Lattice NRQCD Study of Thermal Sommerfeld Factor

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based on
SK, M. Laine (ITP, U of Bern), arXiv:1602.08105

Outline

Thermal Sommerfeld Effect in QGP

- chemical equilibriation rate of heavy quark in Quark-Gluon Plasma (QGP)

$$\Gamma_{\text{chem}} \simeq \frac{8\pi\alpha_s^2}{3M^2}\left(\frac{MT}{2\pi}\right)^{3/2} e^{-M/T}\left[\frac{\bar{S}_1}{3}+\left(\frac{5}{6}+Nf\right)\bar{S}_8\right] .$$

- Sommerfeld effect enhances the Born matrix elements

$$|\mathcal{M}_{\text{resummed}}|^2 = \mathcal{S}|\mathcal{M}_{\text{tree}}|^2$$

for the color singlet

$$\mathcal{S}_1 = \frac{X_1}{1-e^{-X_1}}, X_1 = C_F\frac{g^2}{4v}$$

and for the color octet

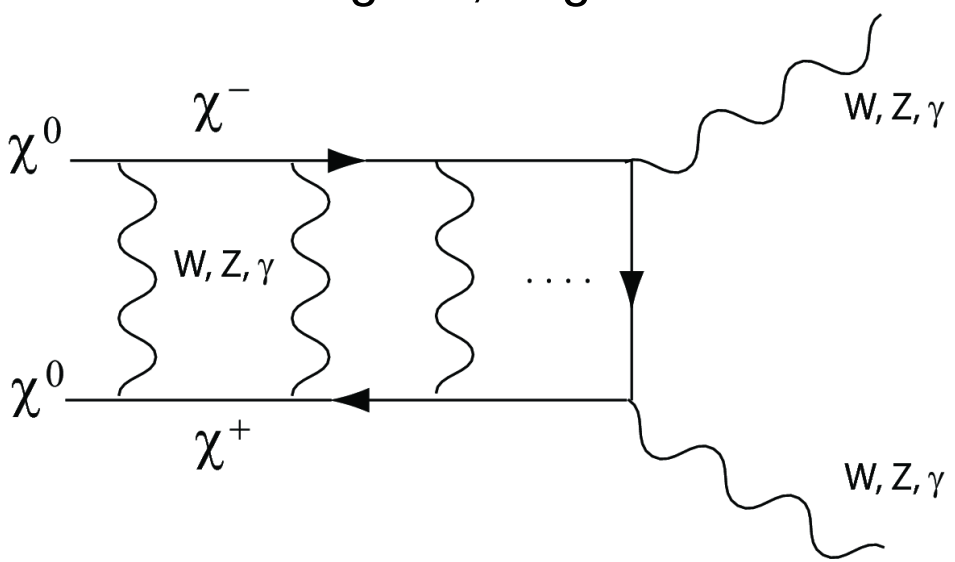
$$\mathcal{S}_8 = \frac{X_8}{e^{X_8}-1}, X_8 = (\frac{N_c}{2}-C_F)\frac{g^2}{4v}$$

Thermal Sommerfeld Effect in QGP

- enhancement of co-annihilation for slowly moving particle :

$$\sim \frac{\alpha}{v}$$

per one rung in a ladder diagram, large for small v



How to calculate: Usual way

- consider Boltzmann equation*

$$\partial_t n \simeq -c(n^2 - n_{eq}^2) = \dot{n}_{\text{loss}} + \dot{n}_{\text{gain}}$$

with $\dot{n}_{\text{loss}} = -cn^2$

- in equilibrium, $n(t) = n_{\text{eq}}$ and $\delta\dot{n} \simeq -2cn\delta n$

$$\Gamma_{\text{chem}} = \frac{\delta\dot{n}}{n}|_{\text{eq}} = -2\frac{\dot{n}_{\text{loss}}}{n_{eq}}$$

- then

$$\Gamma_{\text{chem}} = \frac{2}{2N_c \int_{\mathbf{k}} f_F(E_k)} \int\int (2\pi)^4 \delta^4(P_1 + P_2 - K_1 - K_2) f_F(E_{k_1}) f_F(E_{k_2})$$

$$\left(\frac{1}{2} \sum |M_1|^2 \left[1 + f_B(\varepsilon_{p_1})\right] \left[1 + f_B(\varepsilon_{p_2})\right] + N_f \sum |M_2|^2 \left[1 - f_F(\varepsilon_{p_1})\right] \left[1 - f_F(\varepsilon_{p_2})\right] \right)$$

