Frame-Covariant Formulation of Inflation in the Slow-Roll Approximation in Scalar-Curvature Theories

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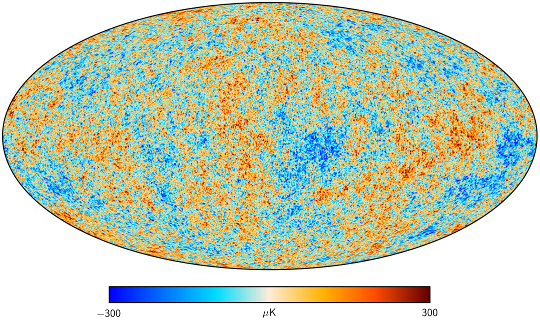
Outline

Introduction

- Inflation originally proposed to explain *flatness* and *horizon* problems:
  - $k = 0$ is unstable fixed point; if Universe is close to flat now, it must have been flatter in the past
  - The Universe is homogeneous at length scales larger than the Hubble horizon at surface of last scattering
- A period of accelerated expansion:
  - flattens out the Universe; no need to fine-tune initial curvature
  - ensures that length scales which now appear to be too far away from each other were in causal contact at early times

Introduction



- Inflation is also generic explanation for the origin of anisotropies in the cosmic microwave background (CMB)
- No definitive driving mechanism for inflation; numerous models (quintessence, modified gravity, string-inspired models...)

Scalar-Curvature Theories

- Action $S = S[g_{\mu\nu}, \varphi, f(\varphi), k(\varphi), V(\varphi)]$ for wide class of inflation models in the *Jordan frame*

$$S = \int d^4x\sqrt{-g}\left[-\frac{f(\varphi)}{2}R + \frac{1}{2}k(\varphi)g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi)\right]$$

- Minimally-coupled models: $f(\varphi) = M_P^2$, $k(\varphi) = 1$)
- *Conformal transformation* and *reparametrisation* of the inflaton (collectively *frame transformation*) lead to the *Einstein frame*:

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega(x)^2 g_{\mu\nu},$$
$$\varphi \to \tilde{\varphi} = \tilde{\varphi}(\varphi), (d\tilde{\varphi}/d\varphi)^2 = K(\varphi),$$
$$S[g_{\mu\nu}, \varphi, f(\varphi), k(\varphi), V(\varphi)] = S[\tilde{g}_{\mu\nu}, \tilde{\varphi}, \tilde{f}(\tilde{\varphi}), \tilde{k}(\tilde{\varphi}), \tilde{V}(\tilde{\varphi})].$$

- *Frame problem*: are frame transformations physical or not?
- Aim to answer this by developing formalism applicable to general scalar-tensor theories

Classical Dynamics

- Einstein field equations modified due to non-minimal coupling $f(\varphi) \neq M_P^2$
- Assume homogeneous inflation and Friedman-Walker-Robertson metric

$$g_{\mu\nu} = \mathrm{diag}(N_L^2, -a^2, -a^2, -a^2)$$

- N_L is general lapse function, used to define the Hubble parameter $H = \dot{a}/a = (N_L)^{-1} d \ln a/dt$
- Density and pressure are replaced by their non-minimal (NM) extensions:

$$\frac{\rho^{(\mathrm{NM})}}{M_P^2} = \frac{\rho}{f} - \frac{3H\dot{f}}{f}, \qquad \frac{p^{(\mathrm{NM})}}{M_P^2} = \frac{p}{f} - \frac{3H\dot{f}}{f} - \frac{\ddot{f}}{f}$$

- Acceleration, Friedman, and continuity equations are unchanged

Cosmological perturbations

- Observable anisotropies seeded by primordial perturbations of the metric and the inflaton field
- Scalar, vector, and tensor modes (SVT) decouple to first order

$$g_{\mu\nu}dx^\mu dx^\nu = (1+2\phi)N_L^2 dt^2 + 2a(\partial_i B + B_i)N_L dt\, dx^i \\ - a^2[(1+2\psi)\delta_{ij} + \partial_i\partial_j A + \partial_i A_j + \partial_j A_i + h_{ij}]dx^i dx^j,$$

