Engineering F-Theory GUTs
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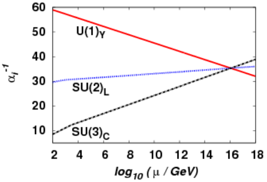
Motivation

- How can we hope to say anything about particle physics given the complexity of string vacua?
- Apparent unification of couplings in MSSM at $M_{GUT} \sim 10^{16}$ GeV



$$\rightarrow M_{SUSY} \ll M_{GUT} \ll M_{Planck}$$

$\Rightarrow$ particle physics captured by SUSY gauge theory that isn't sensitive to details of quantum gravity

Decoupling

- Decoupling the dynamics gauge theory from gravity provides an attractive way to constrain the problem of the complexity of the string vacua.
- It is important to note that the existence of a GUT is in principle compatible with the existence of a decoupling limit.

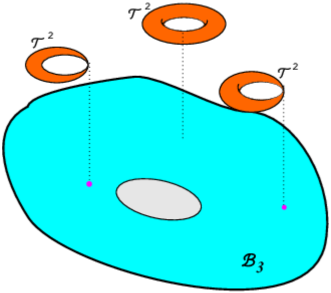
The Idea

If we were to compactify type II-B theory on some manifold $\mathcal{B}$ then this could be expressed in terms of F-theory compactified on a larger manifold $\mathcal{A}$ with specific properties that $\mathcal{A}$ has the structure of a fibre bundle and is an elliptic fibration of the manifold $\mathcal{B}$.

$$\pi : CY_{n+1} \rightarrow B_n$$

More precisely

$$\pi : CY_4 \rightarrow B_3$$



Type IIB has an exact strong-weak coupling duality, taking

$$g_s \to \frac{1}{g_s}$$

which in terms of the axio-dilaton

$$\tau = C_0 + ie^{-\phi}$$

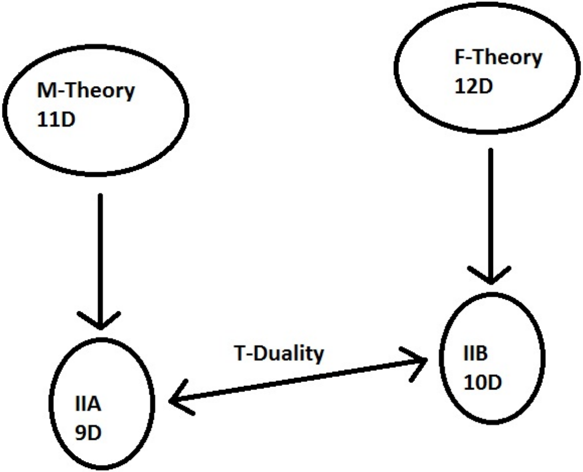
is

$$\tau \to -\frac{1}{\tau}.$$

The low energy effective action of Type IIB has a $SL(2, \mathbb{Z})$ symmetry.

The M-Theory Duality

One of the most convenient ways to think about F-theory is via the duality with M-theory.



The Inhomogeneous Weierstrass Form

Inhomogeneous ($z = 1$) Weierstrass form for elliptic $K3$

$$P_W = y^2 - x^3 - fx - g = 0$$

where x, y, and z are parameters of the fibration and $f, g \in \mathbb{C}$ specify the shape of the elliptic curve.

A general mathematical fact is that a hypersurface described by the equation $P_W = 0$ becomes singular whenever

$$P_W = 0, \qquad \mathrm{d}P_W = 0.$$

The structure of the roots of this cubic polynomial is encoded in the discriminant Δ.

$\Delta(\tau)$: The Discriminant

$$\Delta(\tau) \equiv (2\pi)^{12}\eta(\tau)^{24}$$

where

$$\eta(\tau) \equiv e^{\frac{i\pi\tau}{12}} \prod_{n>1} \left(1 - e^{2\pi i n\tau}\right).$$

$\Delta(\tau)$ is known as the cusp form of weight 12

$$\Delta\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{12}\Delta(\tau)$$

Note. The presence of 24 can be connected to the Leech lattice, which has 24 dimensions.

Why A Singularity Requires $\Delta = 0$?

$$\Delta = 27g^2 + 4f^3$$

where f and g are homogeneous polynomials.
The derivative with respect to y only vanishes at $y = 0$. So, we are only interested in values of x which satisfy both

$$x^3 + fx + g = 0, \qquad 3x^2 + f = 0$$

$$\Rightarrow 4f^3 + 27g^2 = \Delta = 0$$

Engineering GUTs

We can now engineer GUT models in two steps:

- We have to identify the GUT divisor S given by the equation $z = 0$ in B within CY_4.
- We need to impose the GUT group. This amounts to explicitly imposing the factorization conditions on the Tate model meaning that we have to remove all those monomials which do not satisfy the factorization constraints.