How to calculate: New way, $\Gamma_{\rm chem}$ as a transport coefficient

- chemical equilibriation as a transport coefficient (D. Bödeker, M. Laine, JHEP07 (2012) 130, 01 (2013) 037)

- treat the approach to the equilibrium as a Langevin process

$$\delta\dot{n}(t) = -\Gamma_{\rm chem}\delta n(t) + \xi(t)$$

$$\langle\langle \xi(t)\xi(t')\rangle\rangle = \Omega_{\rm chem}\delta(t-t'), \qquad \langle\langle \xi(t)\rangle\rangle = 0$$

where $\delta n(t)$ is the deviation from the equilibrium and $\xi(t)$ is a stochastic noise

$$\delta n(t) = \delta n(t_0)e^{-\Gamma_{\rm chem}(t-t_0)} + \int_{t_0}^{t} dt' e^{\Gamma_{\rm chem}(t'-t)}\xi(t')$$

How to calculate: New way, $\Gamma_{\rm chem}$ as a transport coefficient

- consider a correlator (classical)

$$
\begin{aligned}
\Delta_{cl}(t,t') &= \lim_{t_0\to-\infty}\langle\langle \delta n(t)\delta n(t')\rangle\rangle \\
&= \lim_{t_0\to-\infty}\int_{t_0}^{t} dt_1 e^{\Gamma_{\rm chem}(t_1-t)}\int_{t_0}^{t'} dt_2 e^{\Gamma_{\rm chem}(t_2-t')}\langle\langle \xi(t_1)\xi(t_2)\rangle\rangle \\
&= \Omega_{\rm chem}\lim_{t_0\to-\infty}\int_{t_0}^{t} dt_1 e^{\Gamma_{\rm chem}(t_1-t)}\int_{t_0}^{t'} dt_2 e^{\Gamma_{\rm chem}(t_2-t')}\delta(t_1-t_2) \\
&= \frac{\Omega_{\rm chem}}{2\Gamma_{\rm chem}} e^{-\Gamma_{\rm chem}|t-t'|}
\end{aligned}
$$

and

$$
\partial_t\partial_{t'}\Delta_{cl}(t,t') = -\frac{\Omega_{\rm chem}}{2}\Gamma_{\rm chem}e^{-\Gamma_{\rm chem}|t-t'|} + \Omega_{\rm chem}\delta(t-t')
$$

How to calculate: New way, $\Gamma_{\rm chem}$ as a transport coefficient

- in frequency domain,

$$\tilde{\Delta}_{cl}(\omega) \equiv \int_{-\infty}^{\infty} dt\ e^{i\omega(t-t')}\Delta_{cl}(t,t') = \frac{\Omega_{\rm chem}}{\omega^2+\Gamma_{\rm chem}^2}$$

and

$$\omega^2\tilde{\Delta}_{cl}(\omega) = \int_{-\infty}^{\infty} dt\ e^{i\omega(t-t')}\partial_t\partial_{t'}\Delta_{cl}(t,t') = \frac{\omega^2\Omega_{\rm chem}}{\omega^2+\Gamma_{\rm chem}^2}$$

- then,

$$\langle(\delta n)^2\rangle = \lim_{t_0\to-\infty}\langle\langle\delta n(t)\delta n(t)\rangle\rangle = \frac{\Omega_{\rm chem}}{2\Gamma_{\rm chem}}$$

or

$$\Gamma_{\rm chem} = \frac{\Omega_{\rm chem}}{2\langle(\delta n)^2\rangle}$$

How to calculate: New way, $\Gamma_{\rm chem}$ as a transport coefficient

• to access $\Omega_{\rm chem}$, consider correlators (for heavy quark, or non-relativistic QCD, the heavy quark number is replaced by hamiltonian) in imaginary time

$$\Omega(\tau) = \frac{1}{V}\langle \partial_t H(\tau)\partial_t H(0)\rangle_{\rm qm}$$

or

$$\Delta(\tau) = \frac{1}{V}\langle H(\tau)H(0)\rangle_{\rm qm}$$

then

$$\Omega_{\rm chem} = \lim_{\Gamma_{\rm chem}\ll\omega\ll\omega_{UV}} 2T\frac{\rho_\Omega(\omega)}{\omega} \quad \text{or} \quad = \lim_{\omega\ll T} 2T\omega\rho_\Delta(\omega)$$

and

$$\Gamma_{\rm chem} = \frac{\lim_{\omega\to 0^+} 2T\frac{\rho_\Omega(\omega)}{\omega}}{2\chi_f M^2} \quad \text{or} \quad = \frac{\lim_{\omega\ll T} 2T\omega\rho_\Delta(\omega)}{2\chi_f M^2}$$

