The quantum theory of fluids

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BMG & Dave Sutherland, Phys.Rev.Lett. 114 (2015) 071601

Not a typo: fluids not fields

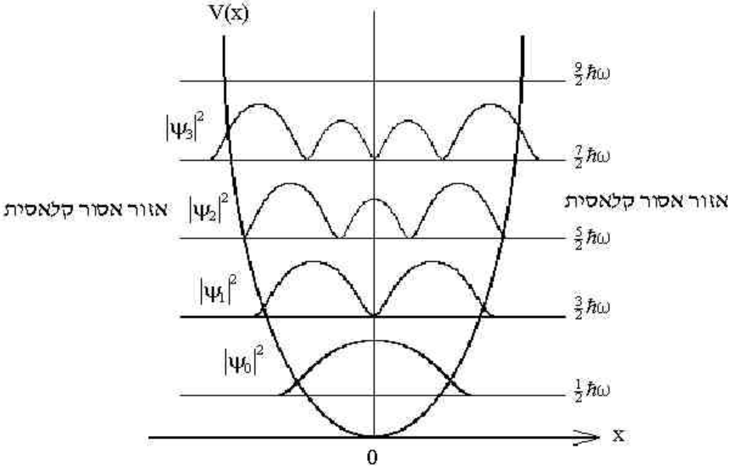
Does $\exists$ a consistent quantum theory of a (perfect, compressible) fluid?

Classical fluids $\subset$ classical fields, so:

- quantization an obvious thing to do,
- but isn't it trivial?

SHO: $L = \dot{q}^2 + q^2 \implies E = n + \frac{1}{2}, n \in \mathbb{Z}^+$

אנרגיה כוללת:



Fluids are special: $\exists$ vortices

Homework exercise: carry out an experiment . . .

$$L = \dot{q}^2 + 0q^2 \implies E = p^2, p \in \mathbb{R}$$

$L = \dot{q}^2 \implies E = p^2, p \in \mathbb{R}$:

- no Fock space
- no S-matrix
- ground state delocalized
- perturbation theory inconceivable

Historical approaches …

Landau 1941: Assume vortices ‘gapped’ $\Longrightarrow$ superfluid

Rattazzi et al. 2011: vortex sound speed $\varepsilon \to 0$

Endlich, Nicolis, Rattazzi, & Wang, 1011.6396

$$L = \dot{q}^2 + \varepsilon q^2 \implies E = \varepsilon(n + \tfrac{1}{2}), n \in \mathbb{Z}^+$$

Everything blows up.

- Conjecture: quantum fluid inconsistent
- Evidence: no fluids at T=0

Endlich, Nicolis, Rattazzi, & Wang, 1011.6396

We claim:

- Conjecture: quantum fluid consistent
- Evidence: computation!
- Also conjecture: quantum fluids unlike classical ones

Why you may care . . .

1. $T \to 0$ limit of normal fluids?
2. Historical successes of quantizing classical fields.
3. An elegant, consistent, & rich QFT

Elegance: We quantize t-dependent diffs $M^d \to M^d$, with $ISO(d,1) \times \mathrm{SDiff}(M^d)$ invariant action

Richness: $\supset$ all classical fluid phenomena!

Consistency: We do not seek a ToE, but rather an EFT

- Non-renormalizable
- Regime in which divergences under control
- Perturbation theory ‘converges’

Outline

- Fluid parameterization
- The classical theory of fluids
- The quantum theory of fluids

Fluid parameterization

- ‘Bathtub’ M^d (e.g. $\mathbb{R}^d$)
- Choose coordinates ϕ at $t = 0$ for fluid particles
- $x_t(\phi)$ is map $M^d \to M^d$ (Lagrange)

- Claim: cavitation and interpenetration cost finite E
- At low enough E, $x_t(\phi)$ is bijective
- Ditto $\phi_t(x)$ (Euler)
- Claim: at large distance ϕ may be assumed smooth

- How to parameterize the group $\mathrm{Diff}(M)$?
- Naïve exp map: $TM \to \mathrm{Diff}(M)$
- But $\mathrm{Diff}(M)$ is not Lie
- exp may not exist (counterexample: $\mathbb{R}$)
- exp may not be locally onto (counterexample: S^1)
- I am (at best) a physicist, so am allowed to just write $\phi = x + \pi$

(

$$\exp \pi = x + \pi + \pi(\partial \pi) + \frac{1}{2!}\pi(\partial \pi(\partial \pi)) + \dots$$

)

$M^d = \mathbb{R}^d$ henceforth

The classical theory of fluids

No one ever writes down the action!