Tate's Algorithm: A Generalization

$\deg(f)$	$\deg(g)$	$\deg(\Delta)$	Singularity Type
0	0	n	A_{n-1}
2	≥ 3	$n+6$	D_{n+4}
≥ 4	5	10	E_8

Figure: Excerpt from the classification of ADE singularities due to Kodaira.

The general inhomogeneous $(z = 1)$ Tate form

$$y^2 = x^3 - a_1xy + a_2x^2 - a_3y + a_4x + a_6$$

Recovering The Weierstrass Form

Define

$$\begin{aligned}
\beta_2 &= a_1^2 + 4a_2 \\
\beta_4 &= a_1 a_3 + 2a_4 \\
\beta_6 &= a_3^2 + 4a_6 \\
\beta_8 &= \frac{\beta_2\beta_6 - \beta_4^2}{4} = \beta_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2
\end{aligned}$$

where

$$\begin{aligned}
f &= -\frac{1}{48}(\beta_2^2 - 24\beta_4) \\
g &= -\frac{1}{864}(-\beta_2^3 + 36\beta_2\beta_4 - 216\beta_6)
\end{aligned}$$

such that

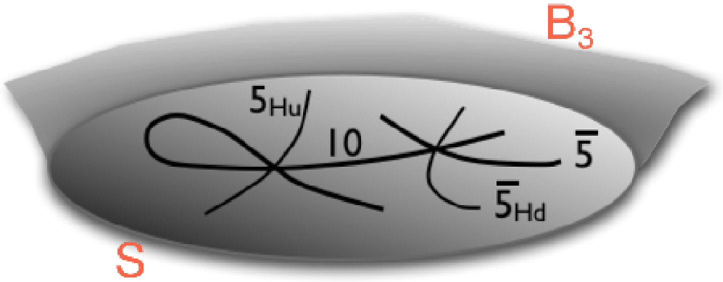
$$\Delta = -\frac{1}{4}\beta_2^2(\beta_2\beta_6 - \beta_4^2) - 8\beta_4^3 - 27\beta_6^2 + 9\beta_2\beta_4\beta_6$$

Model Building Elements

Each lower-dimensional subspace provides an important model building element

Dimension	Ingredient	Complex Codimension	Enhancements
$8D$	Gauge Theory	1	-
$6D$	Matter	2	Rank 1
$4D$	Yukawa Couplings	3	Rank 2

The $SU(5)$ GUT: Pictorially



The six extra dimensions are compactified on B_3 whereas the $SU(5)$ degrees of freedom are localized on the submanifold S_{GUT}. The gauge bosons live on the bulk of S_{GUT} but the chiral multiplets are localized on complex matter curves. At the intersection of two matter curves with a Higgs curve a Yukawa coupling develops.

The $SU(5)$ Example

General inhomogeneous Tate form

$$y^2 = x^3 - a_1 xy + a_2 x^2 - a_3 y + a_4 x + a_6$$

where along the divisor

$$S : z = 0$$

$$\begin{aligned} a_1 &= -b_5 \\ a_2 &= b_4 z \\ a_3 &= -b_3 z^2 \\ a_4 &= b_2 z^3 \\ a_6 &= b_1 z^5 \end{aligned}$$

$$\Rightarrow y^2 = x^3 + b_5 xy + b_4 x^2 z + b_3 yz^2 + b_2 xz^3 + b_1 z^5$$

The $SU(5)$ Example Contd.

The vanishing order of $a_i = b_i z^{n_i}$ characterizes the type of singularity.

$$\begin{aligned} a_1 &\Rightarrow 0 \\ a_2 &\Rightarrow 1 \\ a_3 &\Rightarrow n = 2 \\ a_4 &\Rightarrow n + 1 = 3 \Rightarrow n = 2 \\ a_6 &\Rightarrow 2n + 1 = 5 \Rightarrow n = 2 \end{aligned}$$

The Discriminant - $\deg(\Delta) = 5$

$$\Delta \quad = \quad -z^5\left[P_{10}^4 P_5 + zP_{10}^2(8b_4P_5 + P_{10}R) + \mathcal{O}(z^2)\right]$$

$$P_{10} = b_5$$

$$P_5 = b_3^2 b_4 - b_2 b_3 b_5 + b_1 b_5^2$$

$$R = 4b_1 b_4 b_5 - b_3^3 - b_2^2 b_5$$

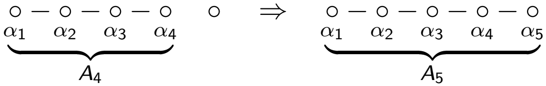
The Matter Curves

- $SU(5)$:

$$P_{10} : b_5 = 0 \Rightarrow \Delta = -z^7(16 b_3^2 b_4^2) \Rightarrow \deg(\Delta) = 7$$

$$P_5 : b_3^2 b_4 - b_2 b_3 b_5 + b_1 b_5^2 = 0 \Rightarrow \Delta = -z^6(b_5^3 R) \Rightarrow \deg(\Delta) = 6$$