How to calculate: New way, $\Gamma_{\rm chem}$ as a transport coefficient

- for non-relativistic QCD), the change in the number of heavy quark/anti-quark can be treated as a "perturbation" since the pair annihilation/creation of heavy quark–anti-heavy quark has "high energy" ($\sim 2M$) and in NRQCD, $p \sim M$ is integrated out

- again, consider an imaginary time correlator

$$\Delta(\tau) = \frac{1}{V}\langle H(\tau)H(0)\rangle_{\rm qm}$$

- and its leading order perturbation by four-fermion vertices

$$O_1(^1S_0) = \psi^\dagger\chi\chi^\dagger\psi, \quad S_M = \frac{f_1(^1S_0)O_1(^1S_0)}{M^2}$$

and

$$O_8(^1S_0) = \psi^\dagger T^a\chi\chi^\dagger T^a\psi, \quad S_M = \frac{f_8(^1S_0)O_8(^1S_0)}{M^2}$$

How to calculate: New way, $\Gamma_{\rm chem}$ as a transport coefficient

• We need to be careful about the analytic structure of these four-fermion vertices since we are interested in imaginary time formulation for finite temperature physics.

• we need to consider, a correlator

$$\Delta(\tau) = \mathrm{Im}\frac{f_1(^1S_0)}{\pi M^2}\varepsilon(\tau)$$

with

$$\varepsilon(\tau) = \int_{\mathbf{x},\mathbf{y}}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\frac{\langle H(\tau,\mathbf{x})H(0,\mathbf{y})(\theta^\dagger\chi)(\tau_1,\mathbf{0})(\chi^\dagger\theta)(\tau_2,\mathbf{0})\rangle}{|\tau_1-\tau_2|}$$

How to calculate: New way, Γ_{chem} as a transport coefficient

- numerator becomes

$$\begin{aligned}
&\int_{\vec{x},\vec{y}} \langle H(\tau,\vec{x})H(0,\vec{y})(\psi^\dagger\chi)(\tau_1,\vec{0})(\chi^\dagger\psi)(\tau_2,\vec{0})\rangle\big|_{\tau_1>\tau_2}\left[\theta(\tau-\tau_1)+\theta(\tau_2-\tau)\right] \\
&= \frac{1}{\mathcal{Z}}\mathrm{Tr}\left[e^{-\beta\hat{\mathcal{H}}}(\hat{\psi}^\dagger\hat{\chi})(\tau_1,\vec{0})(\hat{\chi}^\dagger\hat{\psi})(\tau_2,\vec{0})\int_{\vec{x},\vec{y}}\hat{H}(0,\vec{x})\hat{H}(0,\vec{y})\right] \\
&= \frac{1}{\mathcal{Z}}\sum_{m,n}\langle q\bar{q},m|e^{-\beta\hat{\mathcal{H}}}(\hat{\psi}^\dagger\hat{\chi})(\tau_1,\vec{0})|n\rangle\langle n|(\hat{\chi}^\dagger\hat{\psi})(\tau_2,\vec{0})\int_{\vec{x},\vec{y}}\hat{H}(0,\vec{x})\hat{H}(0,\vec{y})| \\
&= \frac{4M^2}{\mathcal{Z}}\sum_{m,n}e^{-\beta E_m}e^{(\tau_1-\tau_2)(E_m-\varepsilon_n)}\langle q\bar{q},m|\hat{\psi}^\dagger\hat{\chi}|n\rangle\langle n|\hat{\chi}^\dagger\hat{\psi}|q\bar{q},m\rangle
\end{aligned}$$

E_m are the states with heavy quarks, ε_n are states without heavy quarks

How to calculate: New way, Γ_{chem} as a transport coefficient

- with $C_{mn} = \frac{4M^2}{\mathcal{Z}} e^{-\beta E_m} \langle q\bar{q}, m|\hat{\psi}^\dagger \hat{\chi}|n\rangle\langle n|\hat{\chi}^\dagger \hat{\psi}|q\bar{q}, m\rangle$

$$\varepsilon(\tau) = \sum_{m,n} C_{mn} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \frac{e^{(\tau_1-\tau_2)(E_m-\varepsilon_m)}}{\tau_1-\tau_2} \left[\theta(\tau-\tau_1)+\theta(\tau_2-\tau)\right]$$

