Advances in N=4 Supersymmetric Theories

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ENS Paris

Annual UK Theory Meeting
Durham 2015

N=4 super Yang Mills theory

Maximally supersymmetric version of YM theory in 4d

$$\mathcal{L} = \frac{1}{4g^2} \mathrm{tr}\, F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} \mathrm{tr}\, D_\mu \Phi_{AB} D_\mu \Phi_{AB} + \mathrm{tr}\, \bar{\Psi}_A \gamma_\mu D_\mu \Psi_A$$

$$+ \text{ Yukawa and quartic interactions}$$

For short, theory of a massless spin 1 (extended) supermultiplet; everything else follows from it

3 good reasons to like it

Theoretical laboratory : one can explore and identify mathematical and physical structures (at higher loops or strong coupling) more easily than in any other theory

AdS/CFT correspondence : it is one of these few theories for which we believe we know precisely what is the string theory dual (here it is IIB string theory in AdS5 * S5)

Integrability : it is believed to be "exactly solvable", in the 't Hooft planar limit at least, and referred to as the Ising model of 4d gauge theories

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Hence we must solve it!

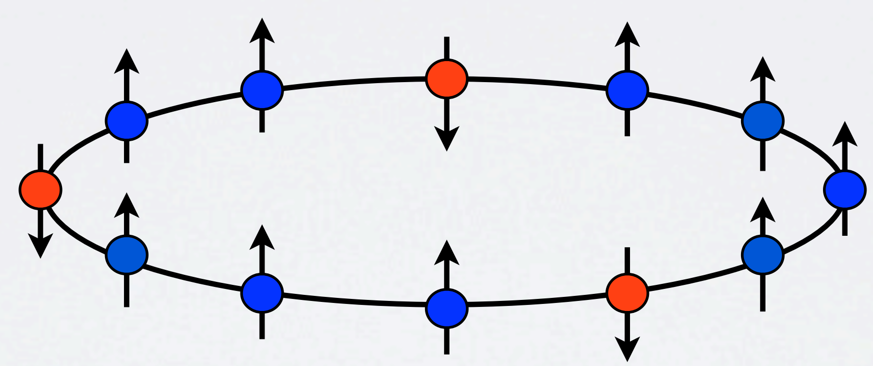
Advances in integrability

Spectrum of scaling dimensions and spin chain

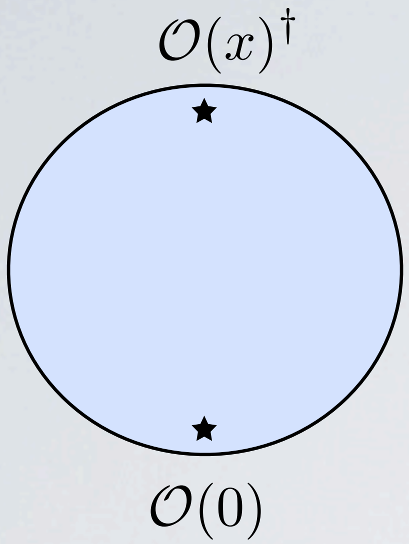
Gluon scattering amplitudes and Wilson loops

Structure constants and string splitting/joining

Scaling dimensions and spin chain



Spectral problem



Spectrum of scaling dimensions of local operators

$$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle = \frac{1}{x^{2\Delta}}$$

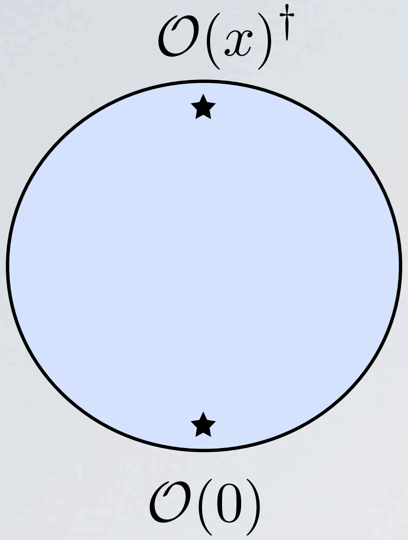
Local (single trace) operator

$$\mathcal{O} = \mathrm{tr}\, \Phi_1 \Phi_2 \dots \Phi_L$$

Mixing problem

Radiative corrections induce mixing of operators

Spectral problem



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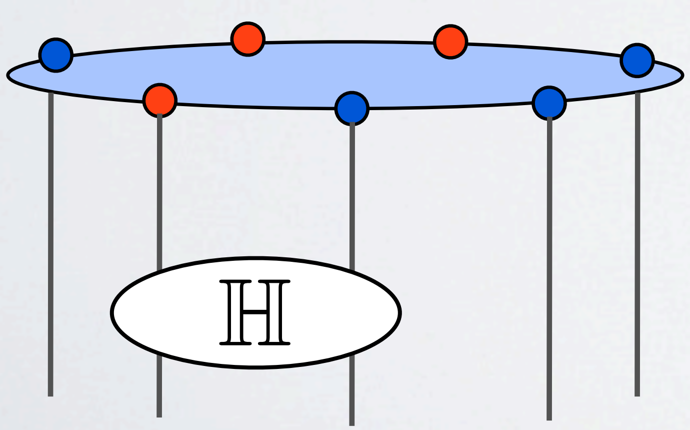
Equivalent to a spin chain problem

't Hooft coupling



$$\Delta = \Delta_{\text{canonical}} + 2g^2 H_{XXX} + O(g^4)$$

Radiative corrections induce mixing of operators = spin chain Hamiltonian

Spectral problem

One-loop dilatation operator

[Minahan,Zarembo'02]
[Beisert,Staudacher'03]

$$H_{XXX} = \sum_{i=1}^{L} (I - P_{ii+1})$$

Werner Heisenberg

Hans Bethe

Heisenberg spin chain is integrable :

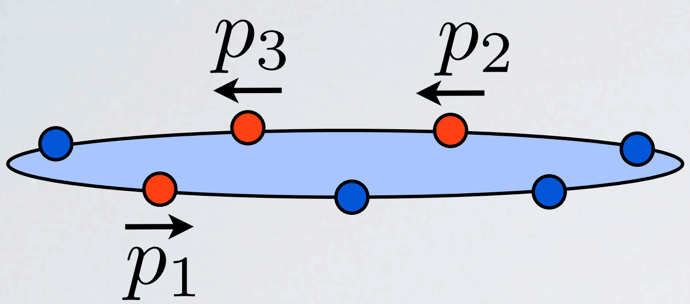
- As many commuting conserved charges as degrees of freedom (i.e., L for SU(2) spin chain)

- Fundamental excitations (magnons) about the ferro vacuum have a factorized S-matrix

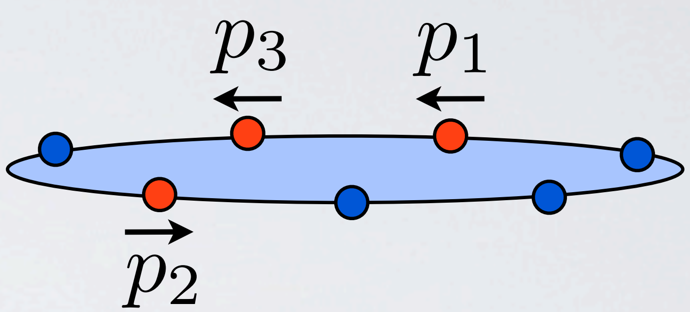
$$\mathbb{S}_{123} = \mathbb{S}_{23}\mathbb{S}_{13}\mathbb{S}_{12}$$

Spectral problem

Bethe wave function



$+S(p_1, p_2)$



$+\ldots$

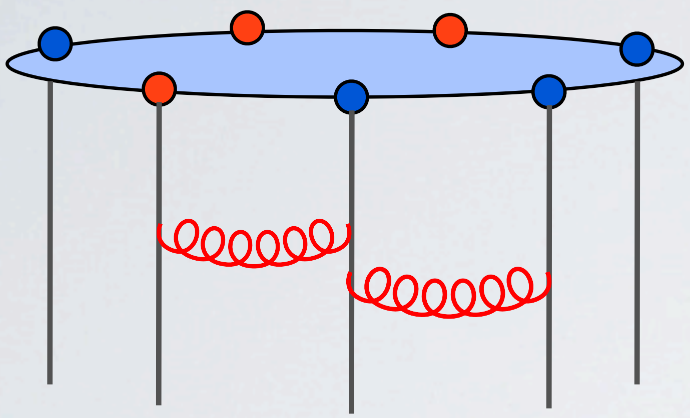
Periodicity conditions gives the Bethe ansatz equations
(ie quantization conditions for the magnon momenta)

$$e^{ip_i L} \prod_{j \neq i} S(p_i, p_j) = 1$$

And the spectrum of energies follows :

$$E = \sum_i E(p_i)$$

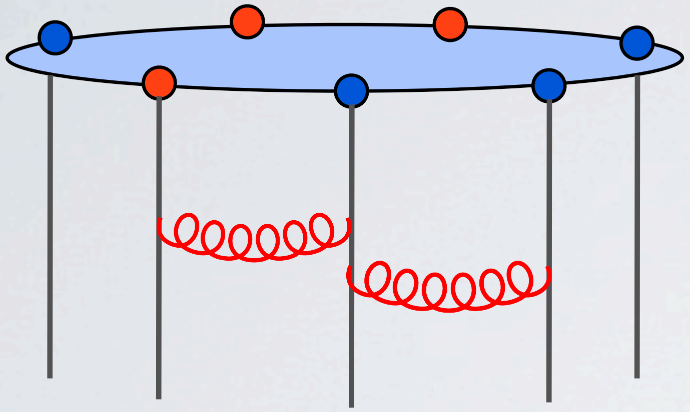
Higher loops?



Increasing loop order = increasing range of the spin chain Hamitonian

Not much is known about the resulting long range spin chain
It is however believed to remain integrable

Higher loops?



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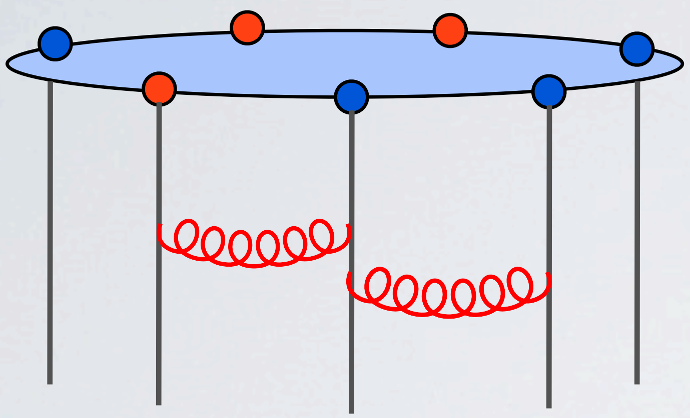
Hint :

[Bena,Pochinski,Roiban'03]

Higher loops?



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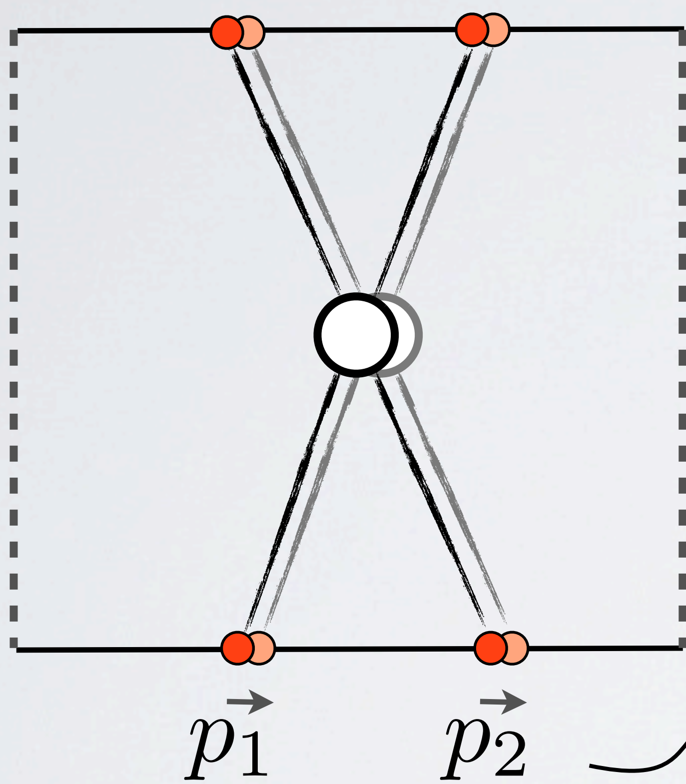
How do we solve for the spectrum of an unknown Hamiltonian?
We simply add loop corrections to our previous ingredients :
energy and S-matrix

Power of symmetry

Residual symmetry group of BMN (ferro) vacuum : [Beisert'05]

$$\underbrace{PSU(2|2)}_{\text{Left}} \times \underbrace{PSU(2|2)}_{\text{Right}} \ltimes \mathbb{R}^3$$

Central extensions : contain **energy** (and coupling constant)



Magnon transforms in bi-fundamental irrep

$$\underbrace{\mathbf{2}|\mathbf{2}}_{\text{Left}} \otimes \underbrace{\mathbf{2}|\mathbf{2}}_{\text{Right}}$$

(Dimension = 16 = 8 bosons + 8 fermions)

Dispersion relation

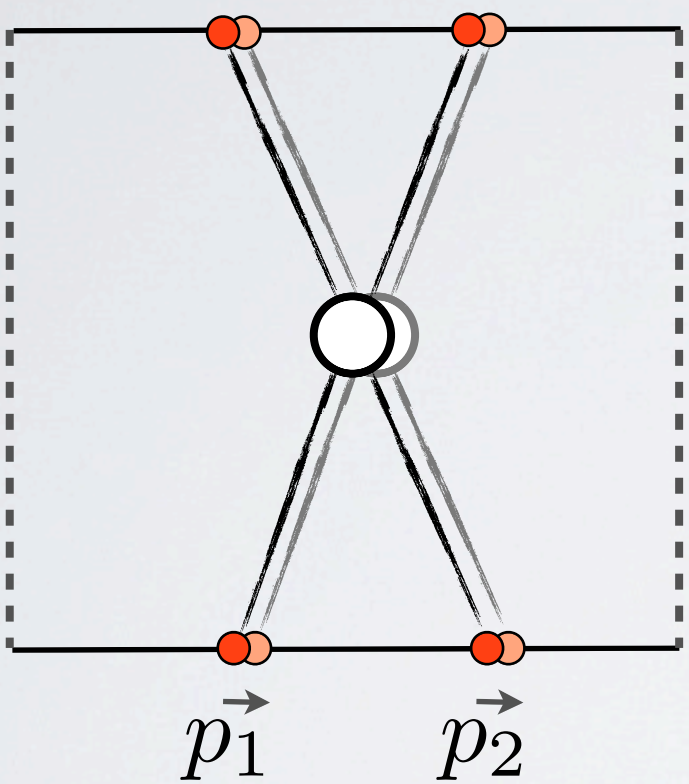
$$E = \sqrt{1 + 16\, g^2 \sin^2\left(\frac{p}{2}\right)}$$

Power of symmetry

Residual symmetry group of BMN (ferro) vacuum : [Beisert'05]

$$PSU(2|2) \times PSU(2|2) \ltimes \mathbb{R}^3$$

Left Right

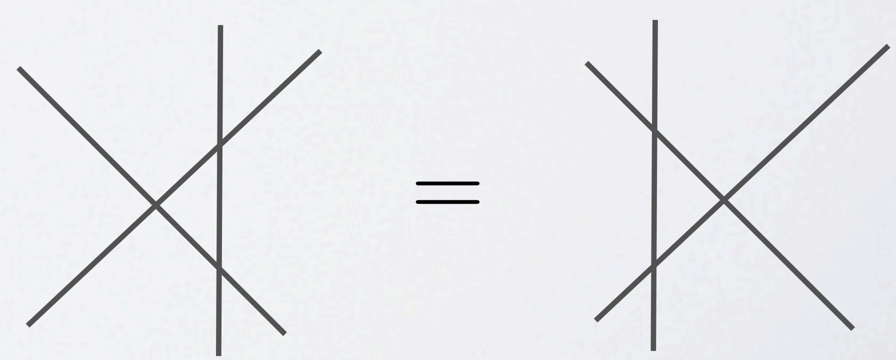


Symmetry fixes S-matrix

$$\mathbb{S}_{12} \sim S^0_{12}\,\mathcal{S}_{12} \times \dot{\mathcal{S}}_{12}$$

up to scalar factor

✓ Fulfills **Yang-Baxter** equation



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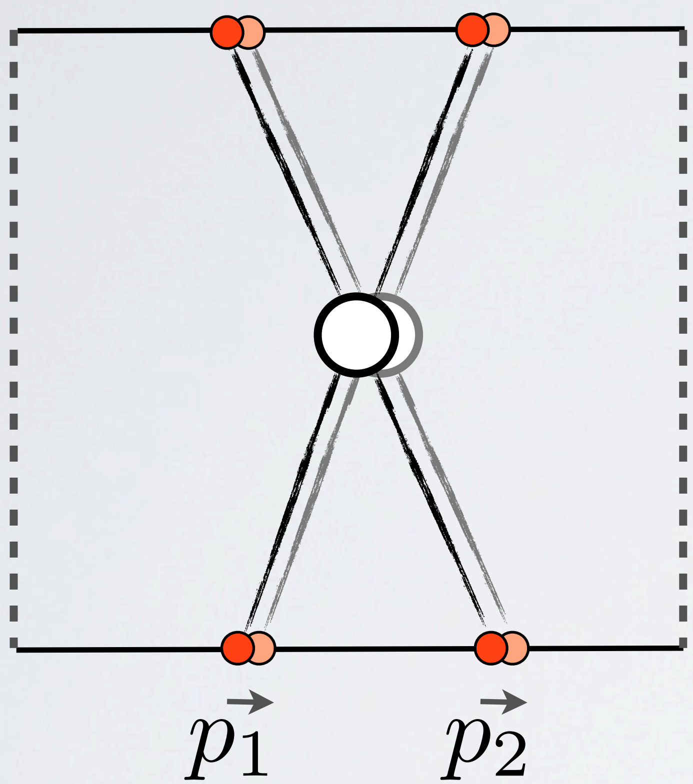

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Symmetry fixes S-matrix

$$\mathbb{S}_{12} \sim S^0_{12}\, \mathcal{S}_{12} \times \dot{\mathcal{S}}_{12}$$

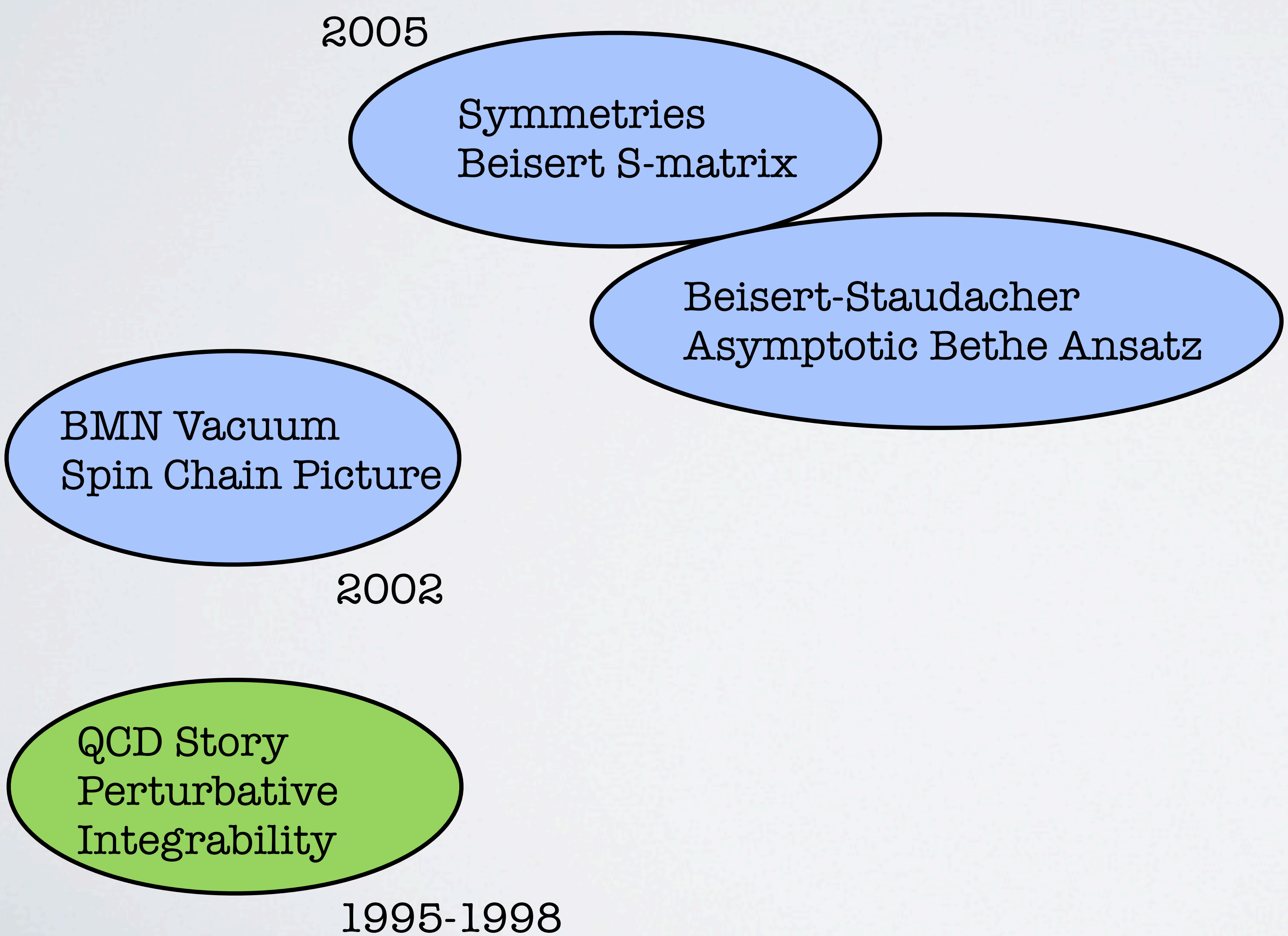
up to scalar factor

✓ Fulfills **Yang-Baxter** equation

✓ Scalar factor constrained by crossing symmetry [Janik'05]

Full solution?

