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Tales of the Unexpected: One-Loop Soft Theorems via Hidden Symmetries

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Outline

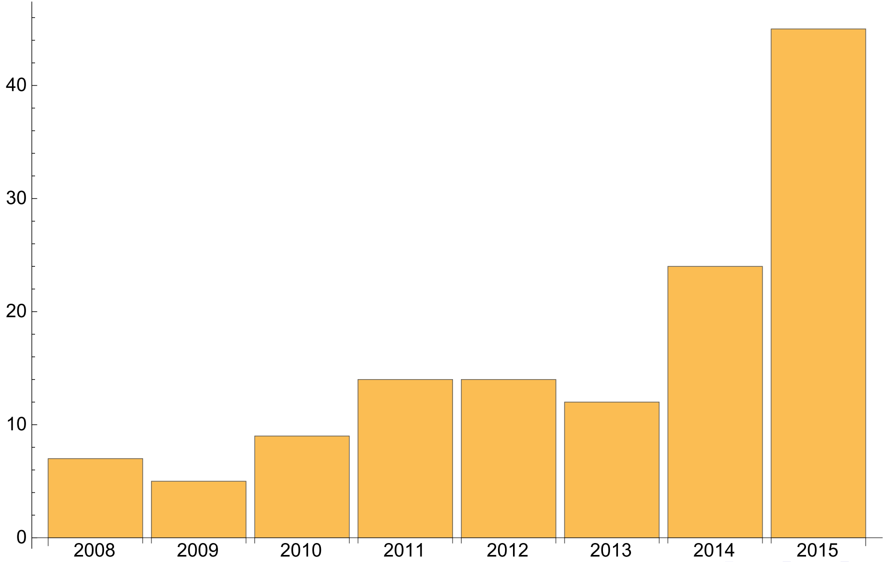
I. Motivation

II. Soft Theorems at Tree Level

III. Soft Theorems at One Loop

IV. Applications

hep-th Papers with 'Soft' in the Title



Solution to the Information Paradox?

Soft Hair on Black Holes

Stephen W. Hawking[†], Malcolm J. Perry[†] and Andrew Strominger[*]

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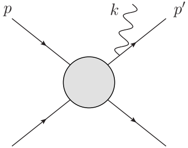
Abstract

It has recently been shown that BMS supertranslation symmetries imply an infinite number of conservation laws for all gravitational theories in asymptotically Minkowskian spacetimes. These laws require black holes to carry a large amount of soft (*i.e.* zero-energy) supertranslation hair. The presence of a Maxwell field similarly implies soft electric hair. This paper gives an explicit description of soft hair in terms of soft gravitons or photons on the black hole horizon, and shows that complete information about their quantum state is stored on a holographic plate at the future boundary of the horizon. Charge conservation is used to give an infinite number of exact relations between the evaporation products of black holes which have different soft hair but are otherwise identical. It is further argued that soft hair which is spatially localized to much less than a Planck length cannot be excited in a physically realizable process,

Origin of IR Divergences

- Massless QFTs have two types of IR divergences
- Brehmsstrahlung processes (soft, collinear)



$$\sim d\sigma_0 \frac{\alpha}{\pi} \log\left(\frac{-E_l^2}{\mu^2}\right) \log\left(\frac{-q^2}{m^2}\right)$$

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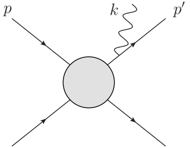
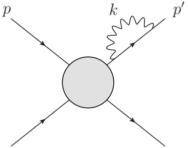


$$\sim d\sigma_0 \frac{\alpha}{\pi} \log\left(\frac{-E_l^2}{\mu^2}\right) \log\left(\frac{-q^2}{m^2}\right)$$

- Massless particles running in loops



$$\sim d\sigma_0 \left[1 - \frac{\alpha}{\pi} \log\left(\frac{-q^2}{\mu^2}\right) \log\left(\frac{-q^2}{m^2}\right)\right]$$

- Divergences cancel in S-matrix across amplitude loop orders

Soft Theorems Ancient...