Viewing The Enhancements



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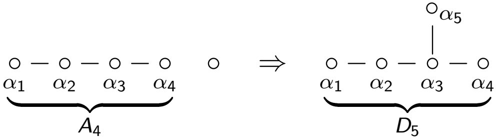


The Discriminant

$$\Delta = -z^5\left[P_{10}^4 P_5 + zP_{10}^2(8b_4P_5 + P_{10}R) + z^2(16b_3^2b_4^2 + P_{10}Q) + \mathcal{O}(z^3)\right]$$

$$P_{10} = b_5$$

$$P_5 = b_3^2 b_4 - b_2 b_3 b_5 + b_1 b_5^2$$

$$R = 4b_1 b_4 b_5 - b_3^3 - b_2^2 b_5$$

The Yukawa Points - $\deg(\Delta) = 8$

$$P_{10} = b_5 = 0 \Rightarrow \Delta = -z^7(16 b_3^2 b_4^2) \Rightarrow b_3 = 0 \text{ or } b_4 = 0$$

Either

$$b_5 = 0 = b_3$$

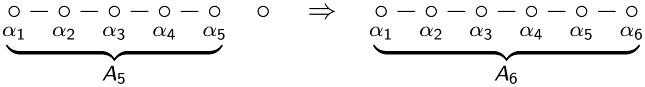
or

$$b_5 = 0 = b_4$$

Note. No contribution from

$$P = 0 \ \cap \ b_5 = 0 \Rightarrow \Delta = 0$$

Viewing The Enhancements



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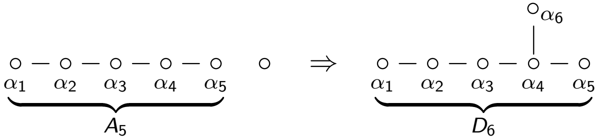

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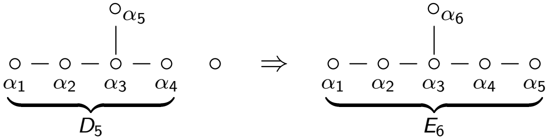

The Gauge Enhancements

	$\deg(\Delta)$	Type	Gauge Group	Object Equation
GUT	5	A_4	$SU(5)$	$S : z = 0$
Matter Curve	6	A_5	$SU(6)$	$P_5 : P = 0$
Matter Curve	7	D_5	$SO(10)$	$P_{10} : b_5 = 0$
Yukawa Point	8	D_6	$SO(12)$	$b_3 = b_5 = 0$
Yukawa Point	8	E_6	E_6	$b_4 = b_5 = 0$
Extra	7	A_6	$SU(7)$	$P_5 = R = 0$, $(b_4, b_5) \neq (0, 0)$

The $SO(10)$ Example

General inhomogeneous Tate form

$$y^2 = x^3 - a_1 xy + a_2 x^2 - a_3 y + a_4 x + a_6$$

where along the divisor

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$$\begin{aligned} a_1 &= -b_5 z \\ a_2 &= b_4 z \\ a_3 &= -b_3 z^2 \\ a_4 &= b_2 z^3 \\ a_6 &= b_1 z^5 \end{aligned}$$

$$\Rightarrow y^2 = x^3 + b_5 xyz + b_4 x^2 z + b_3 yz^2 + b_2 xz^3 + b_0 z^5$$

The $SO(10)$ Example Contd.

The vanishing order of $a_i = b_i z^{n_i}$ characterizes the type of singularity.

$$
\begin{aligned}
a_1 &\Rightarrow 1 \\
a_2 &\Rightarrow 1 \\
a_3 &\Rightarrow n = 2 \\
a_4 &\Rightarrow n + 1 = 3 \Rightarrow n = 2 \\
a_6 &\Rightarrow 2n + 1 = 5 \Rightarrow n = 2
\end{aligned}
$$