Is it that simple?



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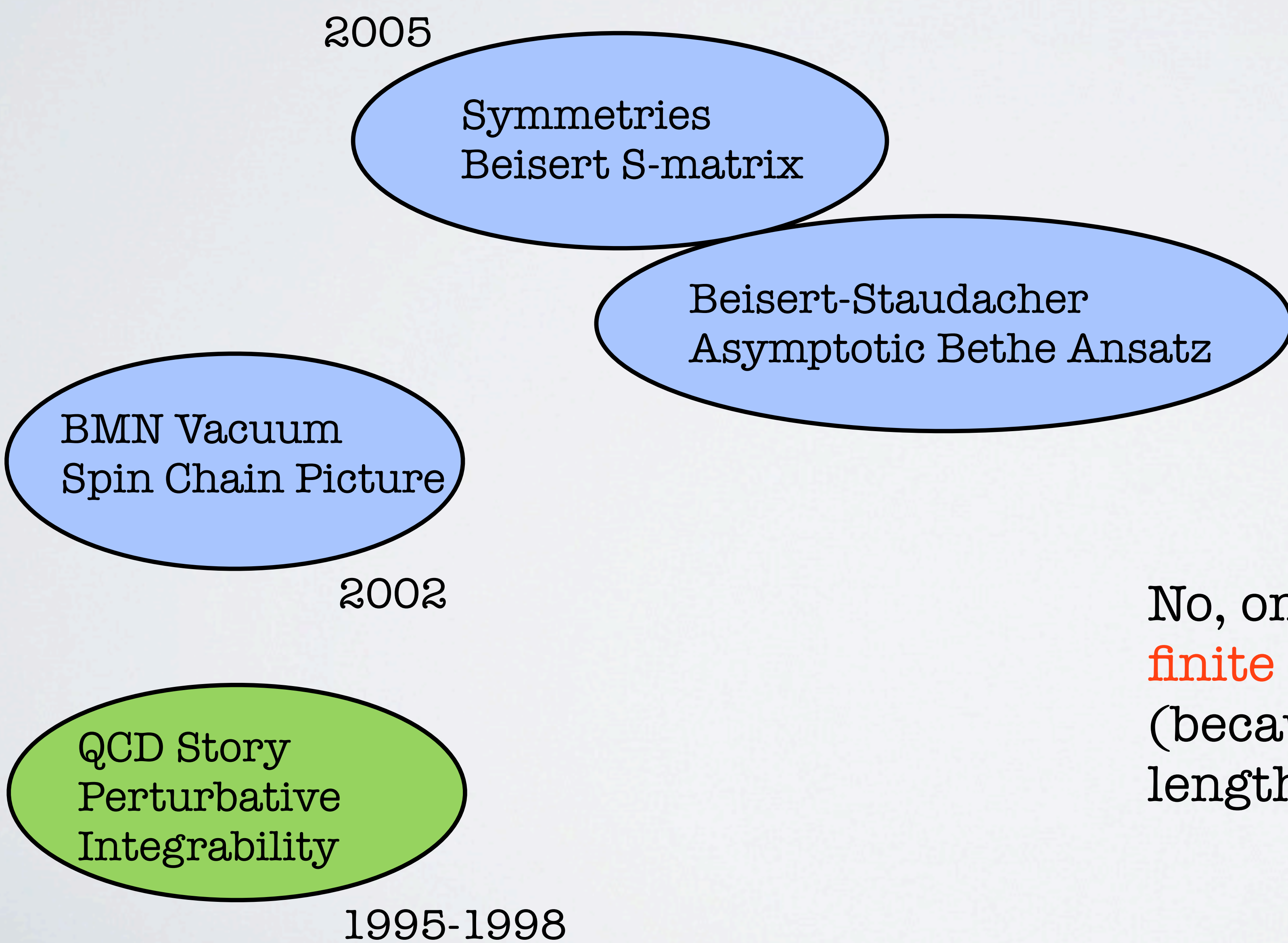
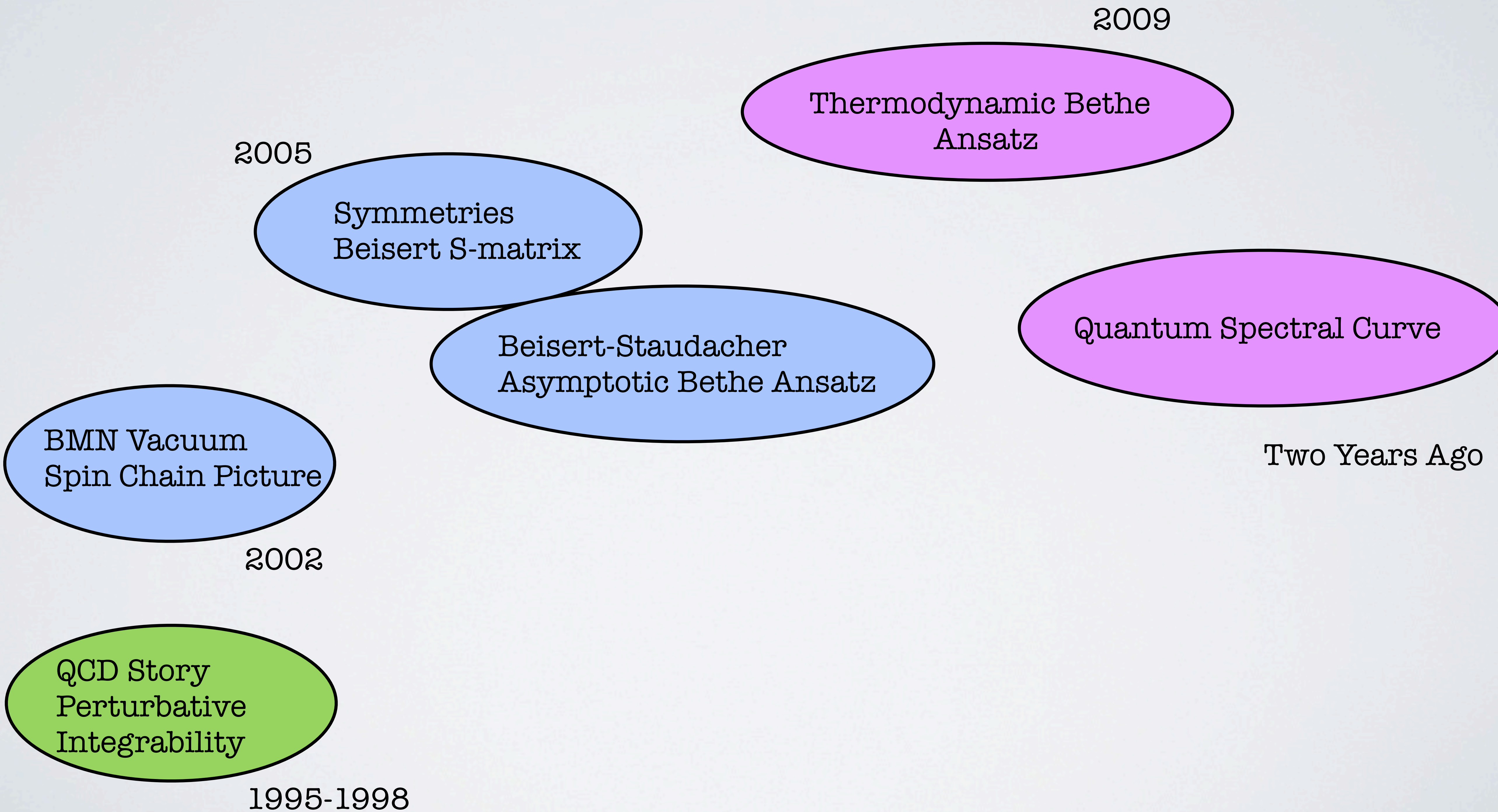


No, one must also account for finite size corrections (because spin chain has finite length)

Full solution



Applications

Scaling dimension of shortest unprotected operator
(so-called Konishi multiplet)

[Marboe,Volin'14]

$$
\begin{aligned}
\Delta \;=\; & 4+12g^2-48g^4+336g^6+g^8\left(-2496+576\,\zeta_3-1440\,\zeta_5\right)\\
&+g^{10}\left(15168+6912\,\zeta_3-5184\,\zeta_3^2-8640\,\zeta_5+30240\,\zeta_7\right)\\
&+g^{12}\left(-7680-262656\,\zeta_3-20736\,\zeta_3^2+112320\,\zeta_5+155520\,\zeta_3\,\zeta_5+75600\,\zeta_7-489888\,\zeta_9\right)\\
&+g^{14}\big(-2135040+5230080\,\zeta_3-421632\,\zeta_3^2+124416\,\zeta_3^3-229248\,\zeta_5+411264\,\zeta_3\,\zeta_5\\
&\quad-993600\,\zeta_5^2-1254960\,\zeta_7-1935360\,\zeta_3\,\zeta_7-835488\,\zeta_9+7318080\,\zeta_{11}\big)\\
&+g^{16}\Big(54408192-83496960\,\zeta_3+7934976\,\zeta_3^2+1990656\,\zeta_3^3-19678464\,\zeta_5-4354560\,\zeta_3\,\zeta_5\\
&\quad-3255552\,\zeta_3^2\,\zeta_5+2384640\,\zeta_5^2+21868704\,\zeta_7-6229440\,\zeta_3\,\zeta_7+22256640\,\zeta_5\,\zeta_7\\
&\quad+9327744\,\zeta_9+23224320\,\zeta_3\,\zeta_9+\frac{65929248}{5}\,\zeta_{11}-106007616\,\zeta_{13}-\frac{684288}{5}\,Z_{11}^{(2)}\Big)\\
&+g^{18}\Big(-1014549504+1140922368\,\zeta_3-51259392\,\zeta_3^2-20155392\,\zeta_3^3+575354880\,\zeta_5\\
&\quad-14294016\,\zeta_3\,\zeta_5-26044416\,\zeta_3^2\,\zeta_5+55296000\,\zeta_5^2+15759360\,\zeta_3\,\zeta_5^2-223122816\,\zeta_7\\
&\quad+34020864\,\zeta_3\,\zeta_7+22063104\,\zeta_3^2\,\zeta_7-92539584\,\zeta_5\,\zeta_7-113690304\,\zeta_7^2-247093632\,\zeta_9\\
&\quad+119470464\,\zeta_3\,\zeta_9-245099520\,\zeta_5\,\zeta_9-\frac{186204096}{5}\,\zeta_{11}-278505216\,\zeta_3\,\zeta_{11}-253865664\,\zeta_{13}\\
&\quad+1517836320\,\zeta_{15}+\frac{15676416}{5}\,Z_{11}^{(2)}-1306368\,Z_{13}^{(2)}+1306368\,Z_{13}^{(3)}\Big)
\end{aligned}
$$

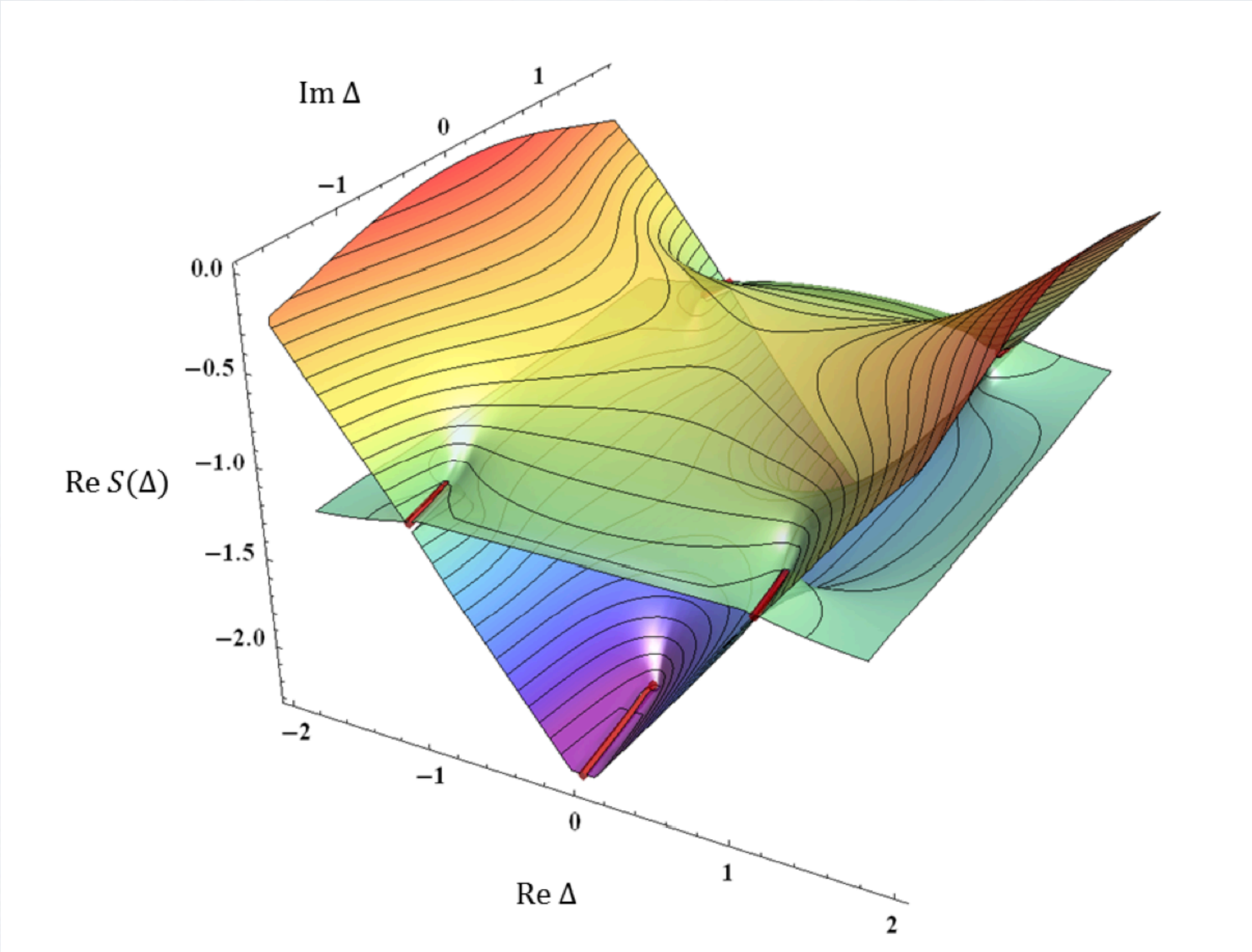
Comments :

- Finite size corrections here starts at 4 loops
- Z.. stand for single valued multiple zeta values

Applications

Scaling dimension of twist two operator for complex spin

[Gromov,Levkovich-Maslyuk,Sizov'15]



Plot of real part of the spin S as a function of the scaling dimension Δ
for 't Hooft coupling = 6.3

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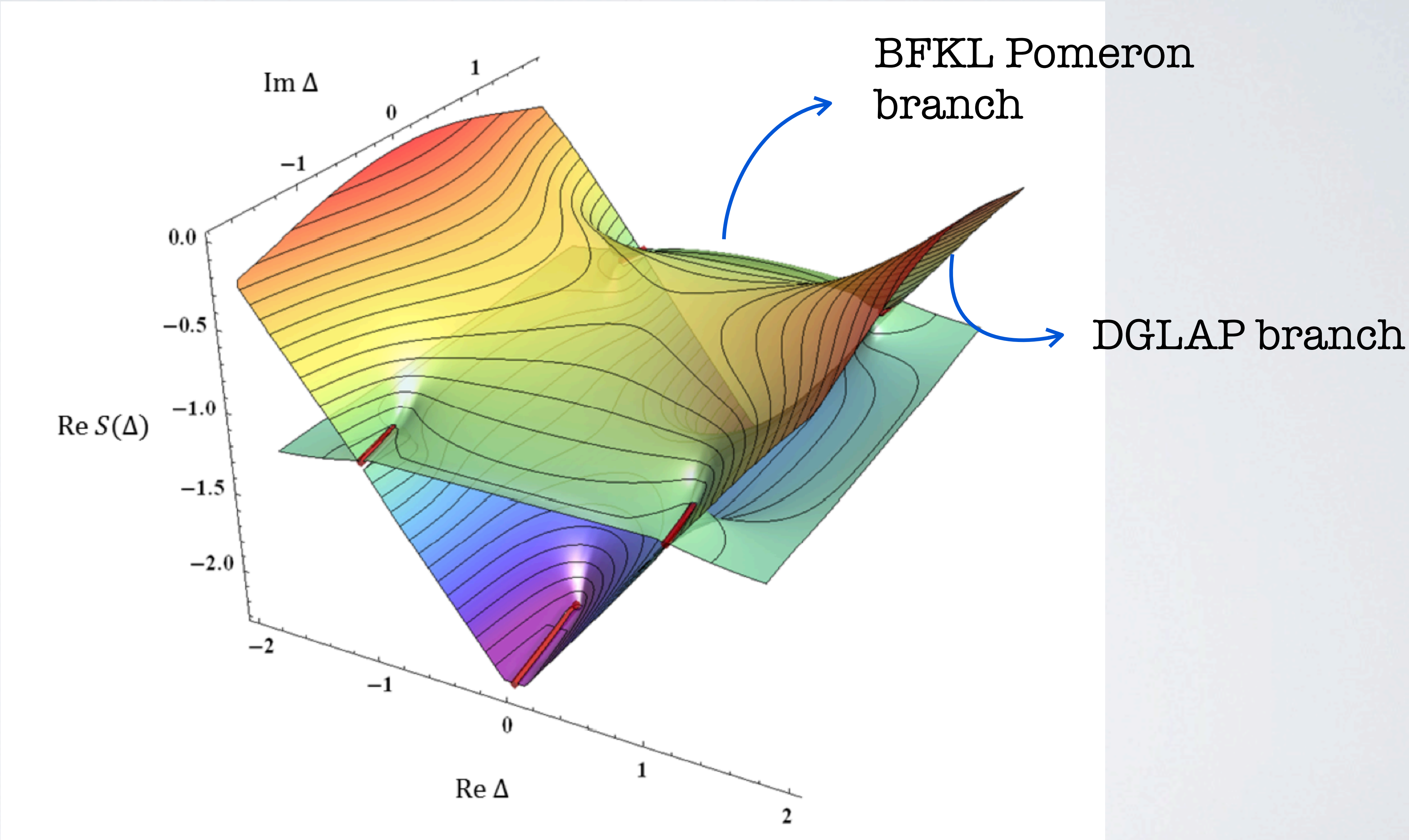