- Leading tree-level soft **universal** in QED and gravity [Weinberg 1964]

$$\mathcal{A}_n^{\text{tree}} \to \frac{1}{\delta^2} S^{(0)} \mathcal{A}_{n-1}^{\text{tree}} + \dots \text{ as } p_n \to 0$$

- Leading tree-level soft universal in YM theories [Berends, Giele 1988]

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- Leading **loop corrections** universal in QCD [Bern et al. 1998]

$$\mathcal{A}_n^{\text{1-loop}} \to \frac{1}{\delta^2} \left[S^{(0)\text{tree}} \mathcal{A}_{n-1}^{\text{1-loop}} + S^{(0)\text{1-loop}} \mathcal{A}_{n-1}^{\text{tree}} \right] + \dots$$

- Leading soft not renormalised in gravity [W 1964, B+ 1998]

$$\mathcal{A}_n^{\text{1-loop}} \sim \mathcal{A}_n^{\text{tree}} \sum_{i \neq j} \left(s_{ij} \frac{\log s_{ij}}{\epsilon} \right)$$

...and Modern

- Subleading tree-level soft behaviour universal in gravity [White]
- **Ward identity** for Virasoro symmetry at null infinity [Cachazo et al.]
- Valid in arbitrary dimension via scattering equations [Schwab et al.]

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$$\mathcal{A}_n^{\text{tree}} \to \frac{1}{\delta^2} S^{(0)} \mathcal{A}_{n-1}^{\text{tree}} + \frac{1}{\delta} S^{(1)} \mathcal{A}_{n-1}^{\text{tree}} + \dots$$

- Symmetry interpretations for YM and QED [Lipstein, Strominger, Lysov, Paterski]

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- Symmetry interpretations for YM and QED [Lipstein, Strominger, Lysov, Paterski]
- Limited knowledge of **subleading behaviour at 1-loop** [Bern, Dixon, Nohle, Neill, Stewart, Larkoski, Broedel, de Leeuw, Plefka, Rosso]

The Question

Is 1-loop subleading soft behaviour universal?

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(in planar $\mathcal{N} = 4$ SYM theory)

Outline

Definition of Soft Operators

- Consider holomorphic soft scaling of a positive helicity gluon

$$|n\rangle \to \delta|n\rangle\ , \quad |n] \to |n]\ , \quad p_n \to \delta p_n$$

- In $\mathcal{N}=4$ we can write a tree-level supersoft theorem [Casali, Liu]

$$\mathcal{A}_n^{\text{tree}} = \left(\frac{1}{\delta^2}S^{(0)} + \frac{1}{\delta}S^{(1)} + \dots\right)\mathcal{A}_{n-1}^{\text{tree}}$$

- Universal soft operators are given by

$$S^{(0)} = \frac{\langle n-1\ 1\rangle}{\langle n-1\ n\rangle\langle n\ 1\rangle}$$

$$S^{(1)} = \frac{|n]}{\langle n-1\ n\rangle}\cdot\frac{\partial}{\partial|n-1]} + \frac{|n]}{\langle n\ 1\rangle}\cdot\frac{\partial}{\partial|1]}$$

Soft Theorem for Stripped Amplitudes

- Amplitudes contain an overall momentum conservation delta function

$$\mathcal{A}_n = A_n \delta^{(4)}(P_n)$$

- Can define various equivalent stripped amplitudes by integrating out

$$A_n^{\overline{(ab)}} = \int d|a]d|b]|\langle a\ b\rangle|\mathcal{A}_n$$

- Soft theorems require **consistent** solution of momentum conservation [Bern, Nohle, Davies]

$$A_n^{\text{tree}\overline{(ab)}} = \left[\left(\frac{1}{\delta^2}S^{(0)} + \frac{1}{\delta}S^{(1)}\right) A_{n-1}^{\text{tree}}\right]^{\overline{(ab)}}$$

Soft Operators from Conformal Symmetry

- $\mathcal{N} = 4$ has a conformal symmetry with boost generator

$$k_{\alpha\dot{\alpha}} A_n = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial |i\rangle^\alpha [i|^{\dot{\alpha}}} + \frac{1}{\delta} \frac{\partial^2}{\partial |n\rangle^\alpha \partial [n|^{\dot{\alpha}}} A_n = 0$$

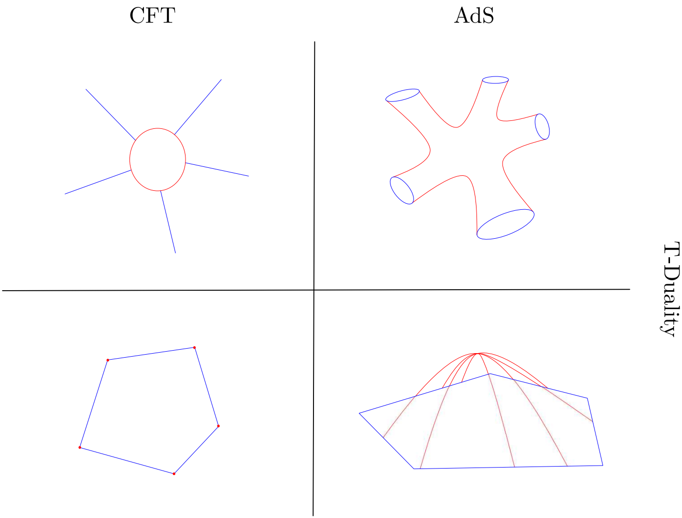
- Substituting the amplitude soft expansion gives constraint equations

