ESTIMATION FOR 2 DIMENSIONAL FREE FERMION NEGATIVITY

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Entanglement Entropy

- Definition: System with Hilbert Space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- Pure State: $|\Psi\rangle$
- $\rho = |\Psi\rangle\langle\Psi|$ $\qquad$ $\rho_A = Tr_B \rho$
- Renyi Entropy:

$$S_A^{(n)} = \frac{1}{1-n} \log Tr \rho_A^n$$

- Entanglement Entropy:

$$S_A = \lim_{n \longrightarrow 1} S_A^{(n)} \qquad S_A = -Tr \rho_A \log \rho_A$$

Negativity

- Definitions: Hilbert Space:

$$\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_B$$

- Partial Transpose:

$$\langle e_i^{(1)} e_j^{(2)} | O^{T_2} | e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} | O | e_k^{(1)} e_j^{(2)} \rangle$$

- Logarithmic negativity

$$\mathcal{E} = \log \left\| \rho_A^{T_2} \right\|_1$$

- Trace Norm: sum of singular values $\|M\|_1 = \sum_{i=1}^{\dim \mathrm{M}} \sigma_i$
- Schatten p-Norm
- $\|M\|_p = \left(\sum_{i=1}^{\dim \mathrm{M}} \sigma_i^p \right)^{1/p}$ σ_i : Eigenvalues of $\left(M^\dagger M\right)^{1/2}$

$$\|M\|_1 = tr\left(\left(M^\dagger M\right)^{1/2}\right) = tr\left(\left(M M^\dagger\right)^{1/2}\right)$$

- $\rho_A^{T_2}$ is Hermitian:

$$\|M\|_1 = \sum_{i=1}^{\dim \mathrm{M}} |\lambda_i|$$

$$\left\|\rho_A^{T_2}\right\|_1 = tr\left|\rho_A^{T_2}\right| = tr\left(\left(\left(\rho_A^{T_2}\right)^2\right)^{1/2}\right)$$

$$\mathcal{E} = \lim_{n_e \to 1} \log Tr\left(\rho^{T_2}\right)^{n_e}$$

- Proof of Negativity is an Entanglement Monotone:

M. B. Plenio, Logarithmic Negativity: A Full Entanglement Monotone That is not Convex," Phys. Rev. Lett. 95 , 090503 (2005)

$$\mathcal{E}\left(\rho\right) \geq p_i \mathcal{E}\left(\rho_i\right)$$

- Negativity of Conformal field theory:

P. Calabrese, J. Cardy and E. Tonni, Entanglement negativity in extended systems: A field theoretical approach," J. Stat. Mech. 1302 , P02008 (2013) [arXiv:1210.5359 [cond-mat.statmech]].

- Single interval

$$Tr\left(\rho_A^{T_2}\right)^{n_o} = c_{n_o}\left(\frac{l}{a}\right)^{-c/6(n_o-1/n_o)} \qquad Tr\left(\rho_A^{T_2}\right)^{n_e} = c^2_{n_{e/2}}\left(\frac{l}{a}\right)^{-c/3(n_e/2-2/n_e)}$$

- Two disjoint interval:

$$Tr\left(\rho_A^{T_2}\right)^{n} = c_n^2\left(\frac{(u_2-u_1)(v_2-v_1)}{(v_1-u_1)(v_2-u_2)(v_2-u_1)(u_2-v_1)}\right)^{c/6(n-1/n)}\mathcal{F}_n(x) \qquad x=\frac{(v_1-u_1)(v_2-u_2)}{(u_2-u_1)(v_2-v_1)}$$

- Free Fermion on Lattice $\longrightarrow$ Spin Chain Nearest Neighbor Hopping

$$\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \rightarrow \begin{pmatrix} \psi_n^{(L)} \\ \psi_n^{(R)} \end{pmatrix}$$

$$H \rightarrow -\frac{i}{2}\sum_{i=1}^{N}\left(\psi_i^{(L)\dagger}\psi_{i+1}^{(L)} - \psi_{i+1}^{(L)\dagger}\psi_i^{(l)} - \psi_i^{(R)\dagger}\psi_{i+1}^{(R)} + \psi_{i+1}^{(R)\dagger}\psi_i^{(R)}\right)$$