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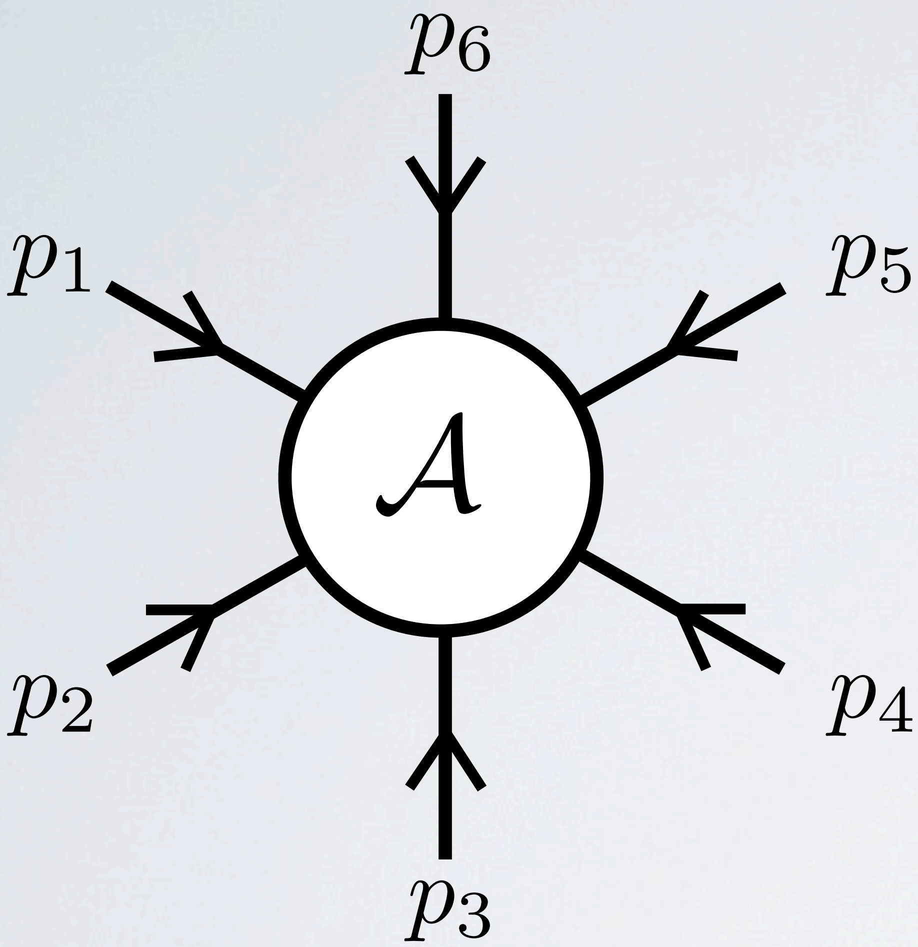
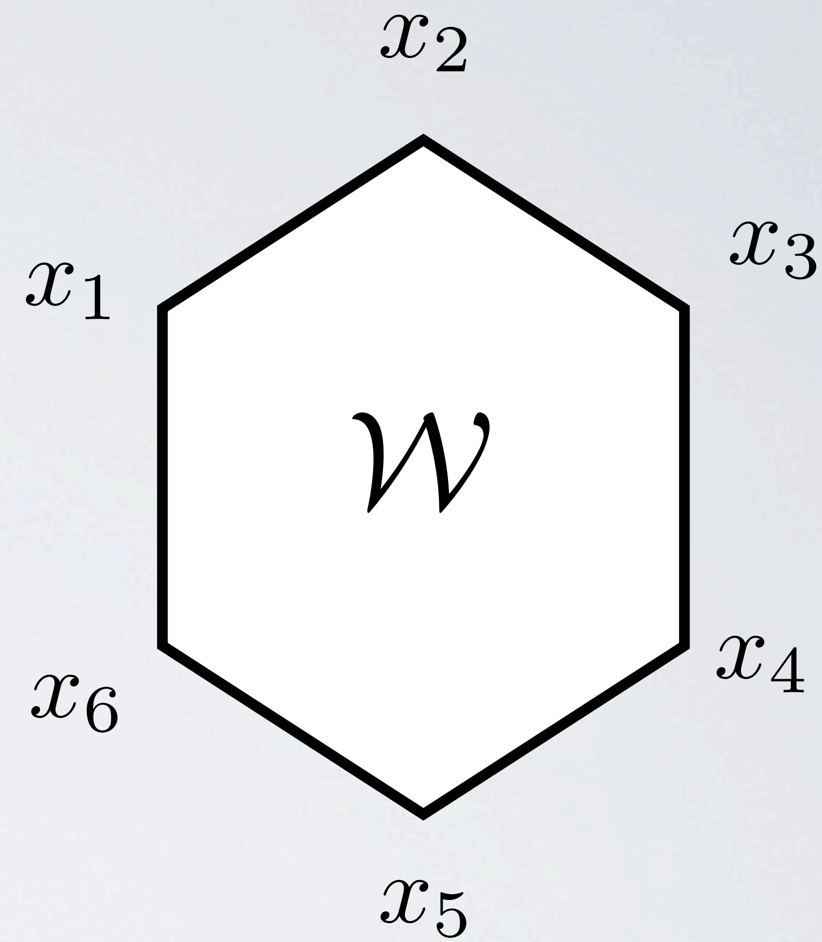
Gluon scattering amplitudes
and
Wilson loops

Scattering amplitudes = Wilson loops



=

$$p_i = x_{i+1} - x_i$$



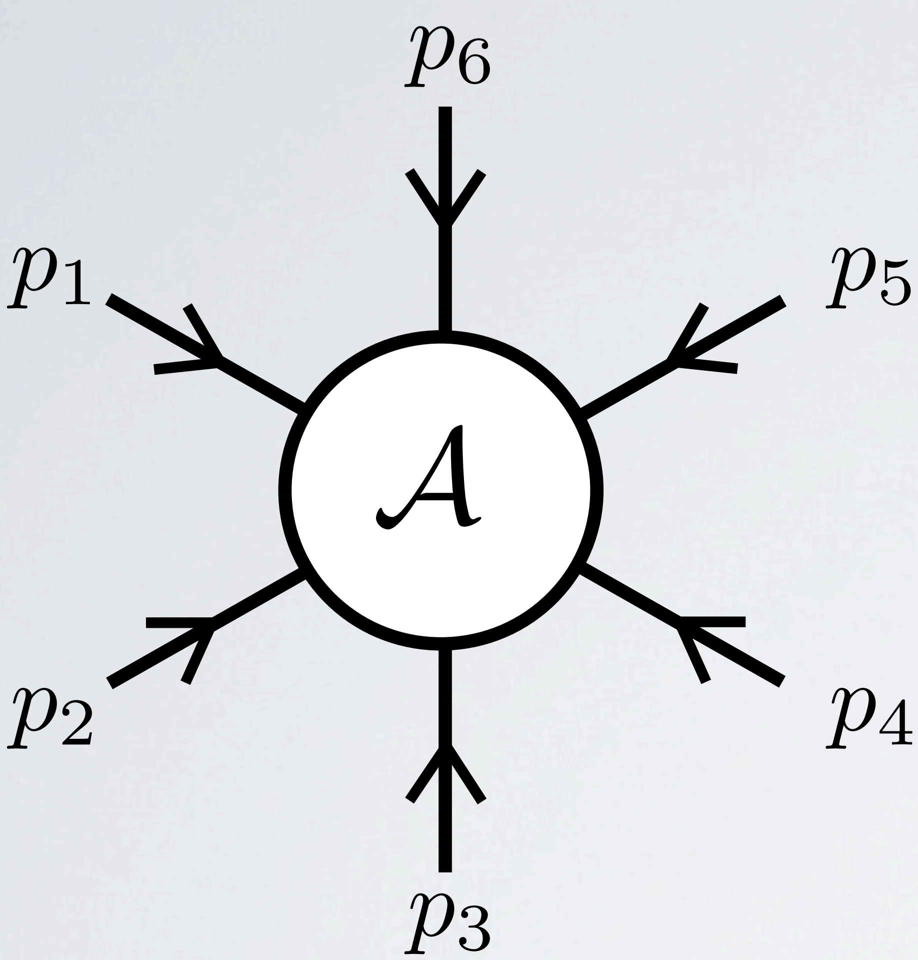
gluon scattering amplitude

[Alday,Maldacena'07]
[Drummond,Korchemsky,Sokatchev'07]
[Brandhuber,Heslop,Travaglini'07]
[Drummond,Henn,Korchemsky,Sokatchev'07]

light-like polygonal Wilson loop

In this theory they are the same

Scattering amplitudes = Wilson loops



$$=$$

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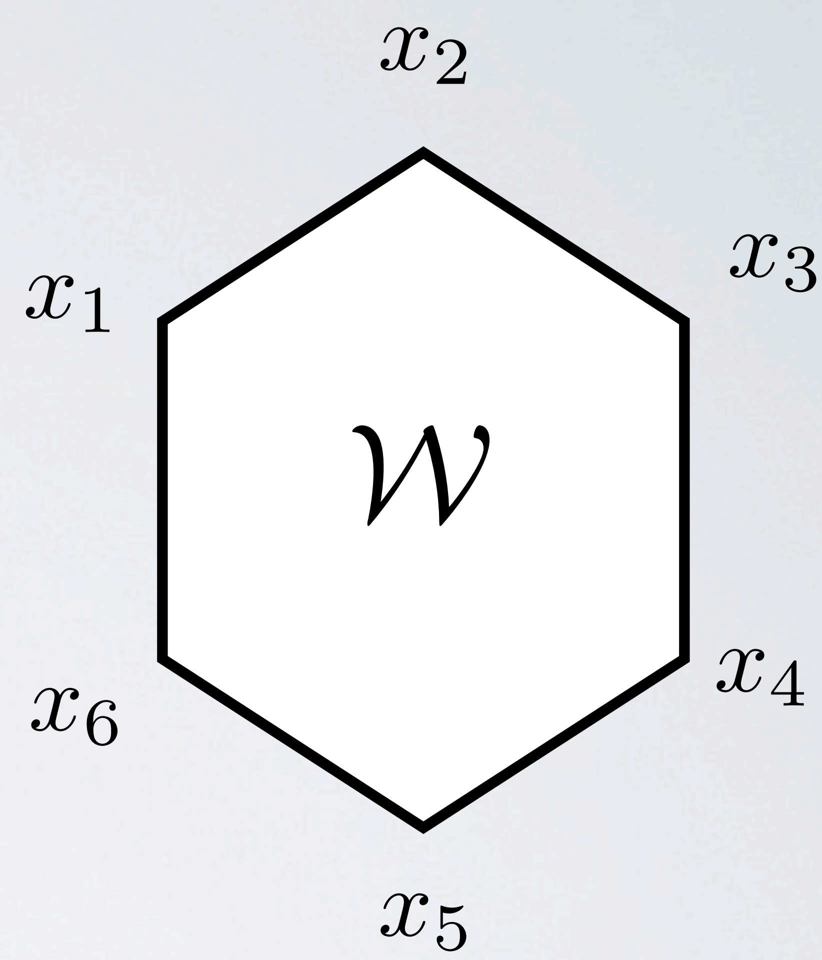

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light-like polygonal Wilson loop

→ *Dual conformal symmetry*

Combining symmetries

Super conformal + dual super conformal gives a Yangian symmetry (one of the hallmark of integrability)

[Drummond, Henn, Plefka'09]

Put severe constraints on the integrand of scattering amplitudes which can be constructed exactly

They lead to a purely geometrical reformulation of these integrands (Grasmannian, Amplituhedron)

[Arkani-Hamed, Bourjaily, Cachazo,Caron-Huot,Goncharov, Postnikov,Trnka'10'12]

Also put constraints on the full (integrated) scattering amplitudes

They lead to a bootstrap for constructing SA without any use of Feynman diagrams (proceeds from knowleged of space of functions + additional physical requirements)

[Dixon,Drummond, Henn'11]
[Dixon,Drummond, vonHippel, Pennington'13]
[Dixon,Drummond, Duhr, Pennington'13]
[Drummond,Papathanasiou, Spradlin'14]

Immediate consequences

Amplitudes are function of cross ratio only (up to divergent part) :

[Drummond,Henn,Korchemsky,Sokatchev'07]

$$\log W_n = \mathrm{BDS}_n + R_n(u_1, \ldots, u_{3n-15})$$

Bern-Dixon-Smirnov ansatz
(contains all IR/UV divergences)

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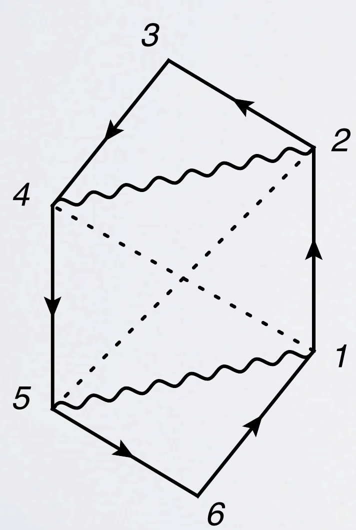
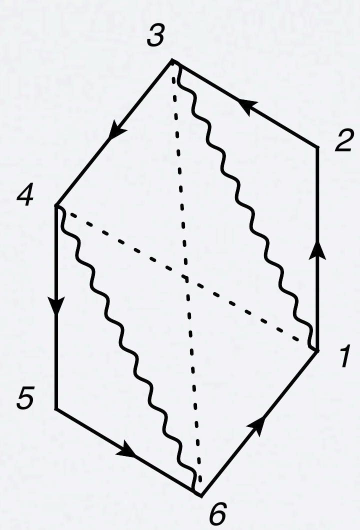
[Drummond,Henn,Korchemsky,Sokatchev'07]

$$\log W_n = \mathrm{BDS}_n + R_n(u_1, \ldots, u_{3n-15})$$

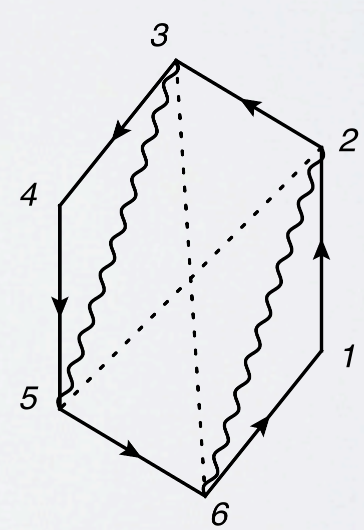
$R_n =$ remainder function $=$ function of 3n-15 cross ratios



$$u_2 = \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2}$$



$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}$$



$$u_3 = \frac{x_{26}^2 x_{35}^2}{x_{25}^2 x_{36}^2}$$

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Amplitudes are function of cross ratio only (up to divergent part) :

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$$\log W_n = \text{BDS}_n + R_n(u_1, \ldots, u_{3n-15})$$

$R_n =$ remainder function $=$ function of 3n-15 cross ratios

In particular

$$R_4 = R_5 = 0$$

(simply because one cannot form cross ratios for 4- and 5-edge null WLs)

4- and 5-gluon amplitudes are thus known exactly and given by the BDS part only!

Immediate consequences

Amplitudes are function of cross ratio only (up to divergent part) :

[Drummond,Henn,Korchemsky,Sokatchev'07]

$$\log W_n = \mathrm{BDS}_n + R_n(u_1, \ldots, u_{3n-15})$$

$R_n =$ remainder function $=$ function of 3n-15 cross ratios

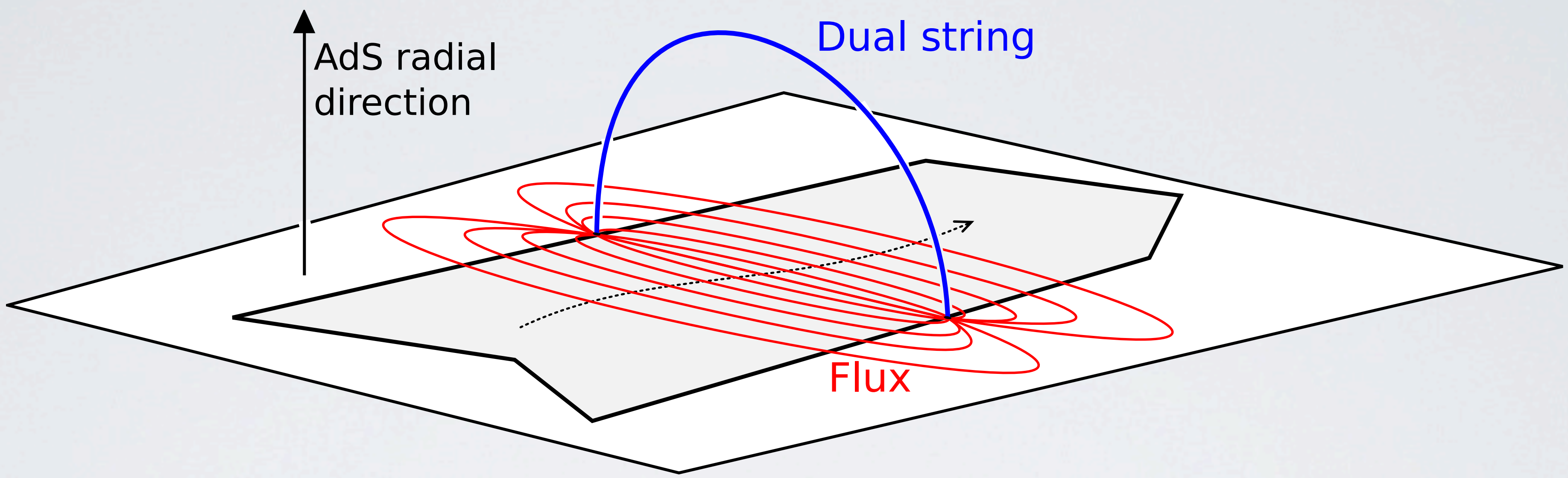
Yet another consequence :

WLs are some sort of non-local Green functions
and one can use the OPE for building big WLs out of smaller ones

Wilson loops at finite coupling

[Alday,Gaiotto,Maldacena,Sever,Vieira'10]

Hamiltonian picture
for OPE



1+1d background : flux tube sourced by two parallel null lines

bottom&top cap excite the flux tube out of its ground state

→ *Sum over all flux-tube eigenstates*

$$\mathcal{W} = \sum_{\text{states }\psi} C_{\text{bot}}(\psi) \times e^{-E(\psi)\tau + ip(\psi)\sigma + im(\psi)\phi} \times C_{\text{top}}(\psi)$$

Wilson loops at finite coupling

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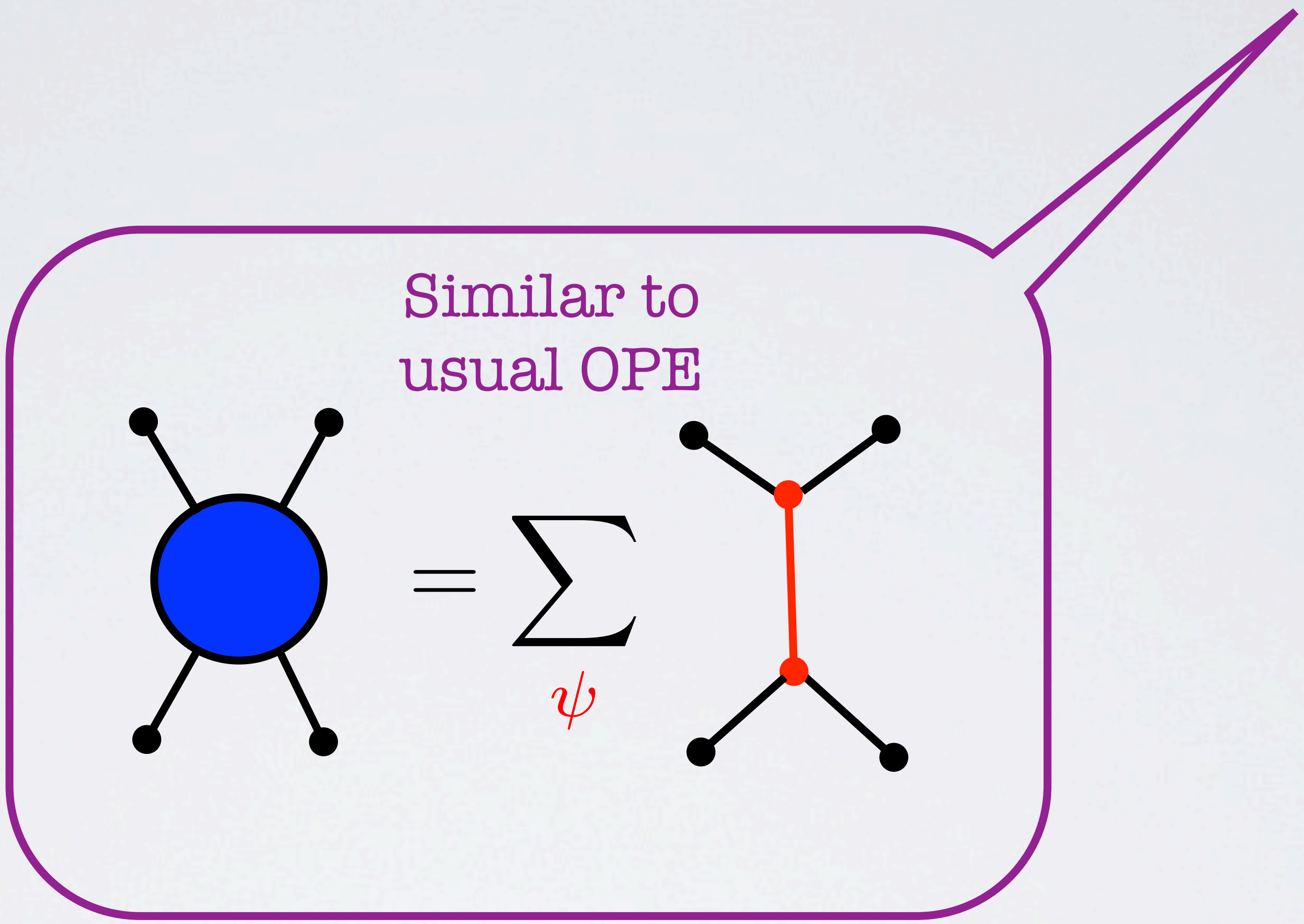


$$\mathcal{W} = \sum_{\text{states } \psi} C_{\text{bot}}(\psi) \times e^{-E(\psi)\tau + ip(\psi)\sigma + im(\psi)\phi} \times C_{\text{top}}(\psi)$$