$$\frac{\partial^2}{\partial |n\rangle^\alpha \partial [n|^{\dot{\alpha}}} \left(S^{(0)} A_{n-1}^{\text{tree}} \right) = 0$$

$$\sum_{i=1}^{n-1} \frac{\partial^2}{\partial |i\rangle^\alpha [i|^{\dot{\alpha}}} \left(S^{(0)} A_{n-1}^{\text{tree}} \right) + \frac{\partial^2}{\partial |n\rangle^\alpha \partial [n|^{\dot{\alpha}}} \left(S^{(1)} A_{n-1}^{\text{tree}} \right) = 0$$

- which **fix** soft operators $S^{(0)}$ and $S^{(1)}$ [Larkoski]
- Difficult to generalize to 1-loop since anomaly unknown

Amplitude/Wilson Loop Duality



Dual Superconformal Symmetry

- Wilson loop has a conformal symmetry...
- ...which is a new **hidden** symmetry of the amplitude!
- Define dual coordinates

$$(x_i - x_{i+1})_{\alpha\dot{\alpha}} = (p_i)_{\alpha\dot{\alpha}} \quad \text{and} \quad (\theta_i - \theta_{i+1})_\alpha^A = \langle i|_\alpha \eta_i^A$$

- The dual conformal boost generator is

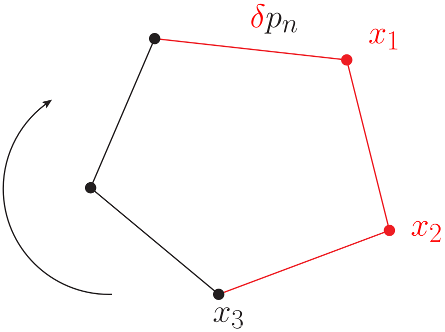
$$K_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \left(x_{i\dot{\alpha}}^{\beta} \langle i|_\alpha \frac{\partial}{\partial |i\rangle^\beta} + x_{i+1\alpha\dot{\beta}} |i]_{\dot{\alpha}} \frac{\partial}{\partial |i]_{\dot{\beta}}} + \theta_{i+1\alpha}^A |i]_{\dot{\alpha}} \frac{\partial}{\partial \eta_i^A} \right)$$

- Tree-level superamplitudes are covariant

$$K_{\alpha\dot{\alpha}} \mathcal{A}_n^{\text{tree}} = -\left(\sum_{i=1}^{n} x_{i\alpha\dot{\alpha}} \right) \mathcal{A}_n^{\text{tree}}$$

Soft Dependence in Dual Space

- We must determine x_i as a function of p_j
- **Ambiguities**! Which base point? Which way round polygon?
- Related to ambiguity in defining stripped amplitude
- Choose minimal δ dependence compatible with eliminating $|1]$ and $|2]$



Constraint Equations at Tree Level

- Leading soft

$$\left(K_{\alpha\dot{\alpha}}\right)_{\mathcal{O}(\delta^0)} S^{(0)} = \left(\sum_{j=3}^{n-1} p_j\right) S^{(0)}$$

- Subleading soft

$$-2\frac{|n]\langle 1|}{\langle n\ 1\rangle} + S^{(0)} \left(K_{\alpha\dot{\alpha}}\right)_{\mathcal{O}(\delta^1)} + \left[\left(K_{\alpha\dot{\alpha}}\right)_{\mathcal{O}(\delta^0)}, S^{(1)}\right] = \left(\sum_{j=3}^{n-1} p_j\right) S^{(1)}$$

- Suffice to **fix** soft operators $S^{(0)}$ and $S^{(1)}$

Outline

Definition of Subleading Soft Anomaly

- One-loop amplitudes have IR divergences; we need regulator ϵ
- Ansatz for soft limit

$$\mathcal{A}_n^{\text{1-loop}} = \frac{1}{\delta^2}\left(S^{(0)\text{tree}}\mathcal{A}_{n-1}^{\text{1-loop}} + S^{(0)\text{1-loop}}\mathcal{A}_{n-1}^{\text{tree}}\right) + \frac{1}{\delta}\left(S^{(1)\text{tree}}A_{n-1}^{\text{1-loop}} + S^{(1)\text{1-loop}}\mathcal{A}_{n-1}^{\text{tree}}\right) + \dots$$