In fact very elegant:

- Fields $\phi(x, t)$
- S invariant under Poincaré transformations on x
- and sdiffs of ϕ
- $\implies \mathscr{L} = -w_0 f(\sqrt{B})$, where $B = \det \partial_\mu \phi^i \partial^\mu \phi^j$.

Endlich, Nicolis, Rattazzi, & Wang, 1011.6396

Herglotz, 1911

Soper, Classical Field Theory, 2008

Then find

- $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\eta_{\mu\nu}$ is conserved
- $\rho = w_0 f$
- $p = w_0(\sqrt{B}f' - f)$
- $u^\mu = \frac{1}{2\sqrt{B}}\varepsilon^{\mu\alpha\beta}\varepsilon_{ij}\partial_\alpha\phi^i\partial_\beta\phi^j$. $(d = 2)$

Endlich, Nicolis, Rattazzi, & Wang, 1011.6396

Herglotz, 1911

Soper, Classical Field Theory, 2008

d=2 henceforth
(mostly)

The quantum theory of fluids . . .

Consider small fluctuations about the classical vacuum:

$\phi = x + \pi \ldots$

$$\mathcal{L} = \frac{1}{2}(\dot{\pi}^2 - c^2[\partial\pi]^2) - \frac{(3c^2+f_3)}{6}[\partial\pi]^3 + \frac{c^2}{2}[\partial\pi][\partial\pi^2] + \frac{(c^2+1)}{2}[\partial\pi]\dot{\pi}^2 - \dot{\pi}\cdot\partial\pi\cdot\dot{\pi} - \frac{(f_4+3c^2+6f_3)}{24}[\partial\pi]^4$$

$$+\frac{(c^2+f_3)}{4}[\partial\pi]^2[\partial\pi^2] - \frac{c^2}{8}[\partial\pi^2]^2 + \frac{(1-c^2)}{8}\dot{\pi}^4 - c^2[\partial\pi]\dot{\pi}\cdot\partial\pi\cdot\dot{\pi} - \frac{(1-3c^2-f_3)}{4}[\partial\pi]^2\dot{\pi}^2 + \frac{(1-c^2)}{4}[\partial\pi^2]\dot{\pi}^2 + \frac{1}{2}\dot{\pi}\cdot\partial\pi\cdot\partial\pi^T\cdot\dot{\pi} + \ldots,$$

$$\mathcal{L} = \frac{1}{2}(\dot{\pi}^2 - c^2[\partial\pi]^2) - \frac{(3c^2+f_3)}{6}[\partial\pi]^3 + \frac{c^2}{2}[\partial\pi][\partial\pi^2] + \frac{(c^2+1)}{2}[\partial\pi]\dot{\pi}^2 - \dot{\pi}\cdot\partial\pi\cdot\dot{\pi} - \frac{(f_4+3c^2+6f_3)}{24}[\partial\pi]^4$$
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- a mess
- derivatively coupled: goldstone bosons
- Poincaré non-linearly realized

$$\mathcal{L} = \frac{1}{2}(\dot{\pi}^2 - c^2[\partial\pi]^2)$$

- $c = \sqrt{f_2}$ is speed of sound for $[\partial\pi] \neq 0$
- $[\partial\pi] = 0 \implies$ gapless vortex modes
- Free particles, not harmonic oscillators!
- No ‘easy’ way out: $[\partial\pi] = 0 \implies$ only $\dot{\pi}$ terms

free particles $\Longrightarrow$

- no Fock space
- no S-matrix
- no perturbation theory

Correlators in d space dimensions:

- $\langle \pi_L(x)\pi_L(0)\rangle = \int d\omega d^d k \frac{e^{i(\omega t - k\cdot x)}}{\omega^2 - c^2 k^2} = \text{good}$
- $\langle \pi_T(x)\pi_T(0)\rangle = \int d\omega d^d k \frac{e^{i(\omega t - k\cdot x)}}{\omega^2} = \text{evil}$

ϕ is not physical
sdiffs are a redundancy

cf.

- gauge theories
- $2d$ sigma models

Jevicki 77

McKane & Stone 80

David 80, 81

Elitzur 83

Remark: $T_{\mu\nu}, \rho, p$, and u^{μ} are all sdiff invariants

Let's compute some correlators of invariants, and see what we get . . .