Pentagon way : main ideas

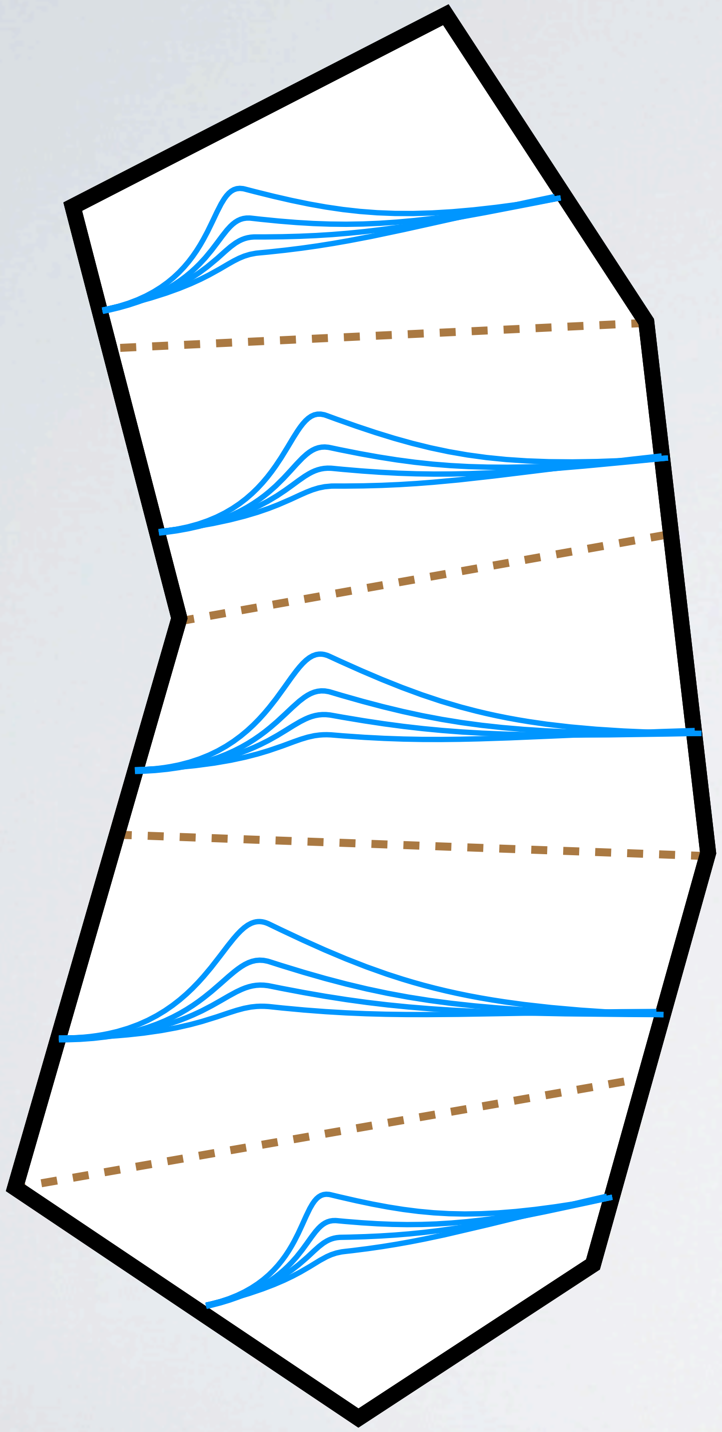
[BB,Sever,Vieira'13]

Remember : use small objects
to build bigger ones

Here smallest objects : squares and pentagons
(no cross ratios = fixed by conformal symmetry)

Analogy with OPE data for local operators :
Square = 2pt function = spectral data
Pentagon = 3pt function = coupling

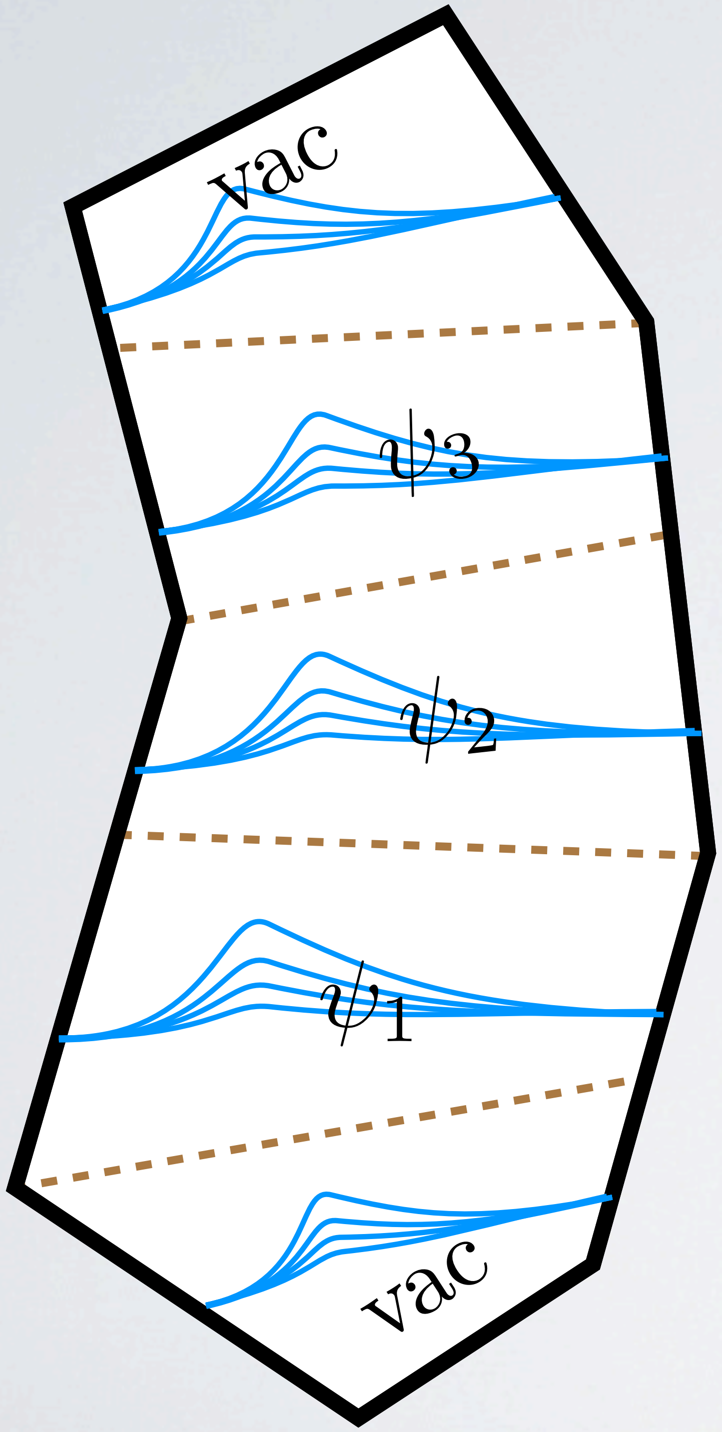
Implementation



Step 1:
Pick a polygon and
divide it into squares

and think about each square
as hosting the flux tube
in a particular state

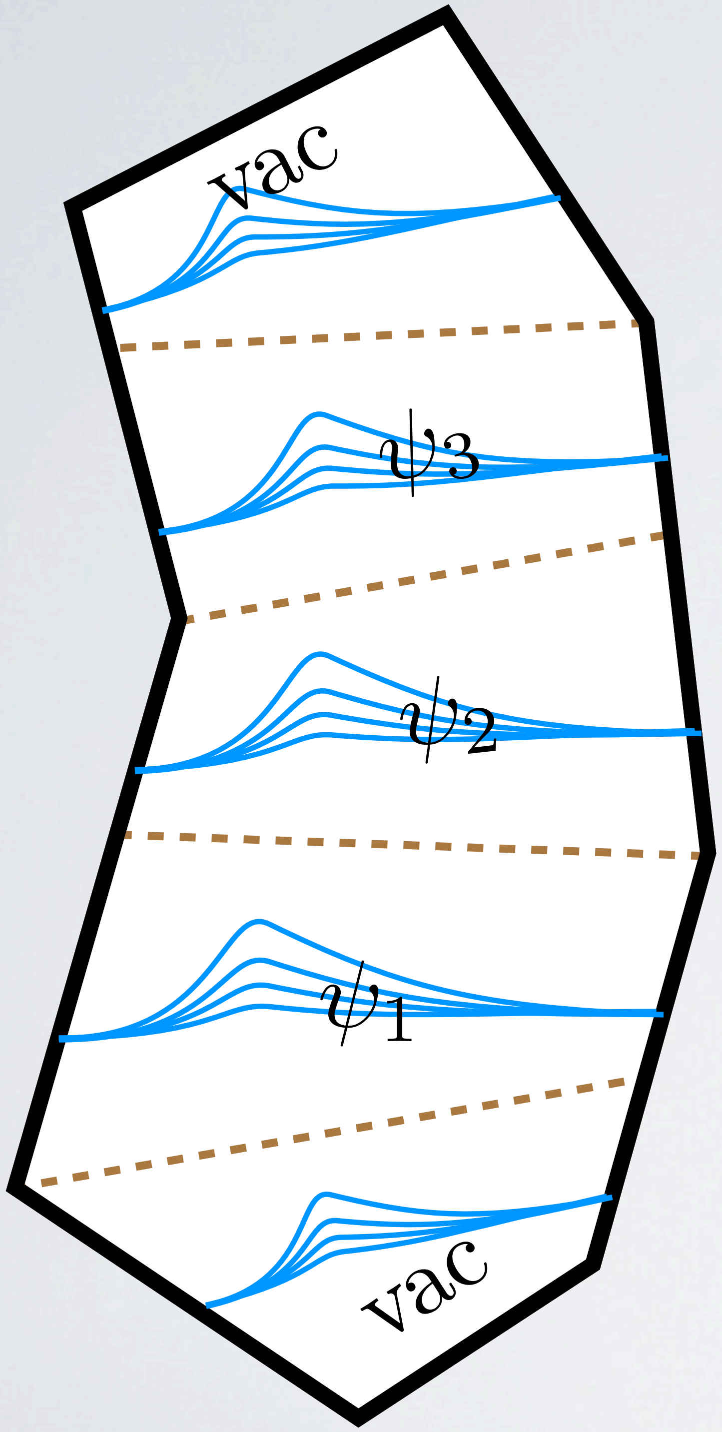
Implementation



Step 1:
Pick a polygon and divide it into squares

Step 2:
Decompose the flux tube state over a basis of eigenstates (w.r.t symmetries of the square)

Implementation

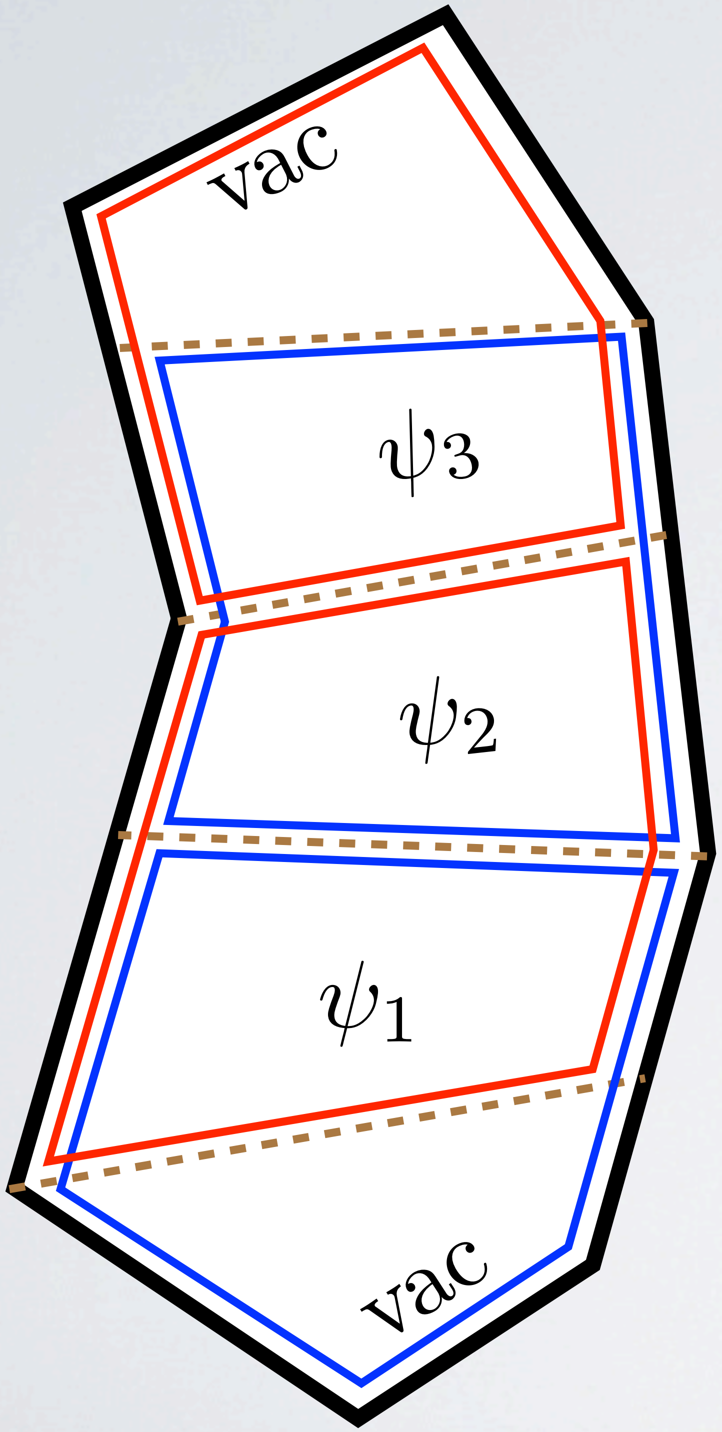


energy / momentum / angular momentum

$$= \sum_{\psi_i} \left[\prod_i e^{-E_i \tau_i + i p_i \sigma_i + i m_i \phi_i} \right] \times \text{upcoming factor}$$

Propagating phase : kinematics/geometry sits here

Implementation

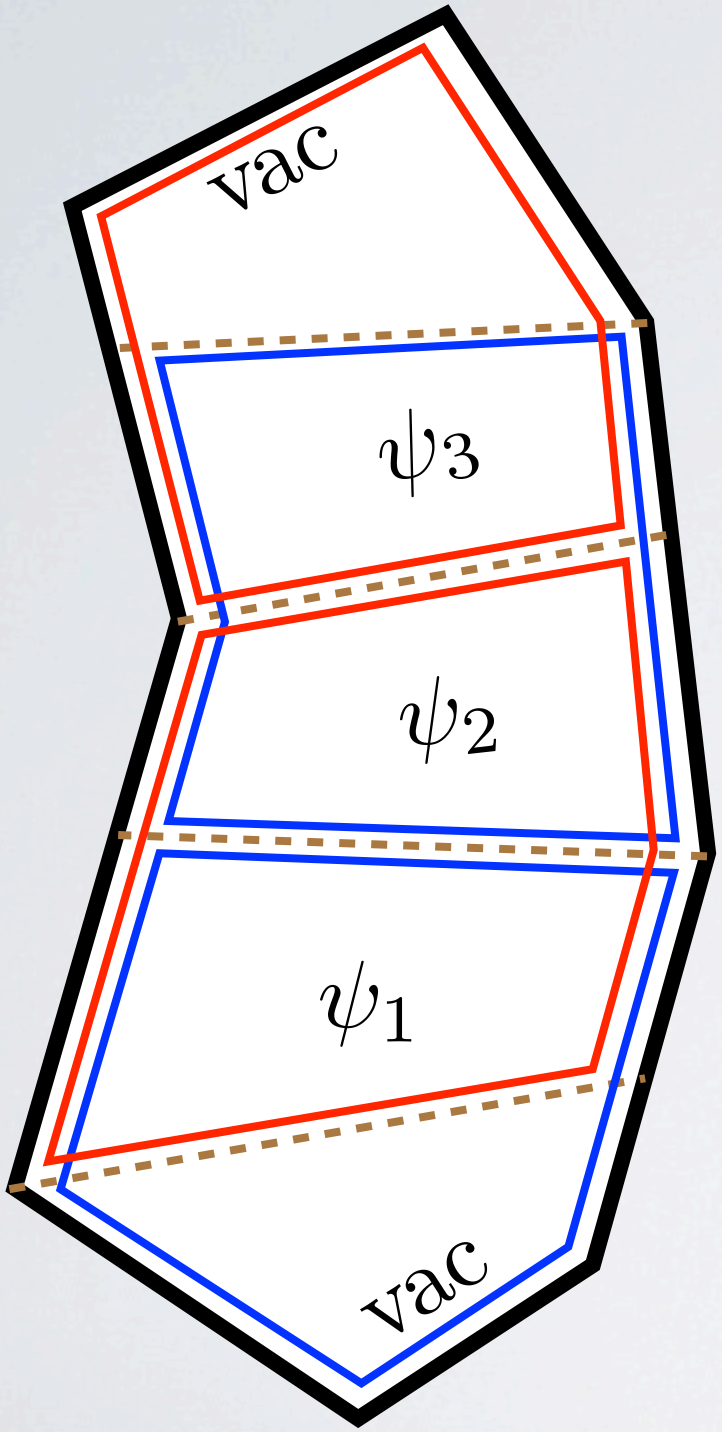


$$= \sum_{\psi_i} \left[\prod_i e^{-E_i \tau_i + i p_i \sigma_i + i m_i \phi_i} \right] \times$$

$$P(0|\psi_1) P(\psi_1|\psi_2) P(\psi_2|\psi_3) P(\psi_3|0)$$

Pentagon transition :
measures the amplitude for
a transition from one state
to another

Pentagon way



$$= \sum_{\psi_i} \left[\prod_i e^{-E_i\tau_i + ip_i\sigma_i + im_i\phi_i} \right] \times$$

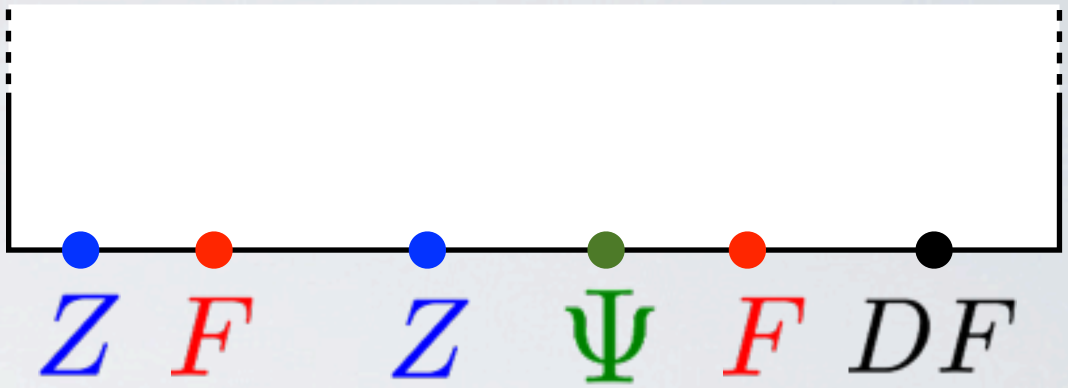
$$P(0|\psi_1)P(\psi_1|\psi_2)P(\psi_2|\psi_3)P(\psi_3|0)$$