- Define the subleading soft anomaly Z by

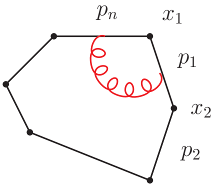
$$S^{(1)\text{1-loop}} = F^{(0)}S^{(1)\text{tree}} + \frac{1}{\epsilon}Z_{-1} + Z_0 + \mathcal{O}(\epsilon)$$

- IR divergent piece Z_{-1} known and universal [Bern, Nohle, Davies]

The Question

Is Z_0 universal?

IR Divergences and Dual Conformal Anomaly



- 1-loop IR divergences in amplitude correspond to...
- ...**cusp** divergences in Wilson loop
- This suggests **simple** dual conformal anomaly

$$K_{\alpha\dot{\alpha}}\mathcal{A}_n^{\text{1-loop}} = \frac{2}{\epsilon}c_\Gamma\left(\sum_{i=1}^{n} x_{i\alpha\dot{\alpha}}\left(-[i\ i-1]\langle i-1\ i\rangle\right)^{-\epsilon}\right)\mathcal{A}_n^{\text{tree}} - \left(\sum_{i=1}^{n} x_{i\alpha\dot{\alpha}}\right)\mathcal{A}_n^{\text{1-loop}}$$

- Proved via unitarity [Brandhuber, Spence, Travaglini]
- Use this to derive constraint equations for soft theorems at 1-loop

Soft Anomaly from Dual Conformal Anomaly

- Constraint equation for Z_0

$$\Big[(\mathcal{K}_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^0)} - \sum_{j=3}^{n-1} |j]\langle j|\Big] Z_0$$
$$= 2\Big[Z_{-1}\sum_{j=3}^{n-1}|j]\langle j| - |n-1]\langle n-1|\frac{[n-2\ n]}{\langle n-1\ n\rangle[n-1\ n-2]} + |1]\langle 1|\frac{[2\ n]}{[2\ 1]\langle 1\ n\rangle}\Big]$$
$$+ \left[\frac{|n]\langle 1|}{\langle n\ 1\rangle} + 2\frac{|n]\langle n-1|}{\langle n-1\ n\rangle} - |1]\langle n|\frac{\langle n-1\ 1\rangle[n\ n-1]}{\langle n-1\ n\rangle\langle n\ 1\rangle[1\ n-1]}\right]\log\left(-\frac{(n-1\ 1)}{(n-1\ n)(n\ 1)}\right)$$
$$-2\frac{|n]\langle n-1|}{\langle n-1\ n\rangle}\log(-(n-1\ 1)) + 2|n]\langle n|\frac{\langle n-1\ 1\rangle}{\langle n-1\ n\rangle\langle n\ 1\rangle}\log(-(n\ 1))$$

- Suggests the subleading $\log\delta$ terms

$$Z_0|_{\log\delta} = \left(\frac{(n\ 1)}{(n-1\ 1)} + \frac{(n-2\ n)}{(n-2\ n-1)} - \frac{(n-2\ 1)(n-1\ n)}{(n-2\ n-1)(n-1\ 1)}\right) S^{(0)}\log(-(n-1\ n))$$

- Must verify this with explicit calculations

MHV Amplitudes at One Loop

- Box-function form [Bern et al. 1994]

$$\mathcal{A}_n^{\text{MHV,1-loop}} = \mathcal{A}_n^{\text{MHV,tree}} \left(\sum_{\text{channels}} F^{\text{2me}} + \text{cyclic} \right)$$

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- Dual-conformal-adapted form

$$\frac{\mathcal{A}_n^{\mathrm{1\text{-}loop}}}{\mathcal{A}_n^{\mathrm{tree}}} = \frac{1}{2} \sum_i \sum_{j \notin \{i-2, i-1, i, i+1, i+2\}} \left(-\mathrm{Li}_2(1 - u_{ij}) + \log x_{ij}^2 \log u_{ij} \right)$$

$$+ \sum_i \log(x_{ii-2}^2) \log \left(\frac{x_{i+1i-2}^2}{x_{i+1i-1}^2 \sqrt{x_{ii-2}^2}} \right),$$