- then with $\omega, \varepsilon_m << E_m$

$$\rho_\Delta(\omega) = \frac{\mathrm{Im} f_1(^1S_0)}{M^2} \sum_{m,n} C_{mn} \frac{e^{\beta\omega} - e^{-\beta\omega}}{\omega^2}$$

How to calculate: New way, $\Gamma_{\rm chem}$ as a transport coefficient

- finally,

$$
\begin{aligned}
\Omega_{\rm chem} &= \frac{4{\rm Im}f_1(^1S_0)}{M^2}\sum_{m,n}\mathcal{C}_{mn} \\
&= 16\,{\rm Im}f_1(^1S_0)\,\frac{1}{\mathcal{Z}}\sum_{m,n}e^{-\beta E_m}\,\langle q\bar{q},m|\hat{\psi}^\dagger\hat{\chi}|n\rangle\langle n|\hat{\chi}^\dagger\hat{\psi}|q\bar{q},m\rangle \\
&= 16\,{\rm Im}f_1(^1S_0)\,\frac{1}{\mathcal{Z}}\,{\rm Tr}\left[e^{-\beta\hat{\mathcal{H}}}(\hat{\psi}^\dagger\hat{\chi})(0^+,\vec{0})\,(\hat{\chi}^\dagger\hat{\psi})(0,\vec{0})\right] \\
&= 16\,{\rm Im}f_1(^1S_0)\left\langle(\psi^\dagger\chi)(0^+,\vec{0})\,(\chi^\dagger\psi)(0,\vec{0})\right\rangle
\end{aligned}
$$

How to calculate: New way, $\Gamma_{\rm chem}$ as a transport coefficient

$$
\begin{aligned}
P_1 &\equiv \frac{1}{2N_c}\mathrm{Re}\left\langle G^{\theta}_{\alpha\alpha;ii}(\beta,\vec{0};0,\vec{0})\right\rangle , \\
P_2 &\equiv \frac{1}{2N_c}\left\langle G^{\theta}_{\alpha\gamma;ij}(\beta,\vec{0};0,\vec{0})G^{\theta\dagger}_{\gamma\alpha;ji}(\beta,\vec{0};0,\vec{0})\right\rangle , \\
P_3 &\equiv \frac{1}{2N_c^2}\left\langle G^{\theta}_{\alpha\alpha;ij}(\beta,\vec{0};0,\vec{0})G^{\theta\dagger}_{\gamma\gamma;ji}(\beta,\vec{0};0,\vec{0})\right\rangle .
\end{aligned}
$$

- singlet Sommerefeld factor

$$\bar{S}_1 = \frac{P_2}{P_1^2} .$$

- octet Sommerefeld factor

$$\bar{S}_8 = \frac{N_c^2 P_3 - P_2}{(N_c^2-1)P_1^2} .$$

Lattice setup

- anisotropic Euclidean lattices (i.e., the time direction lattice spacing is different from the space direction lattice spacing, $a_s/a_t = 3.5$), $a_s = 0.1227(8)$ fm

- $N_f = 2 + 1$ light quark flavors ($M_\pi \simeq 400$ MeV, $M_K \simeq 500$ MeV)

- $24^3 \times N_t$ lattices

- T_c = 185 MeV, $a_s M = 2.92$

- lattices used for bottomonium at $T \neq 0$ study (G. Aarts et al, JHEP07 (2014) 097) and electric conductivity of QGP (G. Aarts et al, JHEP-2 (2015) 186)

Lattice setup

$$
\begin{aligned}
G^{\theta}(0,\mathbf{x};\cdot) &= \frac{\delta_{\mathbf{x},\vec{0}}}{a_s^3}\,, \\
G^{\theta}(a_t,\mathbf{x};\cdot) &= \left(1-\frac{a_t\mathcal{H}_0}{2n}\right)^n U_0^{\dagger}(0,\mathbf{x})\left(1-\frac{a_t\mathcal{H}_0}{2n}\right)^n G^{\theta}(0,\mathbf{x};\cdot)\,, \\
G^{\theta}(\tau+a_t,\mathbf{x};\cdot) &= \left(1-\frac{a_t\mathcal{H}_0}{2n}\right)^n U_0^{\dagger}(\tau,\mathbf{x})\left(1-\frac{a_t\mathcal{H}_0}{2n}\right)^n \left(1-a_t\,\delta\mathcal{H}\right) G^{\theta}(\tau,\mathbf{x}
\end{aligned}
$$

where U_0 is a time-direction gauge link. The lowest-order Hamiltonian reads

$$\mathcal{H}_0 = -\frac{\Delta^{(2)}}{2M}\,,$$

where $\Delta^{(2)}$ is a discretized gauge Laplacian.