To compute amplitudes we need :

- The spectrum of flux-tube states ψ
- All the pentagon transitions $P(\psi_1|\psi_2)$

The flux-tube eigenstates

$\psi = N$ particles state



Field insertions along a light-ray:
create/annihilate state on the flux tube

Not so much different from spin chain...
... in fact it is the same problem as before but expanded around a different vacuum

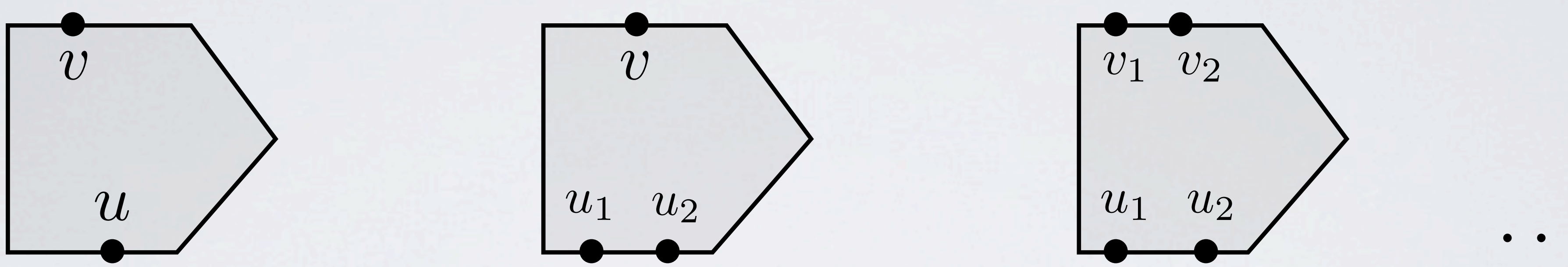
Discretized version of light-ray:
bath of covariant derivatives

$$\mathcal{O} = \mathrm{tr}\,(Z\,DDDD\ldots DDDD\,\overset{\overset{p_1}{\rightarrow}}{F}\,DDDD\ldots DDDD\,\overset{\overset{p_2}{\rightarrow}}{F}\,DDDD\ldots DDDD\,Z)$$

Flux tube states $\longleftrightarrow$ Large spin operators

The pentagon transitions

Field insertions on pentagon WL :

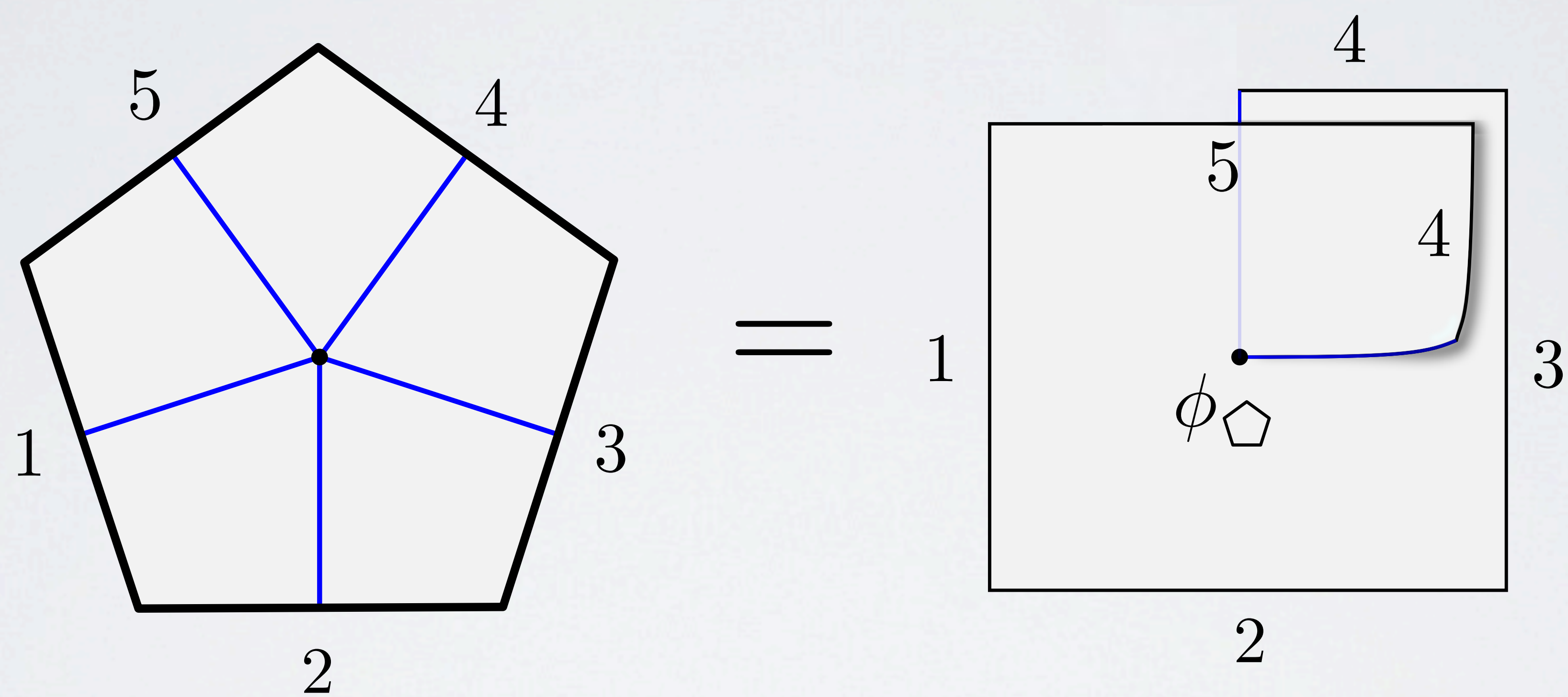


$P(u|v)$ $\quad$ $P(u_1, u_2|v)$ $\quad$ $P(u_1, u_2|v_1, v_2)$

Reminiscent of form factors...

Pentagon as twist operator

In short, a pentagon = 5 quadrants glued together



Geometrical picture :

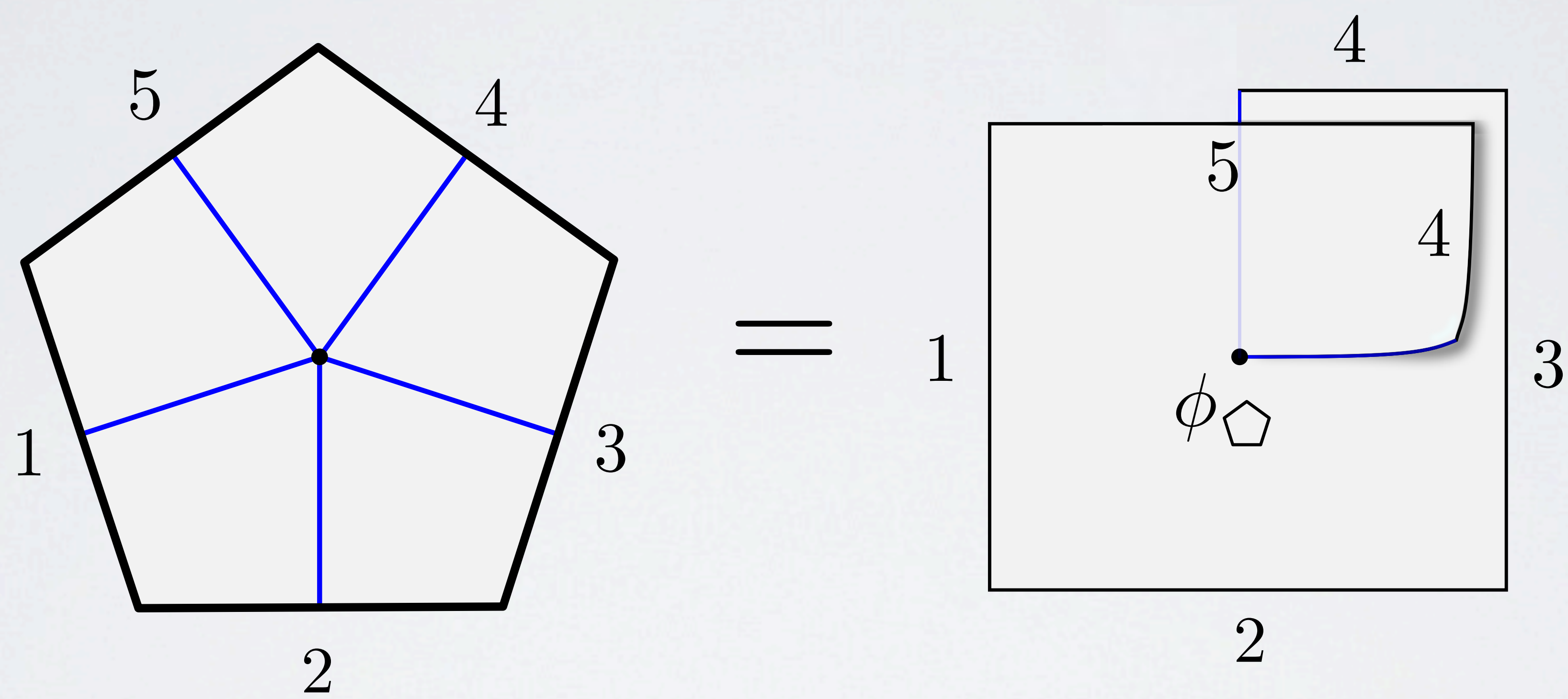
excess angle $= \dfrac{\pi}{2}$

Hamiltonian picture:

twist operator $\phi_{⬠}$

Pentagon as twist operator

In short, a pentagon = 5 quadrants glued together

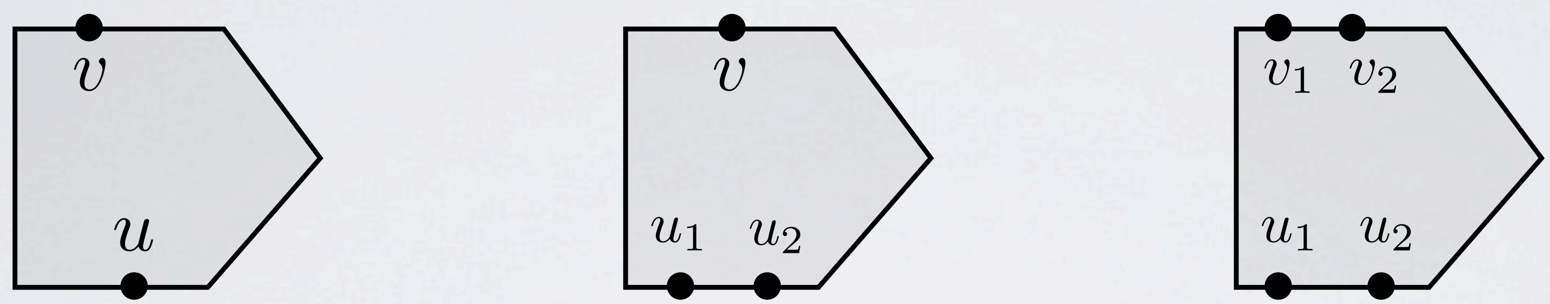


Pentagon transitions as form factors

$$P(u_1, u_2|v_1) = \langle v_1 | \phi_{\pentagon} | u_1, u_2 \rangle$$

The pentagon transitions

Field insertions on pentagon WL :



$P(u|v)$ $\qquad$ $P(u_1, u_2|v)$ $\qquad$ $P(u_1, u_2|v_1, v_2)$ $\qquad \ldots$

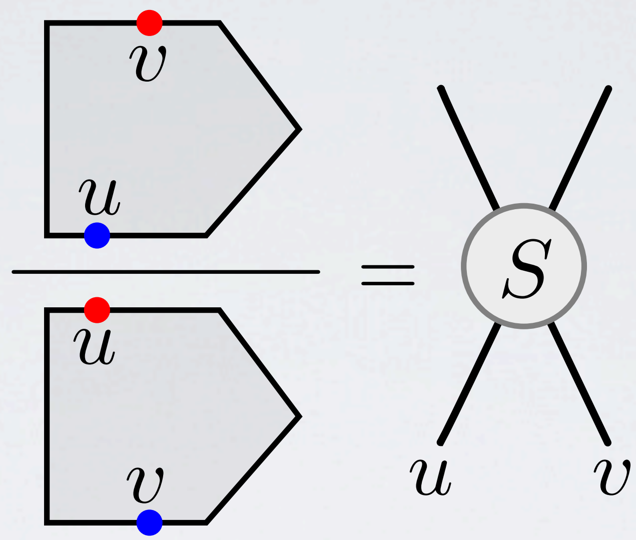
Reminiscent of form factors...

... use integrable bootstrap for finding them

The pentagon transitions

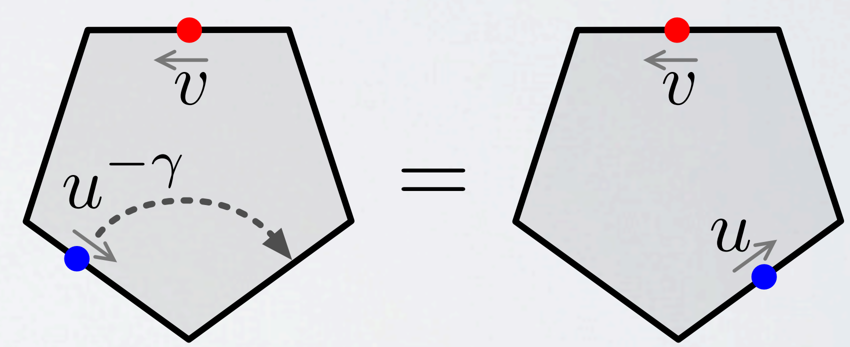
Fundamental axiom :

$$\frac{P(u|v)}{P(v|u)} =$$



Mirror axiom :

$$P(u^{-\gamma}|v) = P(v|u)$$



This is enough to find the transitions in terms of S-matrix :

$$\longrightarrow \quad P(u|v)^2 = \frac{S(u,v)}{(u-v)(u-v+i)S(u^\gamma,v)}$$

All pentagon transitions

$$P_{A|B}(u|v)^2 = \mathcal{F}_{A|B}(u|v) \times \frac{S_{AB}(u,v)}{S_{AB}(u^\gamma,v)}$$

ϕ : scalar

ψ : fermion

F : gluon

[BB,Sever,Vieira'13'14]
[BB,Caetano,Cordova,Sever,Vieira'15]
[Belitsky'14'15]

$$\mathcal{F}_{\phi F}(u|v) = 1\,,$$

$$\mathcal{F}_{\phi\psi}(u|v) = -\frac{1}{(u-v+\frac{i}{2})}\,,$$

$$\mathcal{F}_{\phi\phi}(u|v) = \frac{1}{(u-v)(u-v+i)}\,,$$

$$\mathcal{F}_{FF}(u|v) = \frac{(x^+y^+-g^2)(x^+y^--g^2)(x^-y^+-g^2)(x^-y^--g^2)}{g^2x^+x^-y^+y^-(u-v)(u-v+i)}\,,$$

$$\mathcal{F}_{F\psi}(u|v) = -\frac{(x^+y-g^2)(x^-y-g^2)}{g\sqrt{x^+x^-}y(u-v+\frac{i}{2})}\,,$$

$$\mathcal{F}_{F\bar{\psi}}(u|v) = -\frac{g\sqrt{x^+x^-}y(u-v+\frac{i}{2})}{(x^+y-g^2)(x^-y-g^2)}\,,$$

$$\mathcal{F}_{F\bar{F}}(u|v) = \frac{g^2x^+x^-y^+y^-(u-v)(u-v+i)}{(x^+y^+-g^2)(x^+y^--g^2)(x^-y^+-g^2)(x^-y^--g^2)}\,,$$

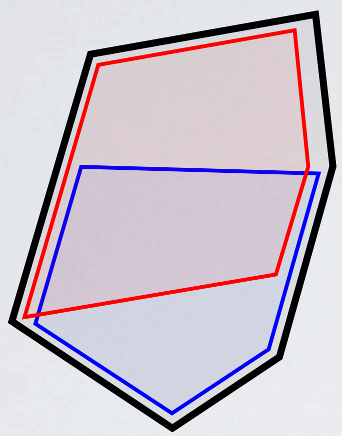
$$\mathcal{F}_{\psi\psi}(u|v) = -\frac{(xy-g^2)}{\sqrt{gxy}(u-v)(u-v+i)}\,,$$

$$\mathcal{F}_{\psi\bar{\psi}}(u|v) = -\frac{\sqrt{gxy}}{(xy-g^2)}\,,$$

Full 6-gluon amplitude

[BB, Sever, Vieira'15]

OPE series :

$$\mathcal{W}_{\text{hex}} =$$  $$= \sum_n \frac{1}{S_n} \int \frac{du_1 \ldots du_n}{(2\pi)^n} \Pi(\{u_i\})$$

Flux tube integrand :

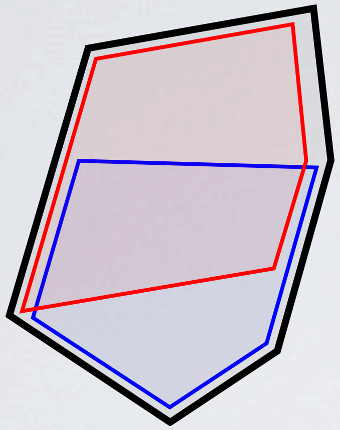
$$\Pi(\{u_i\}) = \Pi_{\text{dyn}} \times \Pi_{\text{mat}}$$

$$\Pi_{\text{dyn}} = \prod_i \mu(u_i)\, e^{-E(u_i)\tau + ip(u_i)\sigma + im_i\phi} \prod_{i<j} \frac{1}{|P(u_i|u_j)|^2}$$

Full 6-gluon amplitude

[BB, Sever, Vieira'15]

OPE series :

$\mathcal{W}_{\text{hex}} =$ 

$$= \sum_n \frac{1}{S_n} \int \frac{du_1 \dots du_n}{(2\pi)^n} \Pi(\{u_i\})$$

Flux tube integrand :

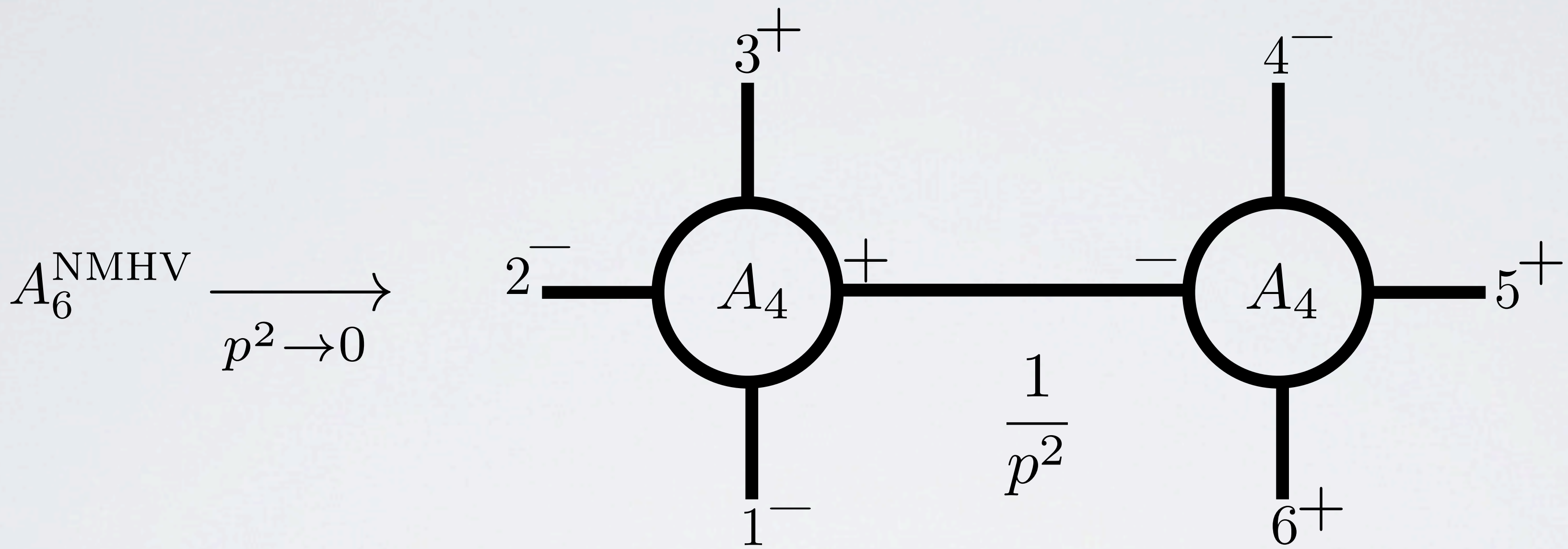
That's it!
(everything here is known at any coupling)

$$\Pi(\{u_i\}) = \Pi_{\text{dyn}} \times \Pi_{\text{mat}}$$

$$\Pi_{\text{dyn}} = \prod_i \mu(u_i)\, e^{-E(u_i)\tau + ip(u_i)\sigma + im_i\phi} \prod_{i<j} \frac{1}{|P(u_i|u_j)|^2}$$