- Reduced Density Matrix of Ground State

$$\rho = |0\rangle\langle 0| \qquad\qquad \rho_A = tr_B\,|0\,\rangle\langle\,0|$$

- ρ_A is Gaussian

$$Tr\left(\rho_A \prod \psi_i^\dagger \prod \psi_j\right) = Tr\left(\rho \prod \psi_i^\dagger \prod \psi_j\right) = \langle 0\left|\prod \psi_i^\dagger \prod \psi_j\right|0\rangle$$

- Trace of Gaussian Matrices:
- Eigenvalue of Γ :

$$\mu_i = \frac{e^{-\lambda_i}}{(1+e^{-\lambda_j})}$$

- Trace of $(\rho_A)^n$:

$$tr\,(\rho_A)^n = \sum (1-\mu_i)^n + \sum \mu_i^n$$

- $(\rho_A)^{T_2}$ is not Gaussian For Free Fermion

- Example: 2-Spin System

$$\rho = \frac{e^{-M_{ij}c_i^\dagger c_j}}{Tr\left(e^{-M_{ij}c_i^\dagger c_j}\right)} \quad i,j=1,2$$

$$\rho^{T_2} = \frac{1}{1+e^{-trM}+tr\left(e^{-M}\right)} \begin{pmatrix} 1 & 0 & 0 & \left(e^{-M}\right)_{21} \\ 0 & \left(e^{-M}\right)_{11} & 0 & 0 \\ 0 & 0 & \left(e^{-M}\right)_{22} & 0 \\ \left(e^{-M}\right)_{12} & 0 & 0 & e^{-trM} \end{pmatrix}$$

$$\left(\rho^{T_2}\right)_{00}\left(\rho^{T_2}\right)_{33} + \left(\rho^{T_2}\right)_{03}\left(\rho^{T_2}\right)_{30} = \left(\rho^{T_2}\right)_{11}\left(\rho^{T_2}\right)_{22}$$

- Whereas for a Gaussian Operator G

$$(G)_{00}(G)_{33} - (G)_{03}(G)_{30} = (G)_{11}(G)_{22}$$

- Definition and Expansion of $O_\pm$:
- Covariance matrix:

$$\Gamma^{\pm} = \left(\begin{array}{cc} \Gamma^{11} & \pm i\Gamma^{12} \\ \pm i\Gamma^{21} & -\Gamma^{22} \end{array} \right)$$

$$O_{\pm} = \sum_{\boldsymbol{\kappa},\boldsymbol{\tau}} \omega_{\boldsymbol{\kappa},\boldsymbol{\tau}} \left(a_{m_1}^{\kappa_1} ... a_{m_{2k}}^{\kappa_{2k}}\right) \left(\left(\pm i a_{n_1}^{\tau_1}\right) ... \left(\pm i a_{n_{2l}}^{\tau_{2l}}\right)\right)$$

- $\rho_A^{T_2}$ As Sum of Gaussian operators:

$$\frac{1}{2}\left(O_+ + O_-\right) = \left(\rho_A^{T_2}\right)_{even}$$

$$\pm\frac{i}{2}\left(O_+ - O_-\right) = \left(\rho_A^{T_2}\right)_{odd}$$

$$\rho_A^{T_2} = \frac{1 \pm i}{2} O_+ + \frac{1 \mp i}{2} O_-$$

V. Eisler and Z. Zimboras, “On the partial transpose of fermionic Gaussian states," New J.Phys. 16 ,123020 (2014).

- Negativity:

$$\frac{1}{2}tr\left(\left(i\left(O_+^2-O_-^2\right)+\{O_+,O_-\}\right)^{1/2}\right)$$

- Moments of Negativity:

$$tr\left(\rho_A^{T_2}\right)^N=2\Re\left(\left(\frac{1-i}{2}\right)^N\right)tr\left(O_+^N\right)+2\Re\left(\left(\frac{1-i}{2}\right)^{N-1}\left(\frac{1+i}{2}\right)\right)tr\left(O_+^{N-1}O_-\right)+\ldots$$

- Examples:
- N=2:

$$tr\left(\rho_A^{T_2}\right)^2=tr\left(O_+O_-\right)$$

- N=4:

$$tr\left(\rho_A^{T_2}\right)^4=-\frac{1}{2}tr\left(O_+^4\right)+tr\left(O_+^2O_-^2\right)+\frac{1}{2}tr\left(\left(O_+O_-\right)^2\right)$$