Lattice setup

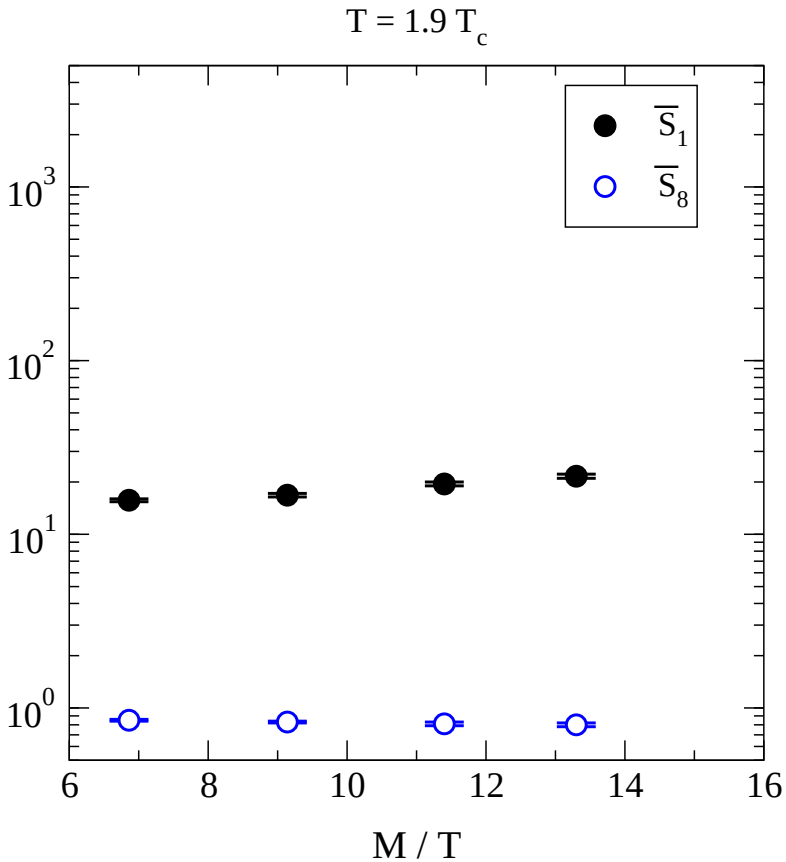
The higher order correction is

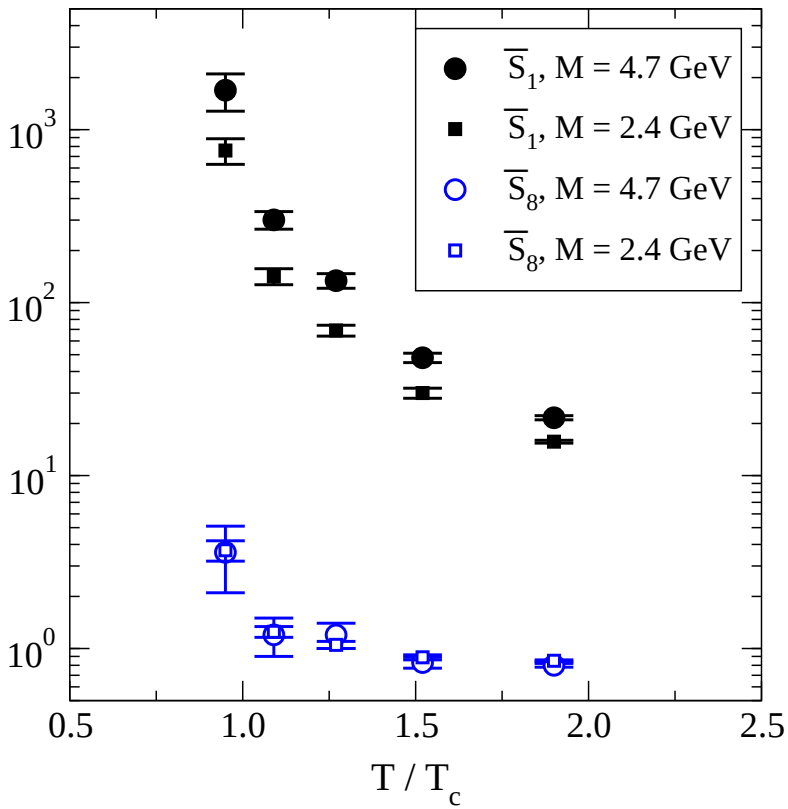
$$\begin{aligned}\delta\mathcal{H} = & -\frac{(\Delta^{(2)})^2}{8M^3} + \frac{ig_0(\nabla\cdot\mathbf{E}-\mathbf{E}\cdot\nabla)}{8M^2} - \frac{g_0\,\sigma\cdot(\nabla\times\mathbf{E}-\mathbf{E}\times\nabla)}{8M^2} \\ & -\frac{g_0\,\sigma\cdot\mathbf{B}}{2M} + \frac{a_s^2\Delta^{(4)}}{24M} - \frac{a_t(\Delta^{(2)})^2}{16nM^2}\,,\end{aligned}$$