Not $p, \rho, \ldots$, but

$$\sqrt{B}u^0 - 1 = [\partial\pi] + \frac{1}{2}([\partial\pi]^2 - [\partial\pi^2]),$$
$$\sqrt{B}u^i = \dot{\pi}^i + [\partial\pi]\dot{\pi}^i - \dot{\pi}^j\partial_j\pi^i,$$

these are quadratic in π in $d = 2$

2-point functions:

$$\langle[\partial\pi][\partial\pi]\rangle = \frac{ik^2}{\omega^2 - c^2k^2},$$

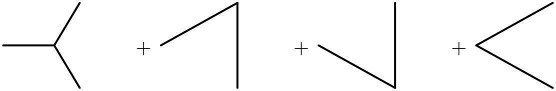
$$\langle\dot{\pi}^i[\partial\pi]\rangle = \frac{i\omega k^i}{\omega^2 - c^2k^2},$$

$$\langle\dot{\pi}^i\dot{\pi}^j\rangle = i\delta^{ij} + \frac{ic^2k^ik^j}{\omega^2 - c^2k^2}.$$

Real space correlators all exist!

3-point functions:

$$\langle \sqrt{B}u^i \sqrt{B}u^j (\sqrt{B}u^0 - 1) \rangle =$$



Many delicate cancellations
Real space correlators all exist

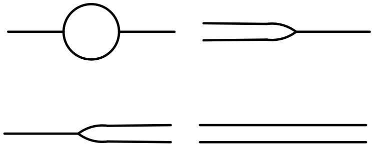
4-point functions also well-behaved

Now consider loops . . .

Now consider loops

- UV and IR divergences
- IR must cancel in invariants
- UV can cancel against counterterms

2-point, 1-loop function:



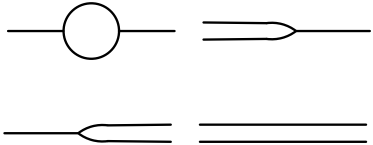
- Vertex factor w_0
- Propagator factor $\frac{1}{w_0}$
- 4 diagrams; 100s of contributions

9 (divergent) master integrals:

$$
\begin{aligned}
\int \frac{\mathrm{d}^d p \mathrm{d}^D P}{(4\pi)^{\frac{d+D}{2}}} \frac{1}{P^2+p^2} \frac{1}{(P+K)^2+(p+k)^2} \frac{1}{p^2} \frac{1}{(p+k)^2} &= \frac{1}{8\pi\epsilon k} + \frac{\alpha}{2\pi k} \\
\int \frac{\mathrm{d}^d p \mathrm{d}^D P}{(4\pi)^{\frac{d+D}{2}}} \frac{1}{(P+K)^2+(p+k)^2} \frac{1}{p^2} &= \frac{1}{8\sqrt{K^2+k^2}} \\
\int \frac{\mathrm{d}^d p \mathrm{d}^D P}{(4\pi)^{\frac{d+D}{2}}} \frac{1}{P^2} \frac{1}{(P+K)^2} \frac{1}{p^2} \frac{1}{(p+k)^2} &= -\frac{1}{K^3 k^2} \\
\int \frac{\mathrm{d}^d p \mathrm{d}^D P}{(4\pi)^{\frac{d+D}{2}}} \frac{1}{P^2+p^2} \frac{1}{(P+K)^2} &= -\frac{3\epsilon}{4K} \\
\int \frac{\mathrm{d}^d p \mathrm{d}^D P}{(4\pi)^{\frac{d+D}{2}}} \frac{1}{P^2+p^2} \frac{1}{(p+k)^2} &= \frac{1}{8\pi\epsilon k} + \frac{\alpha}{2\pi k} \\
\int \frac{\mathrm{d}^d p \mathrm{d}^D P}{(4\pi)^{\frac{d+D}{2}}} \frac{1}{P^2+p^2} \frac{1}{(P+K)^2+(p+k)^2} \frac{1}{p^2} &= \frac{K^2-k^2}{8\pi\epsilon k\left(K^2+k^2\right)^2} + \frac{k}{2\pi\left(K^2+k^2\right)^2} + \frac{\alpha\left(K^2-k^2\right)}{2\pi k\left(K^2+k^2\right)^2} - \frac{2K\tan^{-1}\left(\frac{K}{k}\right)}{\pi\left(K^2+k^2\right)^2} \\
\int \frac{\mathrm{d}^d p \mathrm{d}^D P}{(4\pi)^{\frac{d+D}{2}}} \frac{1}{P^2+p^2} \frac{1}{(P+K)^2+(p+k)^2} \frac{1}{p^2} \frac{1}{(p+k)^2} &= \frac{K^2-k^2}{4\pi\epsilon k^3\left(K^2+k^2\right)^2} + \frac{2\tan^{-1}\left(\frac{k}{K}\right)}{\pi K^3 k^2} + \frac{\alpha\left(K^2-k^2\right)}{\pi k^3\left(K^2+k^2\right)^2} + \frac{4\left(2K^2+k^2\right)\tan^{-1}\left(\frac{K}{k}\right)}{\pi K^3\left(K^2+k^2\right)^2} - \frac{1}{K^3 k^2} - \frac{K^5+2K^3k^2+2Kk^4}{\pi K^3 k^3\left(K^2+k^2\right)^2} \\
\int \frac{\mathrm{d}^d p \mathrm{d}^D P}{(4\pi)^{\frac{d+D}{2}}} \frac{1}{P^2+p^2} \frac{1}{(P+K)^2} \frac{1}{p^2} \frac{1}{(p+k)^2} &= \frac{K^2-k^2}{8\pi\epsilon k^3\left(K^2+k^2\right)^2} + \frac{\tan^{-1}\left(\frac{k}{K}\right)}{\pi K^3 k^2} + \frac{\alpha\left(K^2-k^2\right)}{2\pi k^3\left(K^2+k^2\right)^2} + \frac{2\left(2K^2+k^2\right)\tan^{-1}\left(\frac{K}{k}\right)}{\pi K^3\left(K^2+k^2\right)^2} - \frac{1}{2K^3 k^2} - \frac{K^5+2K^3k^2+2Kk^4}{2\pi K^3 k^3\left(K^2+k^2\right)^2} \\
\int \frac{\mathrm{d}^d p \mathrm{d}^D P}{(4\pi)^{\frac{d+D}{2}}} \frac{1}{P^2+p^2} \frac{1}{(P+K)^2} \frac{1}{(p+k)^2} &= \frac{K^2-k^2}{8\pi\epsilon k\left(K^2+k^2\right)^2} + \frac{k}{2\pi\left(K^2+k^2\right)^2} + \frac{\alpha\left(K^2-k^2\right)}{2\pi k\left(K^2+k^2\right)^2} - \frac{2K\tan^{-1}\left(\frac{K}{k}\right)}{\pi\left(K^2+k^2\right)^2}
\end{aligned}
$$