Application : factorization of amplitude

Multi-particle factorization in massless gauge theory



$$\simeq A_4(1,2,3,p)\frac{F(p,s_{i,i+1})}{p^2}A_4(-p,4,5,6)$$

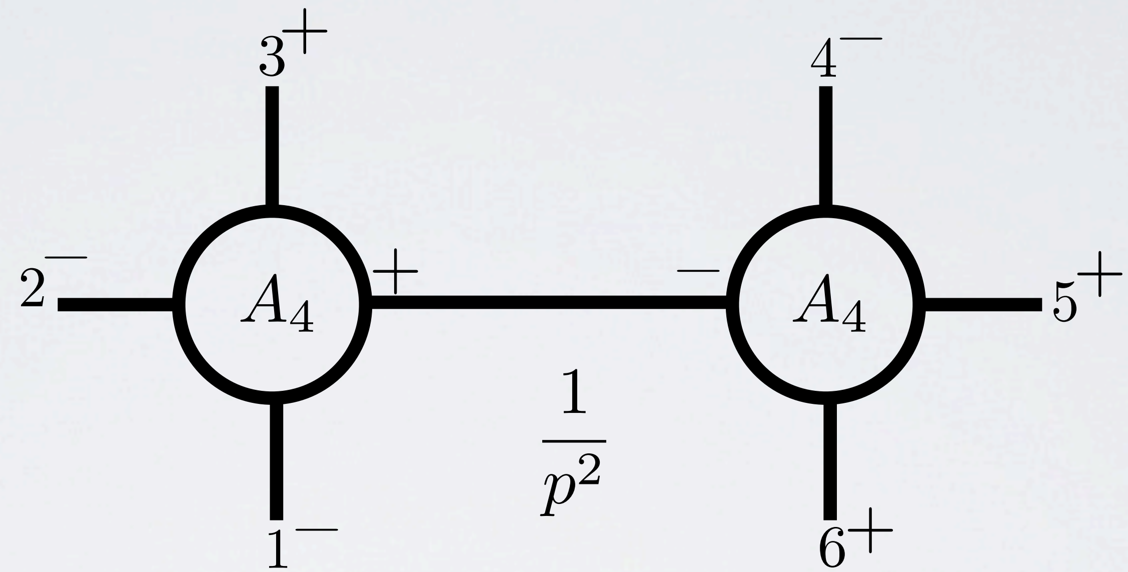
[Dixon,von Hippel'14]

The same factorization function is conjectured to control factorization of bigger amplitudes

Application : factorization of amplitude

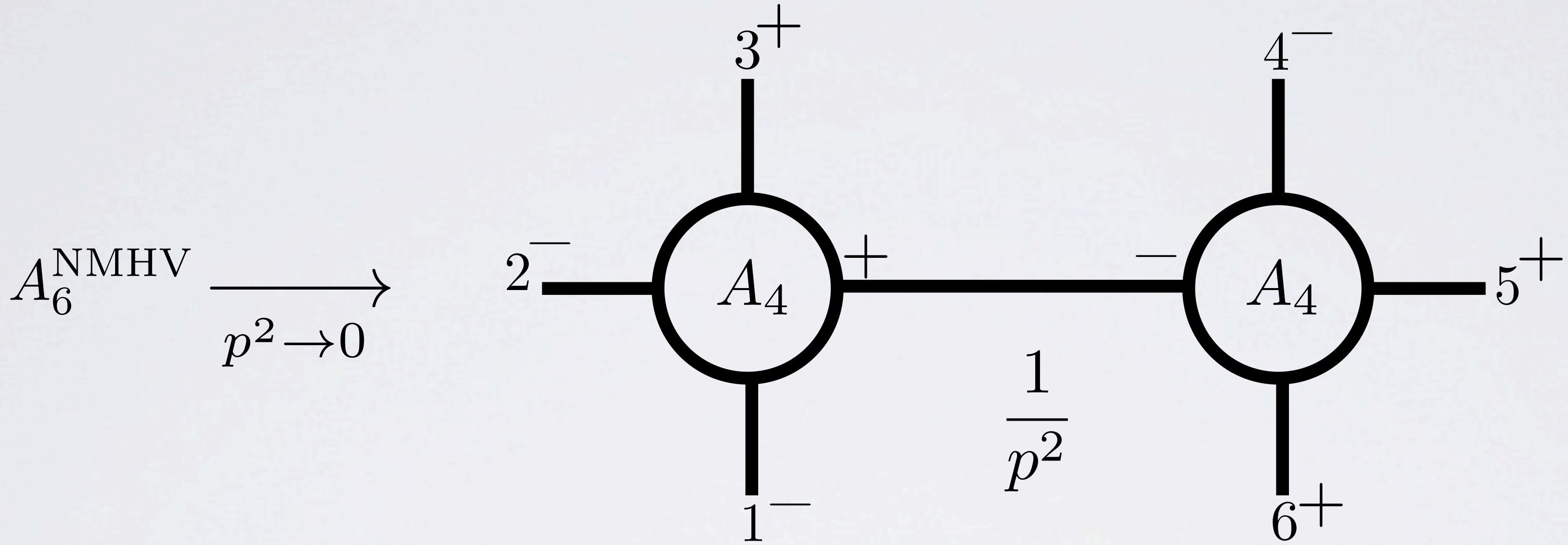
Multi-particle factorization in massless gauge theory

$$A_6^{\text{NMHV}} \xrightarrow[p^2 \to 0]{}$$

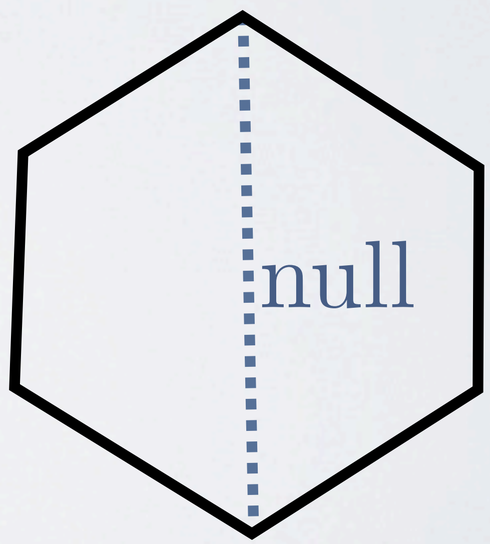


Application : factorization of amplitude

Multi-particle factorization in massless gauge theory



Dual limit for Wilson loops corresponds to two cusps becoming null separated
This limit is within radius of convergency of the OPE



Application : factorization of amplitude

Toy model for the amplitude

[BB,Sever,Vieira - in progress]

$$I \equiv \int_0^\infty du\, e^{-u\,p^2 - \Gamma_{\text{cusp}} \log^2 u}$$

1) At weak coupling

$$I = \frac{1}{p^2} \sum_l g^{2l} \underbrace{\text{Pol}_l(\log p^2)}$$

Perfect match up to 4 loops

[Dixon,von Hippel'14], [Dixon,von Hippel,McLeod'15]

Application : factorization of amplitude

Toy model for the amplitude

[BB,Sever,Vieira - in progress]

$$I \equiv \int_0^{\infty} du\, e^{-u\,p^2 - \Gamma_{\rm cusp} \log^2 u}$$

1) At weak coupling
$$I = \frac{1}{p^2} \sum_l g^{2l}\, \mathrm{Pol}_l(\log p^2)$$

Application : factorization of amplitude

Toy model for the amplitude

[BB,Sever,Vieira - in progress]

$$I \equiv \int_0^\infty du\, e^{-u\,p^2 - \Gamma_{\rm cusp}\log^2 u}$$

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2) At any $g \neq 0$

$$I|_{p^2=0} = \int_0^\infty du\, e^{-\Gamma_{\rm cusp}\log^2 u} < \infty$$

No pole!

Application : factorization of amplitude

Toy model for the amplitude

[BB,Sever,Vieira - in progress]

$$I \equiv \int_0^\infty du\, e^{-u\, p^2 - \Gamma_{\rm cusp} \log^2 u}$$

1) At weak coupling

$$I = \frac{1}{p^2} \sum_l g^{2l}\, {\rm Pol}_l(\log p^2)$$

2) At any $g \neq 0$

$$I|_{p^2=0} = \int_0^\infty du\, e^{-\Gamma_{\rm cusp} \log^2 u} < \infty$$

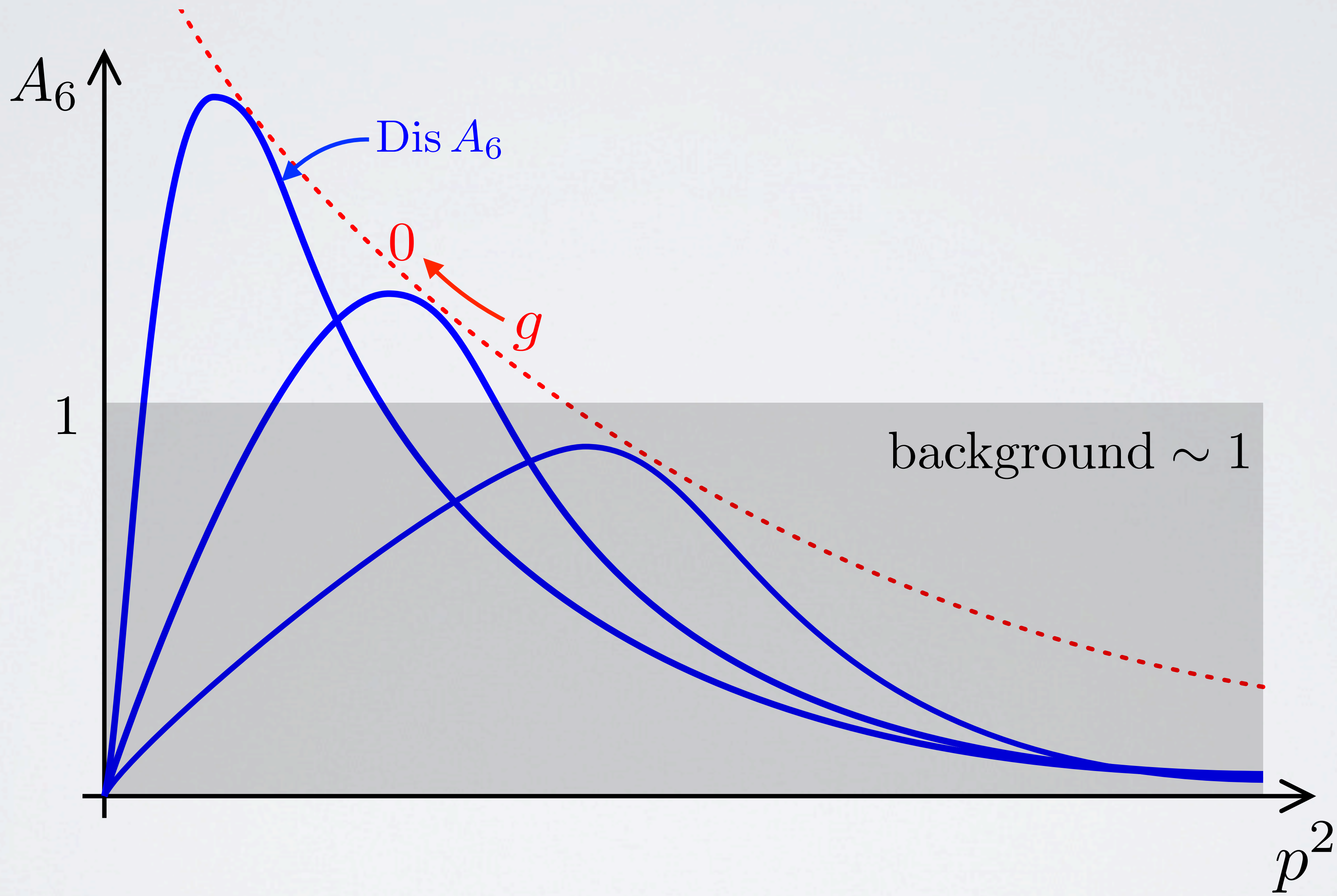
No pole!

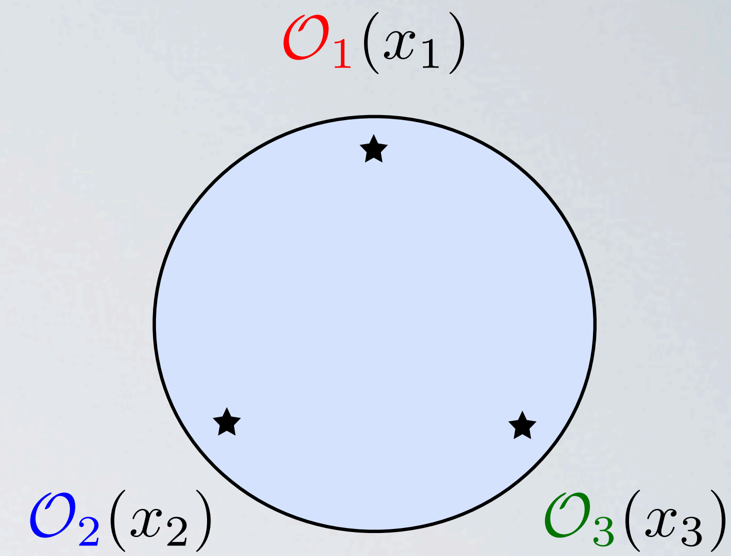
3) There is a discontinuity

$$\text{Dis}\, A_6 \propto e^{-\Gamma_{\rm cusp} \log^2 (p^2)} \neq 0$$

Application : factorization of amplitude

Cartoon of what is happening





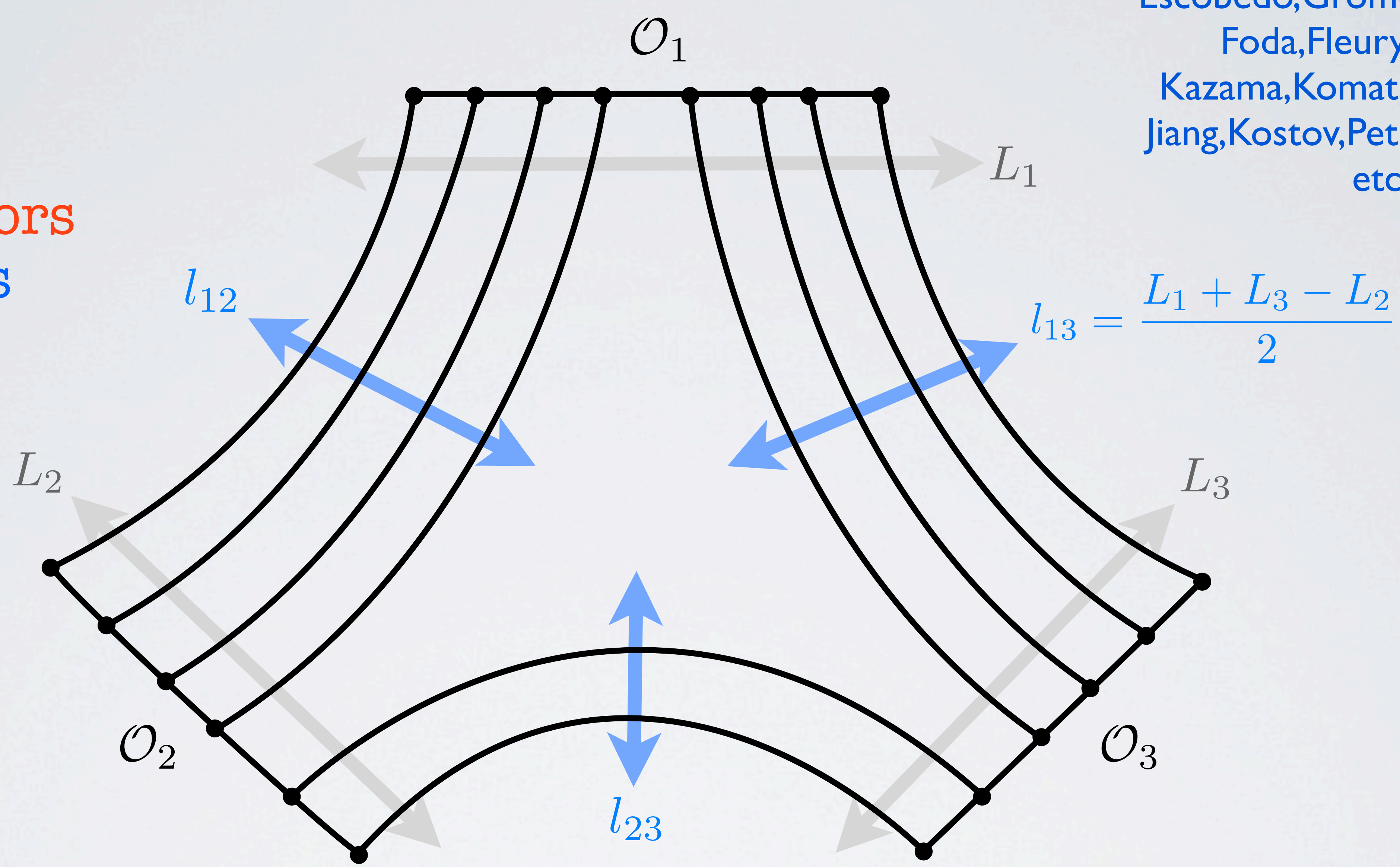
Structure constants and string splitting/joining

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_{12}}x_{23}^{\Delta_{23}}x_{13}^{\Delta_{13}}}$$

Spin chain tayloring

[Many people, see e.g. Escobedo,Gromov,Sever,Vieira, Foda,Fleury,Caetano, Kazama,Komatsu,Nishimura, Jiang,Kostov,Petrovskii,Serban, etc.]

topology :
- 3 operators
- 3 bridges

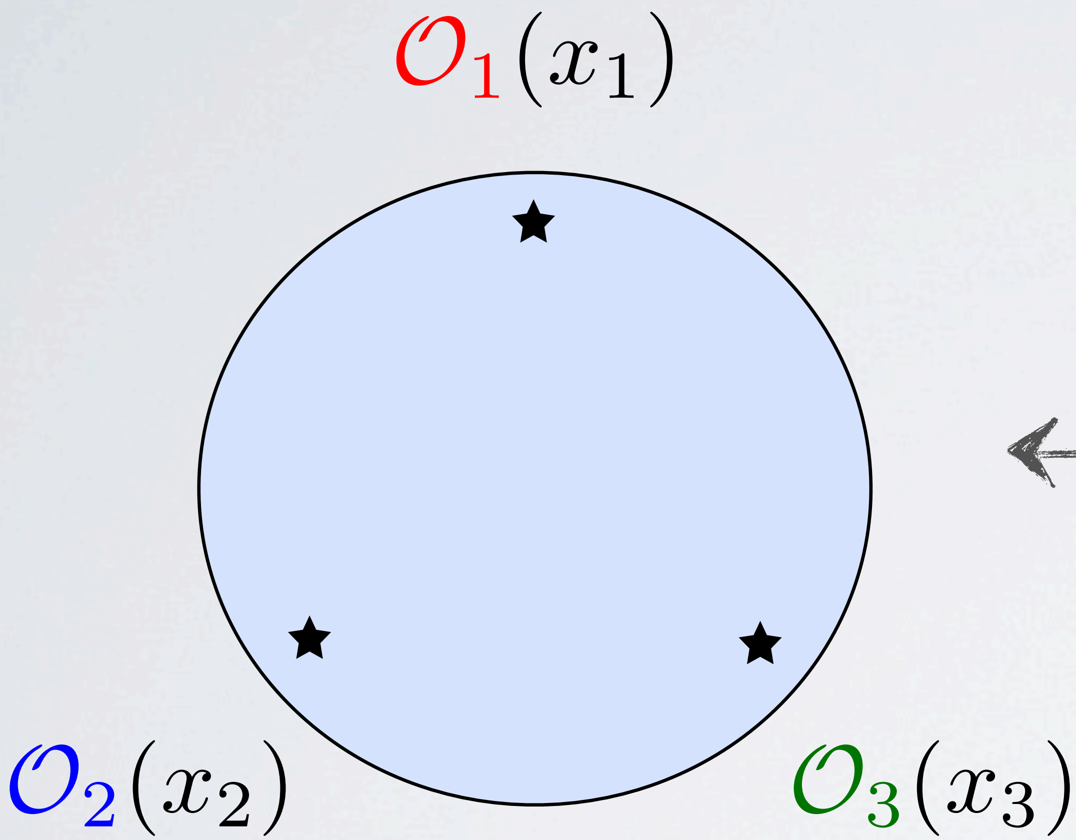


Recipe : cut spin chain states and compute their overlap following the Wick contractions