- The only scalar artifact of inflation is the diffeomorphism-invariant *comoving curvature perturbation*

$$\mathcal{R}_\varphi = \phi - \frac{H}{\dot{\varphi}}\delta\varphi$$

- "Freezes" outside the horizon; perturbations at largest observable scales are inflationary relics Weinberg, Phys. Rev. D, D67, 2003

Cosmological perturbations

- SVT decomposition leads to linearised Einstein field equations

$$\delta G_{\mu\nu} = \delta T^{(NM)}_{\mu\nu}$$

- Comoving curvature perturbation satisfies

$$\frac{1}{N_L^2 a^3 Q_\mathcal{R}} \frac{d}{dt}\left(N_L a^3 Q_\mathcal{R} \dot{\mathcal{R}}\right) + \frac{k^2 \mathcal{R}}{a^2} = 0, \quad Q_\mathcal{R} = \frac{k\dot{\varphi}^2 + \dfrac{3\dot{f}^2}{2f}}{\left(H + \dfrac{\dot{f}}{2f}\right)^2} \equiv \frac{\dot{\varphi}^2}{H^2} Z_\mathcal{R}$$

- Canonicalising the primordial perturbation via $z_\mathcal{R} \equiv a\sqrt{Q_\mathcal{R}}$, $v_\mathcal{R} \equiv z_\mathcal{R}\mathcal{R}$, we derive the generalised *Sasaki-Mukhanov* (SM) equation

$$\frac{d^2 v_{\mathcal{R},k}}{d\eta^2} + \left(k^2 - \frac{1}{z_\mathcal{R}}\frac{d^2 z_\mathcal{R}}{d\eta^2}\right) v_{\mathcal{R},k} = 0$$

Cosmological perturbations

- Power spectrum of perturbations is quantum in origin
- Quantise SM equation by imposing usual commutation relations and the *Bunch-Davies* vacuum (free fields at very early times)
- Solving for the mode functions at large scale limit, we find the two-point function of the primordial perturbations:

$$\langle v_{\mathcal{R},\mathbf{k}_1} | v_{\mathcal{R},\mathbf{k}_2} \rangle = |v_{\mathcal{R},k}|^2 \, \delta(\mathbf{k}_1 + \mathbf{k}_2)$$

- Taking into account the transfer functions which induce a multiplicative multipole contribution, we find the observed scalar power spectrum and tensor-to-scalar ratio:

$$P_{\mathcal{R}} \equiv \frac{k^3}{8\pi^2} \frac{H^4}{Z_{\mathcal{R}} \dot{\varphi}^2}, \qquad r = 16 \frac{Z_{\mathcal{R}}}{Z_T} \frac{\dot{\varphi}^2}{H^2}.$$

Slow-roll inflation

- Define *scalar spectral index* and its running, evaluated at horizon crossing of largest cosmological scales ($k = 0.002$ Mpc^{-1})

$$n_{\mathcal{R}} - 1 = \left. \frac{d \ln P_{\mathcal{R}}}{d \ln k} \right|_{k=aH}, \qquad \alpha_{\mathcal{R}} = \frac{dn_{\mathcal{R}}}{d \ln k}$$

- Slow-roll approximation $\ddot{\varphi} \ll H\dot{\varphi} \ll H^2 \varphi$ implies that Hubble *slow-roll parameters* (HSRP) are *small* and *slowly varying*:

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2}, \qquad \delta_H \equiv \frac{\ddot{\varphi}}{H\dot{\varphi}},$$

$$\kappa_H \equiv \frac{1}{2}\frac{\dot{f}}{Hf}, \qquad \sigma_H \equiv \frac{1}{2}\frac{\dot{E}}{HE}, \qquad E = kf + \frac{3}{2}f_{,\varphi}^2$$

- May express cosmological observables in terms of HSRP:

$$n_{\mathcal{R}} = 1 - 4\epsilon_H - 2\delta_H + 2\kappa_H - 2\sigma_H,$$
$$r = 16(\epsilon_H + \kappa_H)$$

Slow-roll inflation

- HSRP expressions for $n_{\mathcal{R}}$ and r require solving the cosmological equations of motion: unfeasible in all but the simplest of cases
- *Inflationary attractor*: class of solutions which inflationary trajectories approach in phase space
- Equations of motion simplify in the limit of vanishing SRP:

$$H^2 \approx \frac{fU}{3}, \qquad -\frac{3E}{f} H\dot{\varphi} \approx f^2 U_{,\varphi}, \qquad \frac{H}{\dot{\varphi}} \approx -\frac{EU}{f^2 U_{,\varphi}}$$