- N=6:

$$tr\left(\rho_A^{T_2}\right)^6=\begin{array}{l}-\frac{3}{2}tr\left(O_+^5O_-\right)+\frac{3}{2}tr\left(O_+O_-O_+^2O_-^2\right)\\+\frac{3}{4}tr\left(O_+^3O_-^3\right)+\frac{1}{4}tr\left(\left(O_+O_-\right)^3\right)\end{array}$$

- Properties of $O_{\pm}$
- 1 $O_{+}^{\dagger} = O_{-}$
- Partial Transpose of a Hermitian Matrix is Hermitian:
- $\rho_A^{T_2}$ is Hermitian $\rightarrow$ $\left(\rho_A^{T_2}\right)_{even/odd}$ is Hermitian

$\rightarrow$ $O_{+}^{\dagger} = O_{-}$

- 2 $O_{+} = S O_{-} S$

$$S = \begin{pmatrix} I_{even} & 0 \\ 0 & -I_{odd} \end{pmatrix}$$

- Known Results:

- Analytic Expression and Numeric Simulations $tr\left(\left(\left(\rho_A\right)^{T_2}\right)^N\right)$ for Two Disjoint Interval System

A. Coser, E. Tonni and P. Calabrese, “Partial transpose of two disjoint blocks in XY spin chains," arXiv:1503.09114 [cond-mat.stat-mech].

- A. Coser, E. Tonni and P. Calabrese, “Towards entanglement negativity of two disjoint intervals for a one dimensional free fermion," arXiv:1508.00811 [cond-mat.stat-mech].

$$tr\left(\left(\left(\rho_A\right)^{T_2}\right)^N\right) = \left|\frac{(s-t)(u-v)(s-v)(t-u)}{(s-u)(t-v)}\right|^{-\frac{N^2-1}{6N}}\left|\frac{\Theta\left[\boldsymbol{e}\right](\tilde{\tau})}{\Theta(\tilde{\tau})}\right|^2$$

$$\Theta\left[\boldsymbol{e}\right](\tau) = \sum_{\boldsymbol{m}} \exp\left[2\pi i\left(\boldsymbol{m}^T\boldsymbol{e} + \frac{1}{2}\boldsymbol{m}\tau^T\boldsymbol{m}\right)\right]$$

- Analytic Expression of $\mathbb{Z}_N$-symmetric term $tr\left(\left(O_+O_-\right)^{N_e/2}\right)$

- Multiple intervals: $A_1 = \cup_{i=1}^{p}\left(s_i, t_i\right) \quad A_2 = \cup_{i=1}^{q}\left(u_i, v_i\right)$

$$tr\left(\left(O_+O_-\right)^{N/2}\right) = \left(\frac{[S,T]\,[U,V]}{[S,S]\,[T,T]\,[U,U]\,[V,V]\,\epsilon^{+q}}\right)^{-\frac{N^2-1}{6N}} \left(\frac{[S,V]\,[T,U]}{[S,U]\,[T,V]}\right)^{-\frac{N^2+2}{12N}}$$

$$[X,Y] = \left|\prod_{x\in X, y\in Y}(x-y)\right| \qquad [X,X] = \left|\prod_{x,y\in X, x\neq y}(x-y)\right|$$

- Tow disjoint intervals:

$$tr\left(\left(O_+O_-\right)^{N/2}\right) = \left|\frac{(s-t)\,(u-v)\,(s-v)\,(t-u)}{(s-u)\,(t-v)}\right|^{-\frac{N^2-1}{6N}} \left|\frac{\Theta\left[\boldsymbol{e}\right](\tilde{\tau})}{\Theta\left(\tilde{\tau}\right)}\right|^2 \qquad \boldsymbol{e} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \end{pmatrix}$$

- Thomae's Formula:

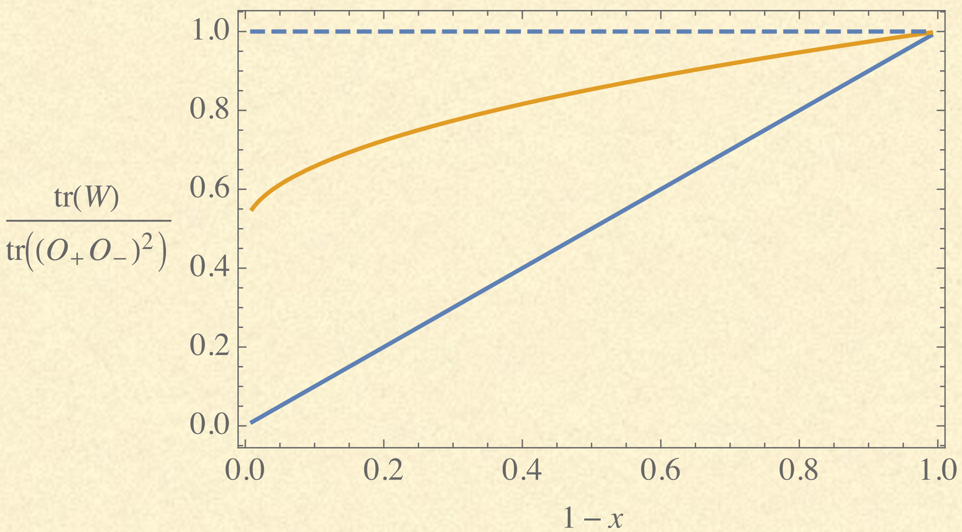
$$\left|\frac{\Theta\left[\boldsymbol{e}\right](\tilde{\tau})}{\Theta\left(\tilde{\tau}\right)}\right|^2 = \left|\frac{(s-v)\,(t-u)}{(s-u)\,(t-v)}\right|^{-\frac{N}{4}}$$

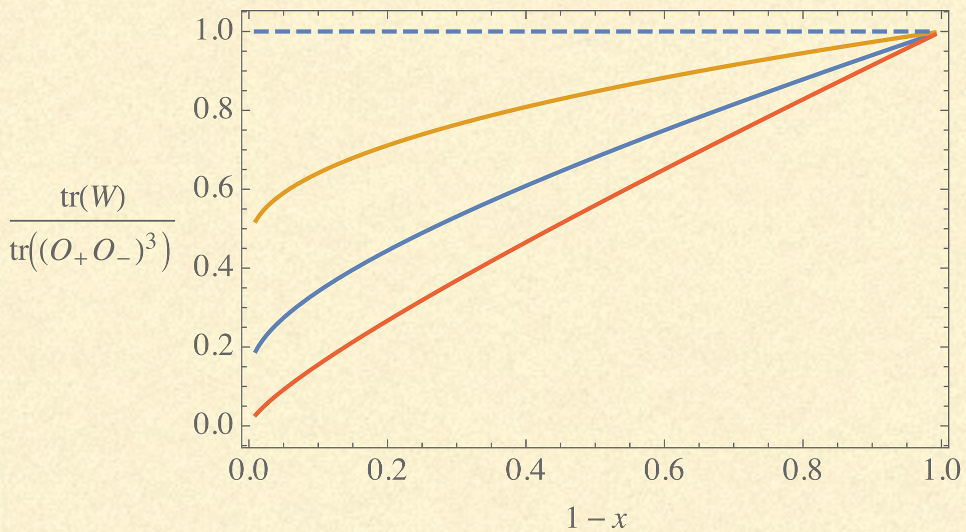
- Esitimations
- 1 lower bound by triangle inequality:

$$tr\left(\left(\rho_A^{T/2}\right)^N\right) \leq 2^{N/2} tr\left((O_+O_-)^{N/2}\right)$$

- 2 bounds from word order

$$tr\left(O_+^N\right) = tr\left[(O_+O_-)^{N/2}\right] X^{N/2} \leq tr\left(O_+^{n_1} O_-^{n_2} \ldots\right) \leq tr\left((O_+O_-)^{N/2}\right)$$





$$\left(1-\frac{1}{2^{N/2-1}}\right) tr\left(O_+^N\right)+\frac{1}{2^{N/2-1}} tr\left(\left(O_+O_-\right)^{N/2}\right) \leq tr\left(\left(\rho_A^{T/2}\right)^N\right) \leq \sqrt{\frac{2^{N+1}}{\pi N}}\left(tr\left(O_+O_-\right)^{N/2}-tr\left(O_+^N\right)\right)$$

Conjectured lower bound

$$tr\left(\left(O_+O_-\right)^{N/2}\right) \leq tr\left(\left(\rho_A^{T/2}\right)^N\right)$$