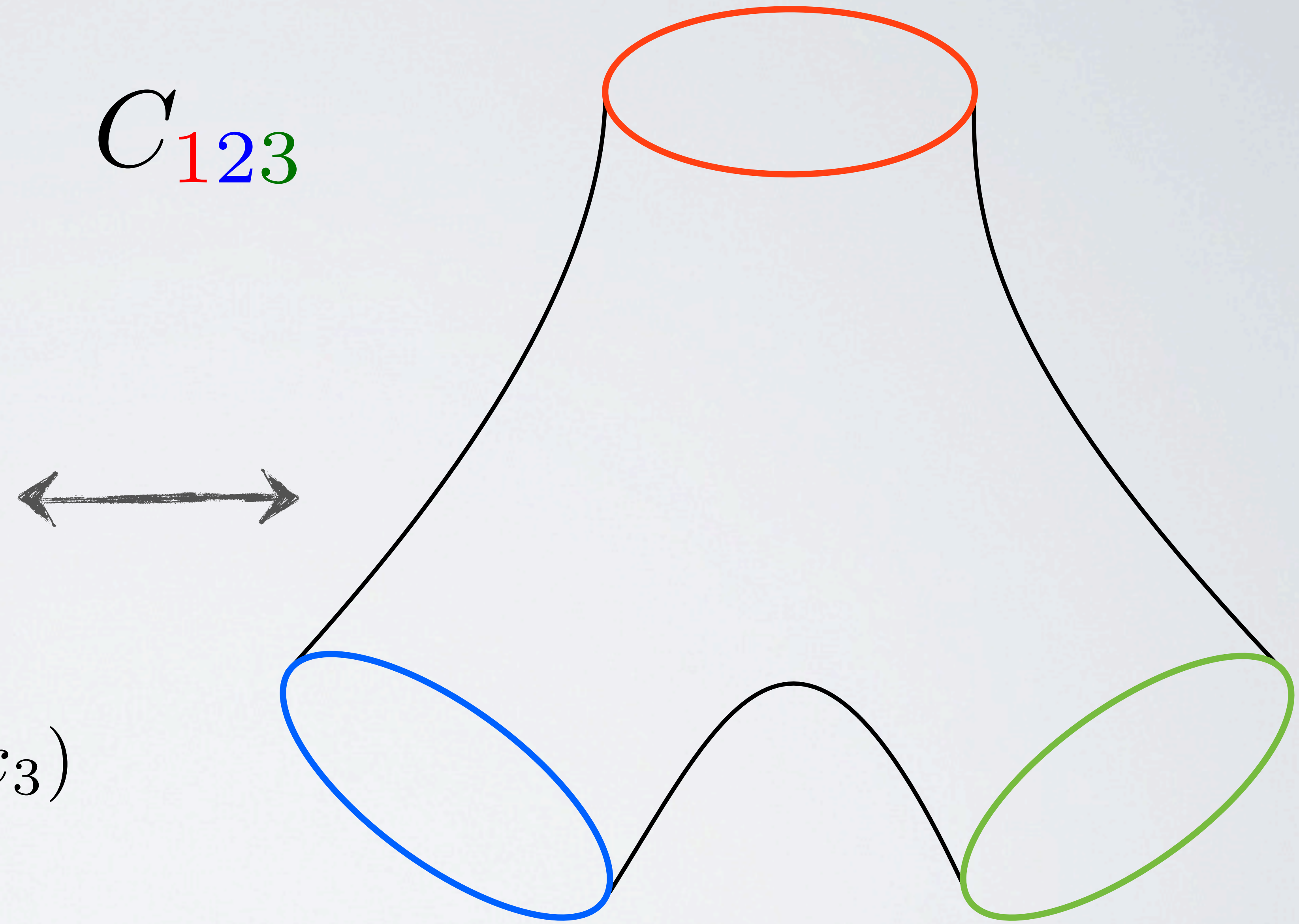
How to go to higher loops? (spin chain wave functions are unknown, as well as correction to splitting vertex)

Inspiration from string

C_{123}



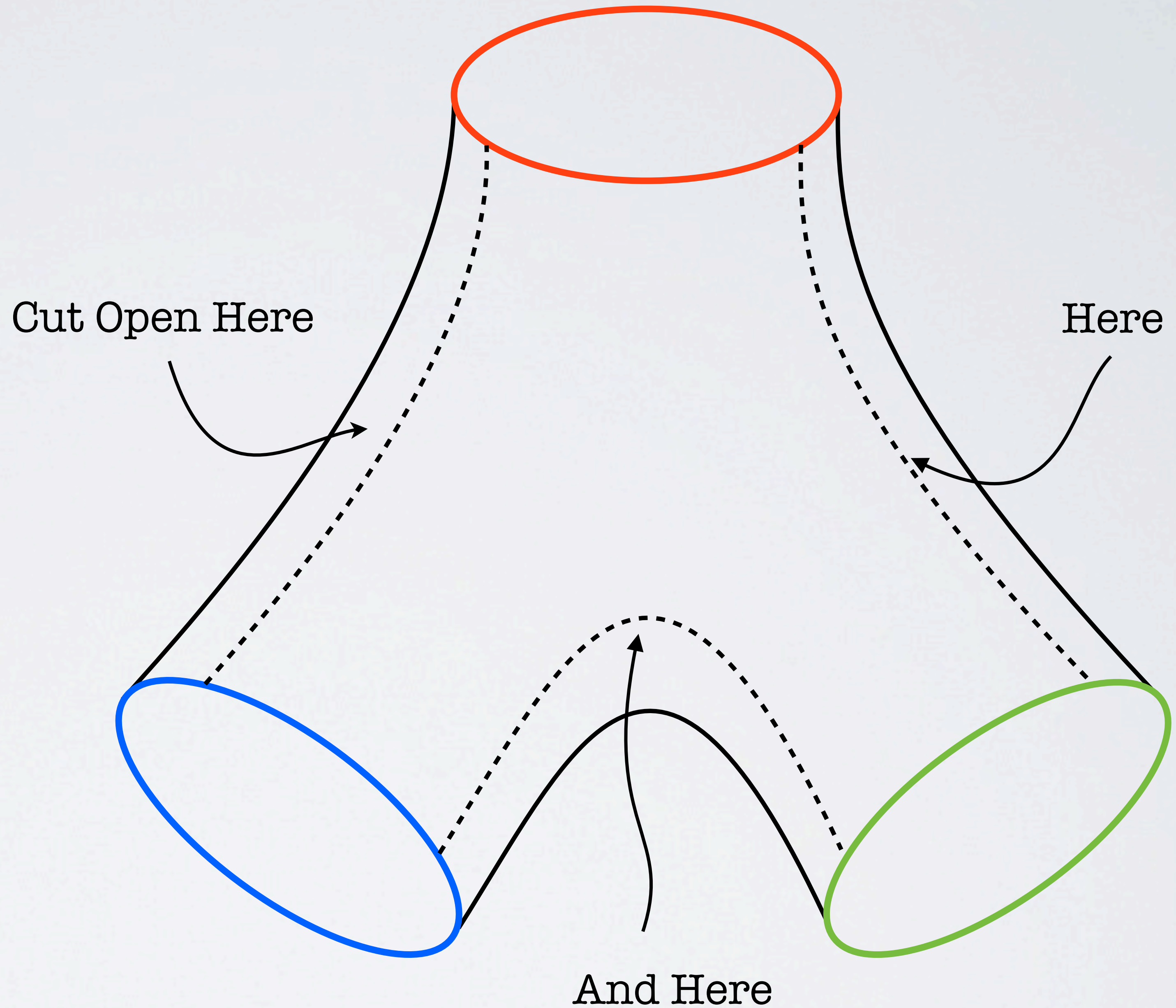
3-punctured sphere



pair of pants

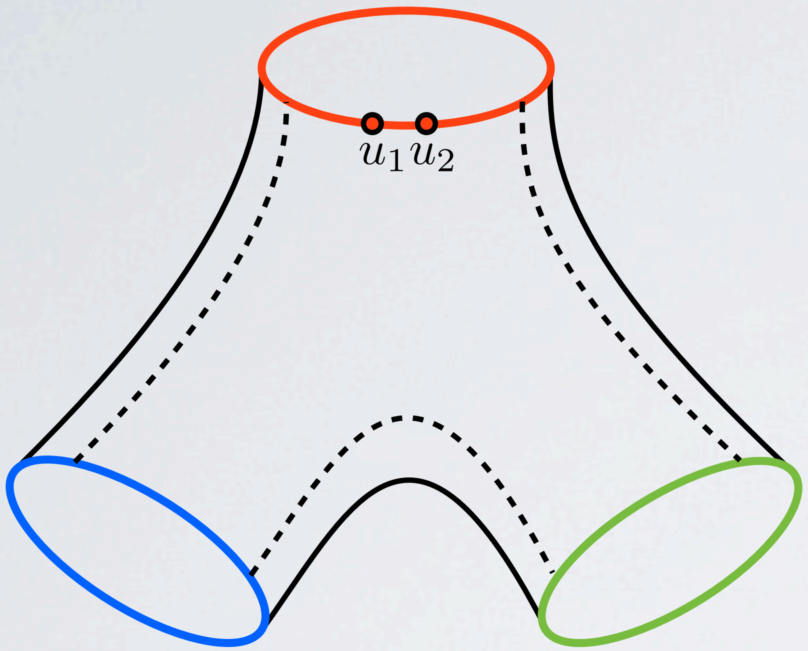
Asymptotic description

Cutting procedure :

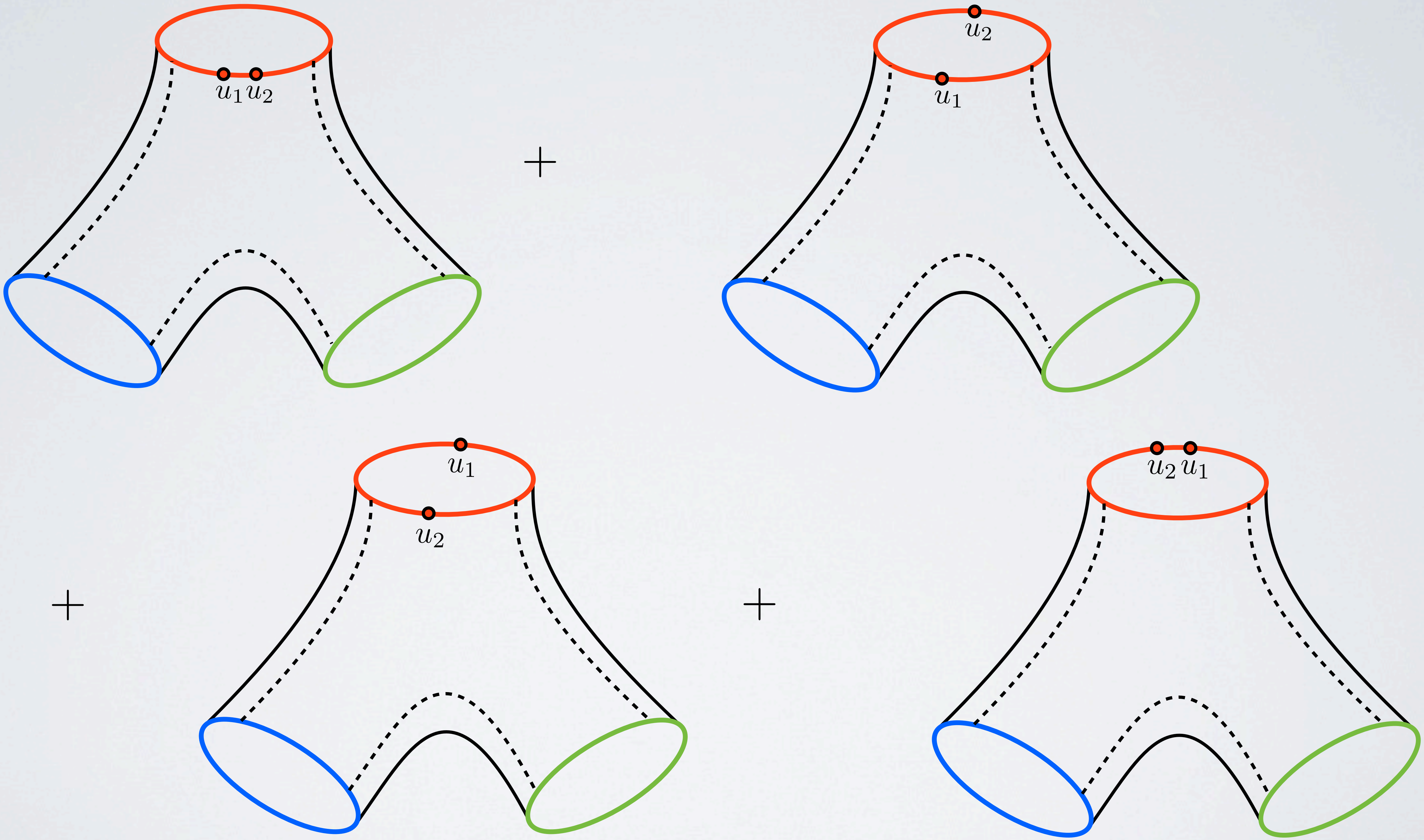


1 pair of pants = 2 hexagons

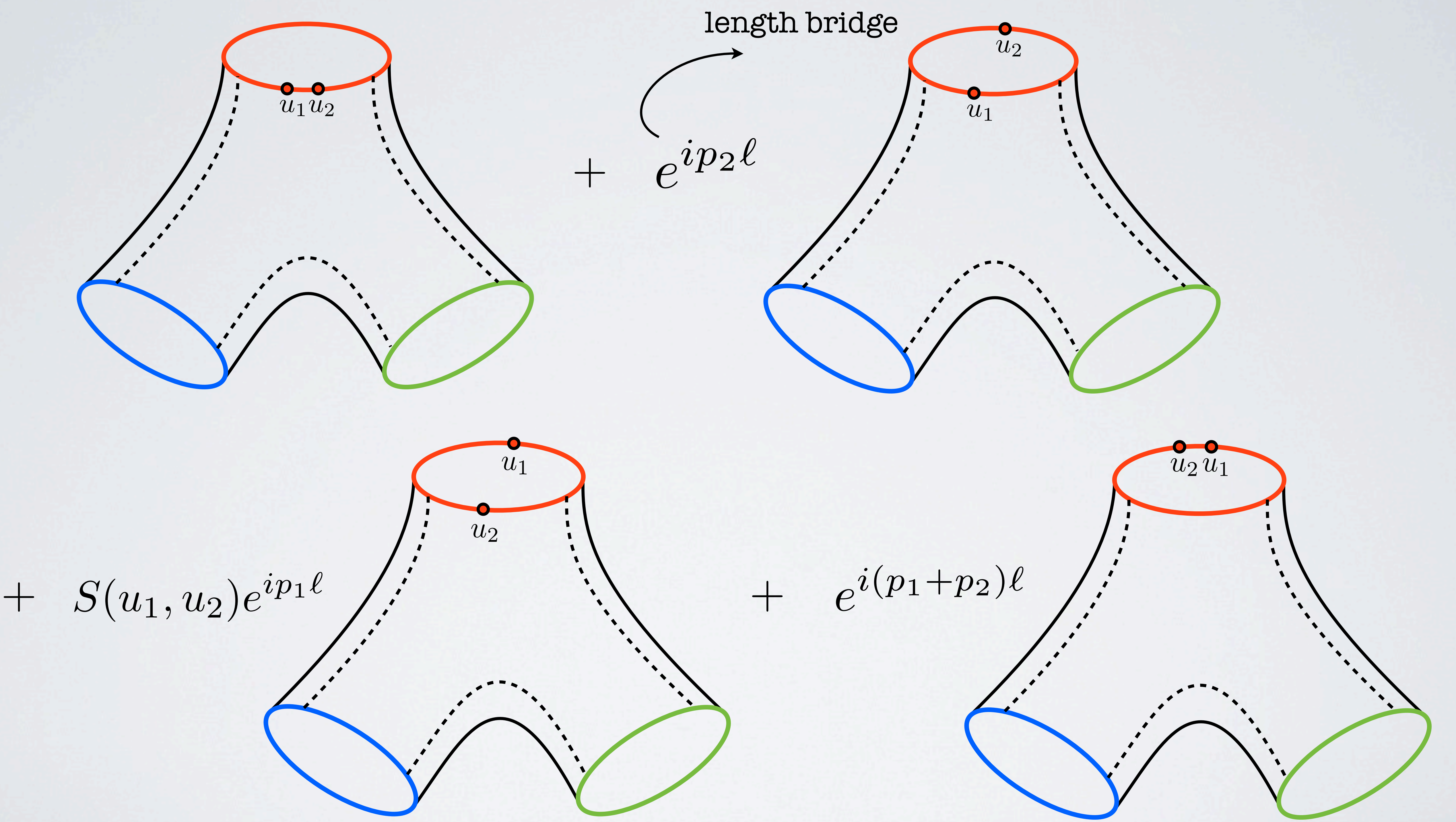
Same but with excitations



Same but with excitations



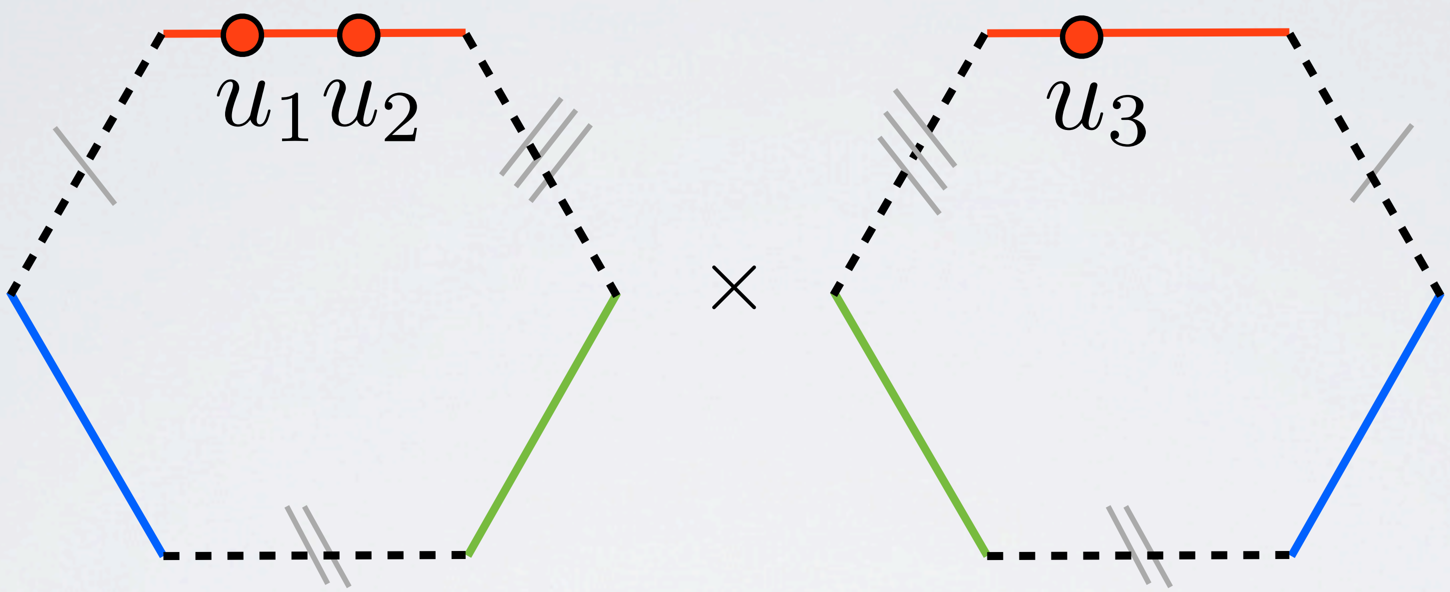
Same but with excitations



Hexagon factorization

Elementary block

[BB,Komatsu,Vieira'15]