The Discriminant - deg(Δ) = 7

$$
\begin{aligned}
\Delta &= -16 b_2^3 b_3^2 z^7 + \big(-27 b_3^4 - 8 b_1^2 b_2^2 b_3^2 + 72 b_2 b_4 b_3^2 \\
&\qquad + 4 b_1 b_2 (9 b_3^2 + 4 b_2 b_4) b_3 + 16 b_2^2 (b_4^2 - 4 b_2 b_6) \big) z^8 \\
&\qquad + \mathcal{O}(z^9) \\
&= z^7 \big[-16 b_2^3 b_3^2 + \big(-27 b_3^4 - 8 b_1^2 b_2^2 b_3^2 + 72 b_2 b_4 b_3^2 \\
&\qquad + 4 b_1 b_2 (9 b_3^2 + 4 b_2 b_4) b_3 + 16 b_2^2 (b_4^2 - 4 b_2 b_6) \big) z \\
&\qquad + \mathcal{O}(z^2) \big]
\end{aligned}
$$

- $SO(10)$:

$$P_{16} : b_2 = 0 \Rightarrow \Delta = -27 b_3^4 z^8$$

$$P_{10} : b_3 = 0 \Rightarrow \Delta = 16 b_2^2 (b_4^2 - 4 b_2 b_6) z^8$$

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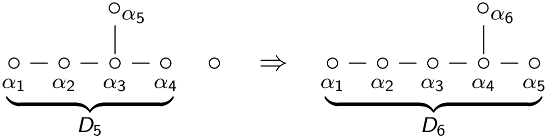

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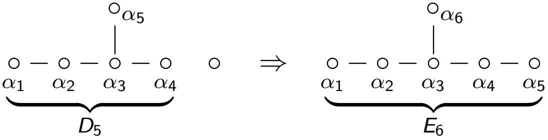

The Discriminant

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&\qquad + 4 b_1 b_2 (9 b_3^2 + 4 b_2 b_4) b_3 + 16 b_2^2 (b_4^2 - 4 b_2 b_6) \Big) z \\
&\qquad + \mathcal{O}(z^2) \Big]
\end{aligned}
$$

The Yukawa Points - $\deg(\Delta) = 9$

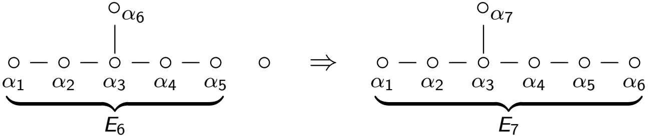
- Either $b_2 = 0 \ \cap \ b_3 = 0$

$$\Rightarrow \Delta = 0$$

- Or

$$b_3 = 0 \Rightarrow \Delta = 16 b_2^2 (b_4^2 - 4 b_2 b_6) z^8 \Rightarrow b_4^2 - 4 b_2 b_6 = 0$$

Viewing The Enhancements



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Matter Curve	8	D_6	$SO(12)$	$P_{10} : b_3 = 0$
Matter Curve	8	E_6	E_6	$P_{16} : b_2 = 0$
Yukawa Points	9	E_7	E_7	$b_2 = b_3 = 0$ $b_3 = b_4^2 - 4b_2 b_6 = 0$

GUT Breaking Methods 1

- Adjoint matter - undesirable phenomenology ✗
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- Discrete Wilson lines on 7-brane world-volume. But discrete symmetries have fixed points leading to singularities on 7-brane world-volume ✗
- GUT breaking by $U(1)$ fluxes ✓

GUT Breaking Methods 2

The preferred GUT breaking method is turning on a non-trivial gauge flux which will break the gauge group.

The use of non-trivial flux as the preferred GUT breaking method is something specific to F-theory.

The Hypercharge Flux

The hypercharge flux is an important ingredient which provides an elegant mechanism for breaking the GUT group.

An Example

Consider the $SU(5)$ GUT.

- The flux leaves the $SU(5)$ symmetry unbroken.

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- The flux leaves the $SU(5)$ symmetry unbroken.
- To break the GUT symmetry by switching on flux along the GUT surface. In order to break $SU(5)$ down to the SM gauge group, this flux is chosen to be proportional to the $U(1)_Y$ within the $SU(5)$.

Work In Progress

- $SO(10)_{GUT} \rightarrow SU(3) \times SU(2)_L \times SU(2)_R \times U(1)$

To Summarize...

- Unification suggests particle physics embedded in non-perturbative regime of type II-B
- Type II-B has non-perturbative $SL(2, \mathbb{Z})$ symmetry rotating the axio-dilaton

$$\tau = C_0 + ie^{-\phi}$$

  $\Rightarrow$ interpret as complex structure of a two-torus.
- Structure of 8D gauge theory (F-theory models) is very rigid
- F-theory on torus fibred:
  $\Rightarrow$ gauge groups from 8D
  $\Rightarrow$ charged matter from 6D
  $\Rightarrow$ Yukawa-type couplings from 4D

THANK YOU!