- Convenient to use momentum twistor variables

Complete Soft Anomaly in MHV Sector

- Compact formula involving bulk terms

$$\frac{\langle n-1\ 1\rangle}{\langle n-1\ n\rangle}\sum_{j=4}^{n-4}\log\left(\frac{y_{n-1j}^2}{y_{1j}^2}\right)\frac{\langle n-2\ n-1\ j-1\ j\rangle\langle n-2\ n-1\ n\ 1\rangle}{\langle n-2\ n-1\ 1\ j-1\rangle\langle n-2\ n-1\ 1\ j\rangle}$$

- **Universal** boundary terms, including the $\log\delta$ piece

$$\left(\frac{(n\ 1)+(n\ 2)}{(1\ 2)}-\frac{s_{n-1,1,2}(n\ 1)}{(n-1\ 1)(1\ 2)}\right)\log(-(n\ 1))$$

- Feed-down contributions, including a $\log\delta$ piece
- **Agreement** with prediction from dual conformal anomaly!

Low-Point Verification

- Numerically verified against soft behaviour of box-function form
- Perfect agreement at $5, 6, 7$-points
- Anomaly from box functions is hard to simplify analytically

# points	# terms Z_0^{dual}	# terms Z_0^{boxes}
5	6	22
6	20	205
7	38	1233

NMHV Amplitudes

- Tree-level NMHV amplitudes equal MHV $\times$ ratio functions

$$\mathcal{A}_n^{\text{NMHV,tree}} = \mathcal{A}_n^{\text{MHV,tree}} \sum_{j,k} R_{1jk}$$

- Ratio functions are dual superconformal invariant [Drummond et al.]
- Each R_{1jk} satisfies the tree-level soft theorem [Bianchi et al.]

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- Ratio functions are dual superconformal invariant [Drummond et al.]
- Each R_{1jk} satisfies the tree-level soft theorem [Bianchi et al.]
- Dual superconformal invariant V functions enter at 1-loop

$$\mathcal{A}_n^{\text{NMHV,1-loop}} = \mathcal{A}_n^{\text{MHV,tree}} \sum_{i,j,k} R_{ijk}^{\text{tree}} V_{ijk}^{\text{1-loop}} + \mathcal{A}_n^{\text{MHV,1-loop}} \sum_{j,k} R_{1jk}^{\text{tree}}$$

$\log\delta$ Anomaly in NMHV Sector

- Surprising and non-trivial **cancellations**
- 6-point anomaly has non-trivial $\log\delta$ piece

$$\frac{1}{2}\frac{\langle n-1\ 1\rangle}{\langle n-1\ n\rangle}R_{524}\frac{\langle 2\ 3\ 4\ 5\rangle\langle 4\ 5\ 6\ 1\rangle}{\langle 1\ 2\ 4\ 5\rangle\langle 3\ 4\ 5\ 1\rangle}\log(-(6\ 1))$$

- 7-point anomaly has non-trivial $\log\delta$ piece

$$\frac{1}{2}\frac{\langle n-1\ 1\rangle}{\langle n-1\ n\rangle}\left[R_{624}\frac{\langle 2\ 3\ 4\ 6\rangle\langle 5\ 6\ 7\ 1\rangle}{\langle 3\ 4\ 6\ 1\rangle\langle 1\ 2\ 5\ 6\rangle}+R_{625}\frac{\langle 2\ 4\ 5\ 6\rangle\langle 5\ 6\ 7\ 1\rangle}{\langle 4\ 5\ 6\ 1\rangle\langle 1\ 2\ 5\ 6\rangle}\right]\log(-(7\ 1))$$

- Evidence for **universality** within NMHV sector
- But no agreement with prediction from dual conformal anomaly

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Within but not between helicity sectors!

(we conjecture)

Outline

A Cornucopia of Applications

- Better resummation in QCD [Bonocore et al.]
- Improved soft-collinear bootstrap [Bourjaily et al.]
- Flat space holography [Adamo, Casali, Skinner, Lipstein, . . .]
- Theory construction [Cheung et al.]
- Black hole entropy [Hawking, Perry, Strominger]

Future Work

- More sectors, higher loops!
- Soft anomaly from one-loop scattering equations [Geyer et al.]
- Relation to soft-collinear effective theory [Larkoski et al.]
- Subleading soft friendly amplitudes [Bern, Dixon, Smirnov]

- Symmetry intepretation of quantum corrections [Strominger et al.]
- Subleading collinear theorems [Stieberger, Kosower]
- Subleading soft behaviour of form factors [Brandhuber et al.]
- Color-kinematics duality $\rightarrow$ gravity [Oxburgh et al.]

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