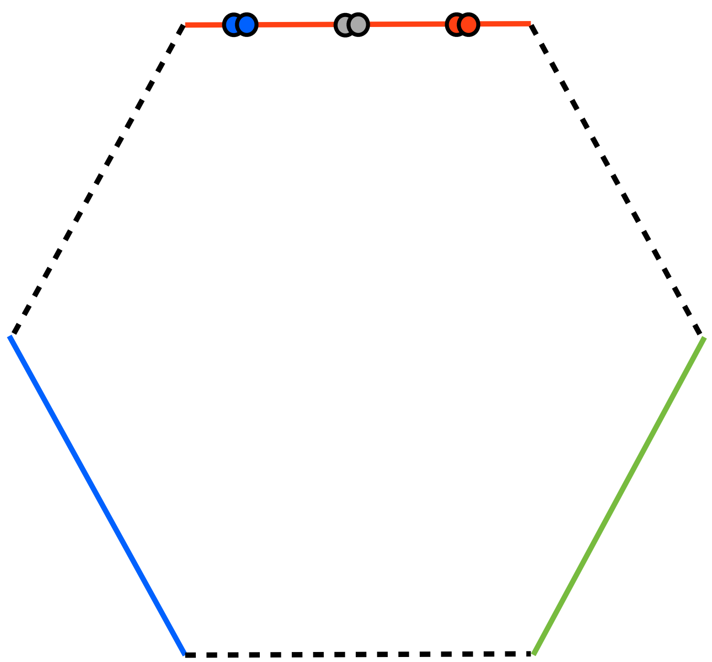


Hexagon form factor : contribution of an hexagon decorated with magnons on its edges

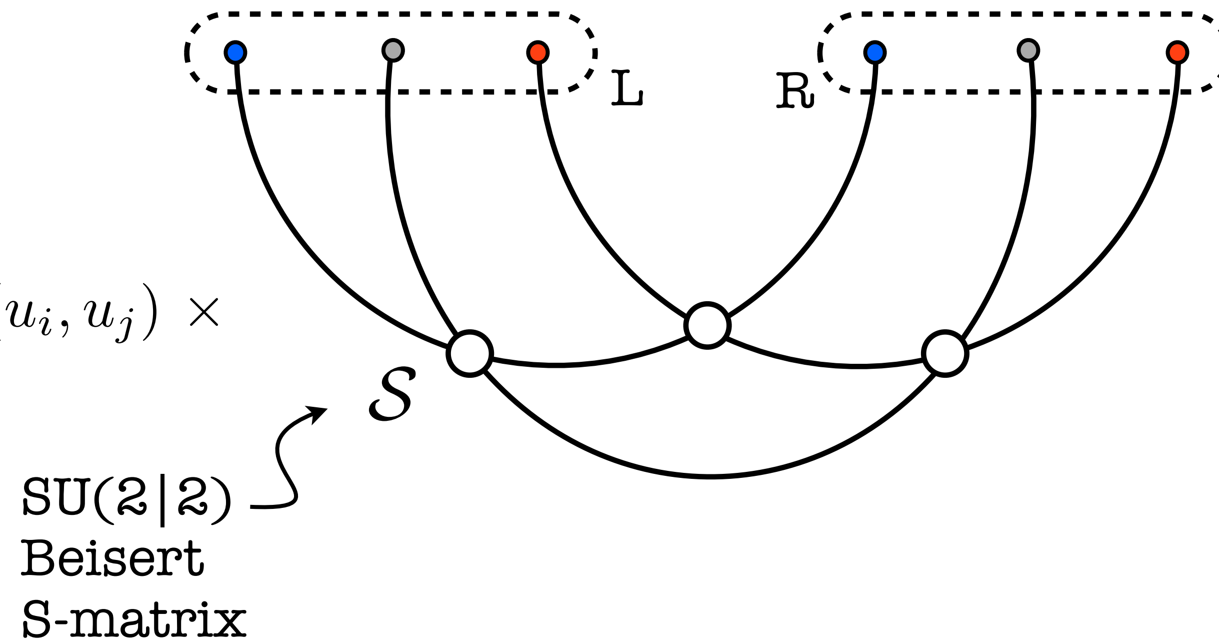
Apply integrable bootstrap again to determine it at finite coupling

N-magnon hexagon

Conjecture (one can actually prove it for low number of magnons):



$$= \prod_{i<j}^{N} h(u_i, u_j) \times$$



$$\mathfrak{h}^{A_1 \dot{A}_1 \cdots A_N \dot{A}_N} = (-1)^{\mathfrak{f}} \prod_{i<j}^{N} h_{ij} \langle \chi_N^{\dot{A}_N} \ldots \chi_1^{\dot{A}_1} | \mathcal{S} | \chi_1^{A_1} \ldots \chi_N^{A_N} \rangle$$

Asymptotic formula

Consider 2 BPS operators and 1 non-BPS operator

i.e.

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = C_{123}^{\bullet\circ\circ} \times \frac{\text{tensor}}{x_{12}^{\Delta_{12}} x_{23}^{\Delta_{23}} x_{31}^{\Delta_{31}}}$$

e.g. $\mathcal{O}_1 = \mathrm{tr} D^S Z^{L_1}$ with the rest BPS

Asymptotic formula

Hexagon prediction :

$$\left(\frac{C_{123}^{\bullet\circ\circ}}{C_{123}^{\circ\circ\circ}}\right)^2 = \frac{\prod_{k=1}^{S}\mu(u_k)}{\det\, \partial_{u_i}\phi_j \prod_{i<j} S(u_i,u_j)} \times \mathcal{A}^2$$

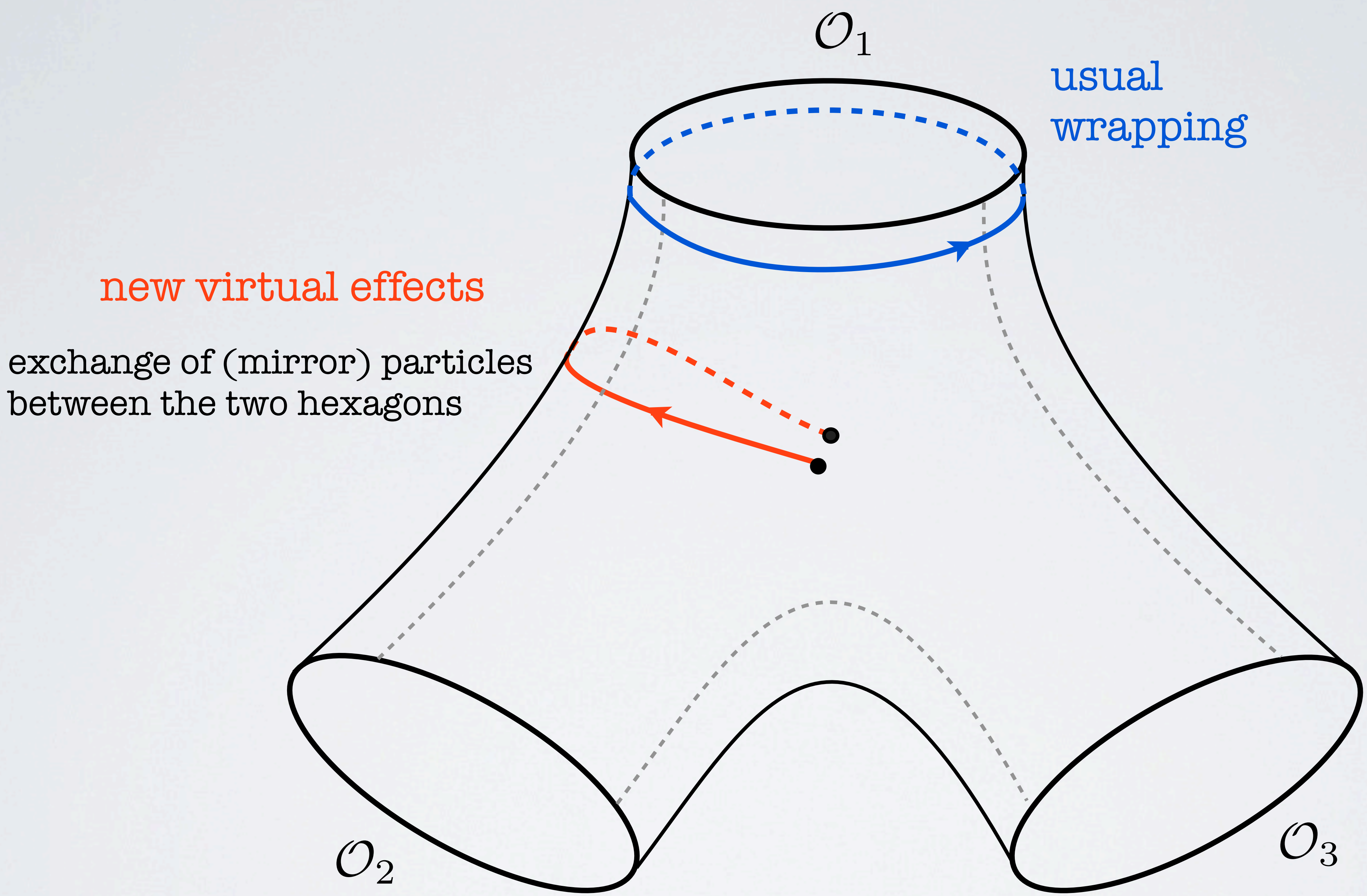
Hexagon part

sum over partitions of Bethe Roots

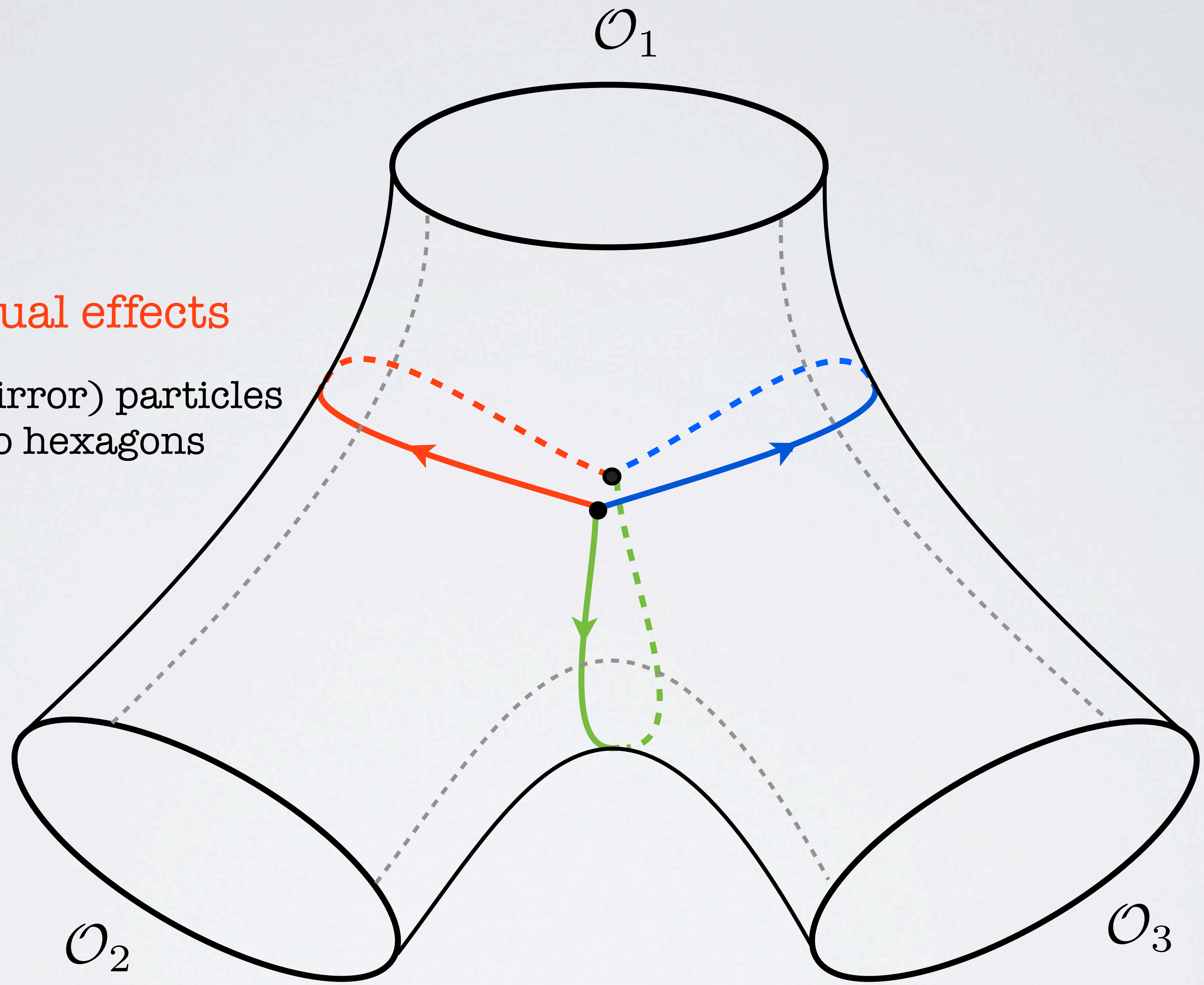
$$\mathcal{A} = \prod_{i<j} h(u_i,u_j) \sum_{\alpha\cup\bar{\alpha}=\mathbf{u}} (-1)^{|\bar{\alpha}|} \prod_{j\in\bar{\alpha}} e^{ip_j\ell} \prod_{i\in\alpha, j\in\bar{\alpha}} \frac{1}{h(u_i,u_j)}$$

Valid to all loops up to finite size effects

Finite size effects



Finite size effects

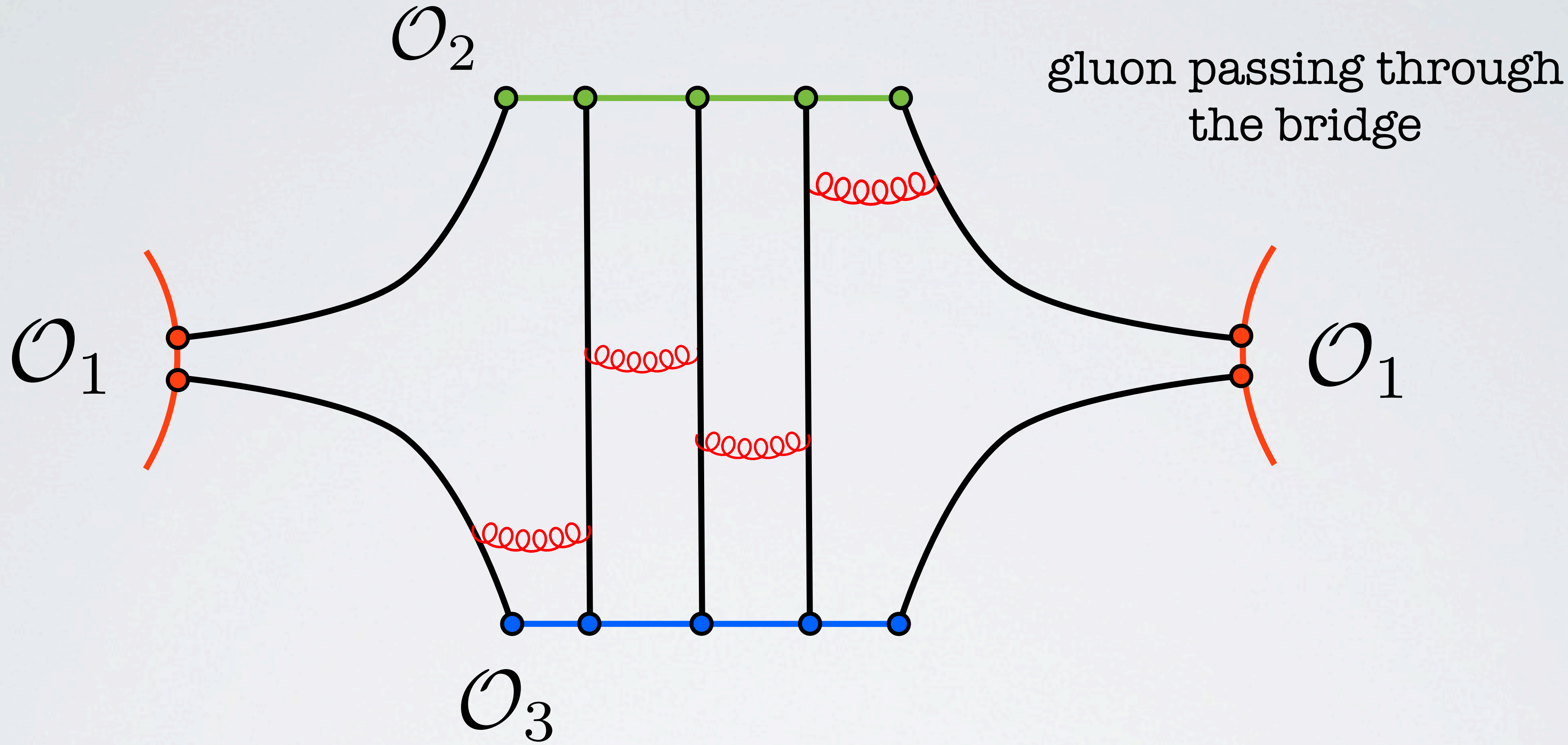


new virtual effects

exchange of (mirror) particles
between the two hexagons

these virtual effects come from the 3 mirror channels (= where we cut)

Finite size effects at weak coupling

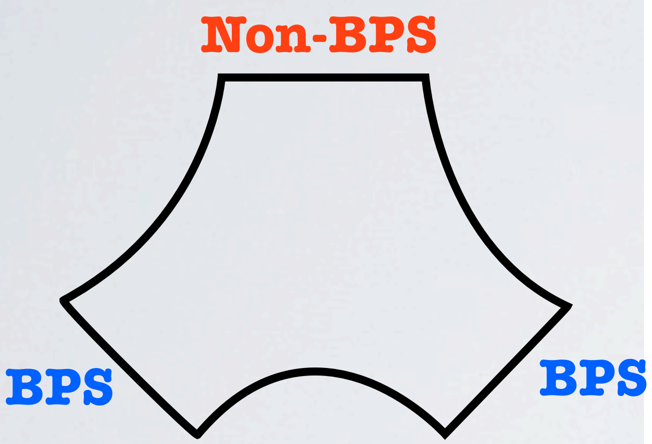


The loop order is given by the size of the bridge

virtual effects = $O(g^{2\ell_{ij}})$ at weak coupling

Comparison with data

$$\left(\frac{C_{123}^{\bullet\circ\circ}}{C_{123}^{\circ\circ\circ}}\right)^2$$



Spin	*"Long" Bridge i.e. length* $\ell = 2$
2	$\frac{1}{6} - 2g^2 + 28g^4 + \dots$
4	$\frac{1}{70} - \frac{205}{882}g^2 + \frac{36653}{9261}g^4 + \dots$
6	$\frac{1}{924} - \frac{553}{27225}g^2 + \frac{826643623}{2156220000}g^4 + \dots$
8	$\frac{1}{12870} - \frac{14380057}{9018009000}g^2 + \frac{2748342985341731}{85305405235050000}g^4 + \dots$
10	$\frac{1}{184756} - \frac{3313402433}{27991929747600}g^2 + \frac{156422034186391633909}{62201169404983234080000}g^4 + \dots$

Spin	*"Short" Bridge i.e. length* $\ell = 1$
2	$\frac{1}{6} - 2g^2 + (28 + 12\zeta_3)g^4 + \dots$
4	$\frac{1}{70} - \frac{205}{882}g^2 + \left(\frac{76393}{18522} + \frac{10}{7}\zeta_3\right)g^4 + \dots$
6	$\frac{1}{924} - \frac{553}{27225}g^2 + \left(\frac{880821373}{2156220000} + \frac{7}{55}\zeta_3\right)g^4 + \dots$
8	$\frac{1}{12870} - \frac{14380057}{9018009000}g^2 + \left(\frac{5944825782678337}{170610810470100000} + \frac{761}{75075}\zeta_3\right)g^4 + \dots$
10	$\frac{1}{184756} - \frac{3313402433}{27991929747600}g^2 + \left(\frac{171050793565932326659}{62201169404983234080000} + \frac{671}{881790}\zeta_3\right)g^4 + \dots$

2-loop finite size effect

perfect agreement
(including zeta's coming from finite size corrections)

Conclusions

Integrability comes with powerful new strategies for computing quantities at any value of the coupling in planar N=4 SYM theory

It allows us to attack increasingly complicated objects and find all-loop expressions (conjectures) for them, like for amplitudes, structure constants, etc.

How far can we go? Can we bootstrap string loops? Can we solve to any order in the 1/N expansion?

How can we prove all these conjectures? Can one understand why is this theory integrable after all?

THANK YOU!