- Define *potential* slow-roll parameters (PSRP); reduce to the corresponding HSRP in the slow-roll limit:

$$\epsilon_U \equiv \frac{1}{2} \frac{fU_{,\varphi}(fU)_{,\varphi}}{EU^2}, \qquad \delta_U \equiv \frac{1}{2} \frac{fU_{,\varphi}(fU)_{,\varphi}}{EU^2} - \left(\frac{f^2 U_{,\varphi}}{EU} \right)_{,\varphi},$$

$$\kappa_U \equiv -\frac{f_{,\varphi}}{2} \frac{fU_{,\varphi}}{EU}, \qquad \sigma_U \equiv -\frac{1}{2} \frac{E_{,\varphi}}{E^2} \frac{f^2 U_{,\varphi}}{U}$$

Frame covariance

- Extract the coupling functions from transformed action after a general frame trasformation
- PSRP transform as

$$\tilde{\epsilon}_U = \epsilon_U - 2\Delta_\Omega, \qquad \tilde{\delta}_U = \delta_U + 2\Delta_\Omega - \Delta_K,$$
$$\tilde{\kappa}_U = \kappa_U + 2\Delta_\Omega, \qquad \tilde{\sigma}_U = \sigma_U + 4\Delta_\Omega + \Delta_K,$$
$$\Delta_X \equiv \frac{1}{2}\frac{f^2 U_{,\varphi}}{EU}\frac{X_{,\varphi}}{X}, \qquad X \equiv \{\Omega, K\}$$

- Inflationary observables are frame-invariant
- Number of e-foldings ($dN = H\,dt$) not frame-invariant to all orders

$$N \to \tilde{N} = N + \ln[\Omega(t)/\Omega(t_{\rm end})]$$

- In potential formalism, N is frame-invariant as long as end-of-inflation condition $\max(\epsilon_U, |\eta_U|) = 1$ is extended to

$$\max(\epsilon_U + \kappa_U, |2(\epsilon_U + \kappa_U) + 2(\delta_U - 3\kappa_U + \sigma_U)|) = 1$$

Specific models

- Induced gravity inflation: $f(\varphi) = \xi\varphi^2$, $V(\varphi) = \lambda(\varphi^2 - 1/\xi)^2$
- Higgs inflation: $f(\varphi) = M_P^2 + \xi\varphi^2$, $V(\varphi) = \lambda(\varphi^2 - v^2)^2$

	Induced gravity inflation	Higgs inflation
$n_{\mathcal{R}}$	$1 - \frac{1}{\xi N} + \frac{6\xi^2 - 17\xi - 5}{8\xi^3 N^2} + O(1/N^3)$	$1 - \frac{2}{N} - \frac{3 + 1/8\xi}{N^2} + O(1/N^3)$
r	$\frac{2(1+6\xi)}{N^2\xi} + \frac{(1+6\xi)^2}{2\xi^2 N^3} + O(1/N^4)$	$\frac{16(1+6\xi)}{8\xi N^2 + N}$
$\alpha_{\mathcal{R}}$	$\frac{2}{N^2}$	$\frac{2}{N^2}$
α_T	$-\frac{3 + 1/(2\xi)}{N^3}$	$-\frac{3 + 1/(2\xi)}{N^3}$

- Possible to incorporate $F(R)$ models in the formalism via a Legendre transform:

$$\Phi = F(R)_{,R}$$

Conclusion

- Frame problem resolved in a natural way; frame invariance of the action is not imposed “by hand”
- Easy-to-use calculational tool for extracting cosmological observables from any scalar-curvature theory:
  - All higher-order runnings of cosmological observables may be calculated without defining additional slow-roll parameters
  - No further approximation beyond inflationary attractor is necessary (such as approximating the high-field potential in the Einstein frame)
- Possible extensions to the formalism:
  - Multi-field non-minimally coupled inflation
  - $F(\varphi, R)$ theories (recast in multi-field form via Legendre transforms)
- Frame covariance: starting point for studying effect of frame transformations to radiative corrections in inflation