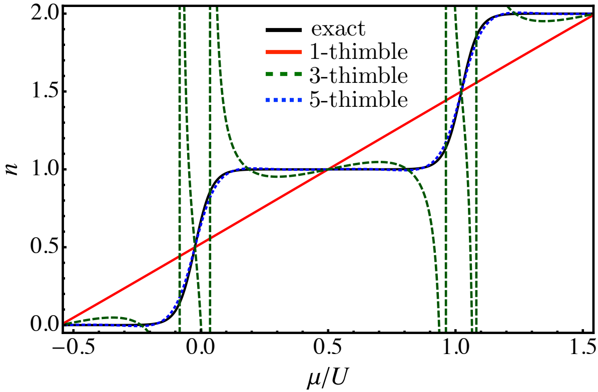


At these solutions, the classical actions become

$$S_0 \simeq -\frac{\beta U}{2}\left(\frac{\mu}{U}+\frac{1}{2}\right)^2,$$
$$\mathrm{Re}\left(S_m - S_0\right) \simeq \frac{2\pi^2}{\beta U} m^2,$$
$$\mathrm{Im}\, S_m \simeq 2\pi m\left(\frac{\mu}{U}+\frac{1}{2}\right).$$

Numerical results

Results for $\beta U = 30$: (1, 3, 5-thimble approx.: $\mathcal{J}_0$, $\mathcal{J}_0 \cup \mathcal{J}_{\pm 1}$, and $\mathcal{J}_0 \cup \mathcal{J}_{\pm 1} \cup \mathcal{J}_{\pm 2}$)



Necessary number of Lefschetz thimbles $\simeq \beta U/(2\pi)$.

(YT, Hidaka, Hayata, NJP 18 (2016) 033002, arXiv:1509.07146[hep-th])

Complex Langevin study of one-site Hubbard model

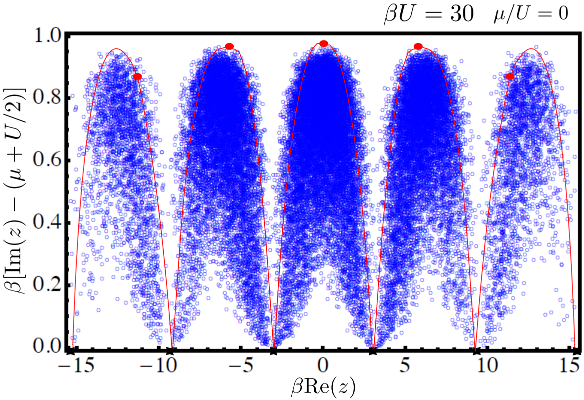
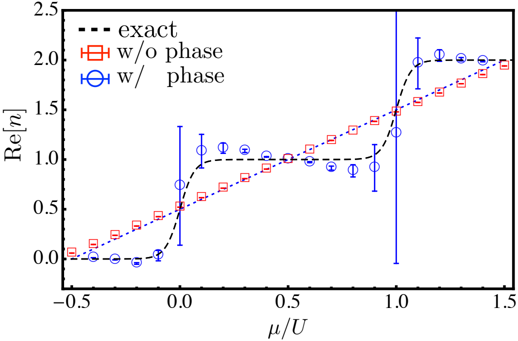


Figure: CL distribution at $\mu/U = 0$

It looks quite similar to Lefschetz thimbles, however the ensemble average cannot reproduce their complex phases.

Complex Langevin simulation

One-site Fermi Hubbard model:



(Hayata, Hidaka, YT, arXiv:1511.02437[hep-lat])

Consequence

The knowledge of LT helps us to understand failures of CL method.

Summary and Conclusion

- Lefschetz-thimble method gives a suitable framework for saddle-point analysis even if $S(\phi)$ takes complex values.
- One-site Hubbard model is a nice toy model to play with the sign problem.
- Interference of complex phases among Lefschetz thimbles plays a pivotal role for the (baryon) Silver Blaze problem.