TABLE I. Master integrals for the 1-loop, 2-point correlator with external momentum k and euclidean energy K, dimensionally regularized with $d = 2 + 2\epsilon$, $D = 1 + 2\epsilon$, to $\mathrm{O}(\epsilon^0)$; $\alpha(k^2) = \frac{1}{2}\log\left(\frac{2e^{\gamma_E}k^2}{\pi}\right)$. The 4th integral appears with a $\frac{1}{\epsilon}$ coefficient in the correlator, and is expanded to $\mathrm{O}(\epsilon^1)$.

2-point, 1-loop function:



- Tree-level: $\frac{1}{p^2}$
- 1-loop: $\int d^{2+1}q \frac{q^6}{(q+p)^8} \sim \sqrt{p^2}$
- All counter-terms are rational functions of p^2
- $\Longrightarrow$ There are no counterterms!
- $\Longrightarrow$ the correlator must be finite!

Finite 2-point, 1-loop function:

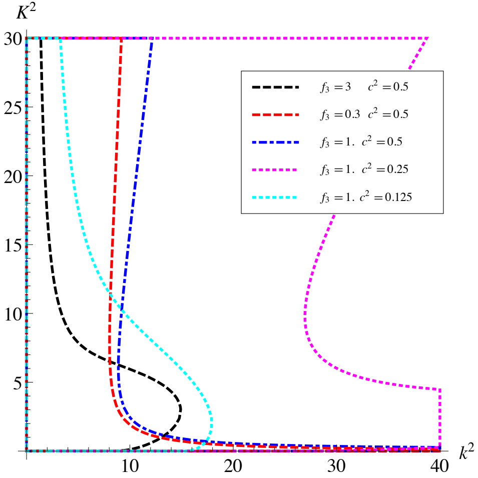
$$\frac{9Kk^6(1+c^4)}{64(K^2+k^2)^2} - \frac{k^4}{1024c^4(K^2+k^2)^{\frac{5}{2}}} \times \Big[c^4(1-c^2)^2(19k^4-4K^2k^2+K^4) - 2f_3c^2(1+c^2)k^2(5k^2+14K^2)+f_3^2(3k^4+8K^2k^2+8K^4)\Big]$$

- IR divergences cancel
- UV divergences cancel
- Does perturbation theory converge?

Does perturbation theory converge?

- a.k.a what is the cut-off?
- not Lorentz-invariant: distance vs. time scales

Ratio of 1-loop to tree amplitudes



Summary

- $\exists$ evidence that quantum fluid theory exists as an EFT
- This theory is very special: $\exists$ vortices
- If it exists, it is of interest to explore the consequences
- What are the quantum analogues of turbulence, shocks, surface waves, Kelvin waves, & c. ?
- What happens when we couple it to EM, &c?
- Nature ought to make use of it somewhere!