Lattice setup

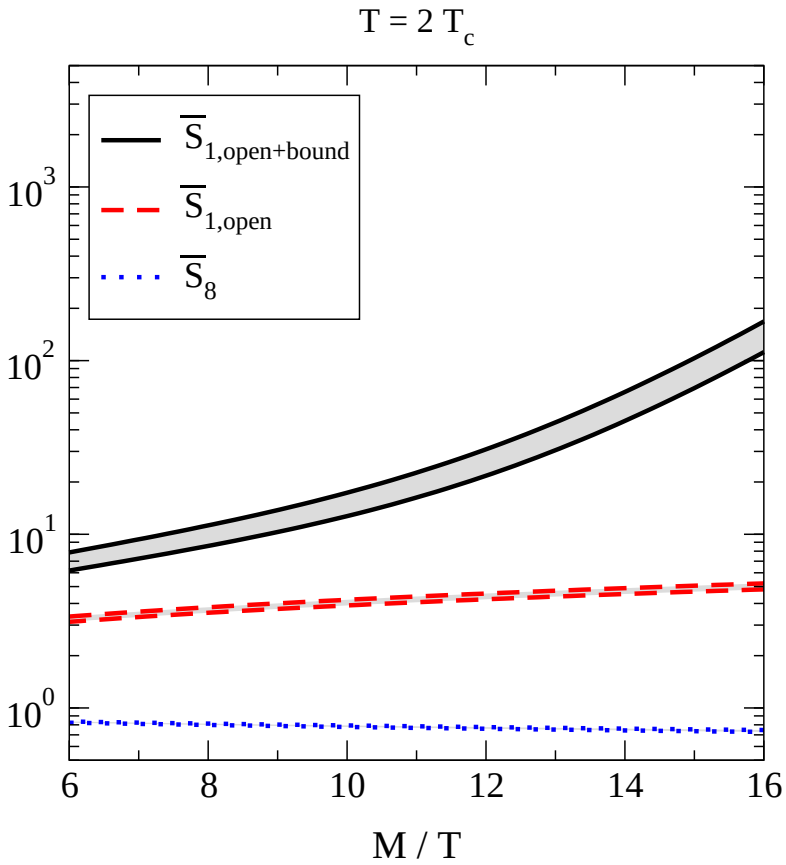
N_t	T/T_c	a_sM	M/T	$10^4 P_1$	$10^5 P_2$	$10^6 P_3$	$\bar{S}_1$	$\bar{S}_8$
32	0.95	2.92	26.7	0.75(7)	0.938(5)	1.06(1)	1690(410)	3.6(15)
28	1.09	2.92	23.4	3.0(1)	2.78(1)	3.19(3)	301(35)	1.2(3)
24	1.27	2.92	20.0	8.0(3)	8.64(3)	10.3(1)	134(13)	1.2(2)
20	1.52	2.92	16.7	24.4(5)	28.5(1)	36.1(4)	48(3)	0.83(6)
16	1.90	2.92	13.3	68.9(9)	102.2(3)	147(1)	21.6(6)	0.80(2)
32	0.95	1.50	13.7	1.31(7)	1.308(8)	1.51(1)	758(128)	3.7(5)
28	1.09	1.50	12.0	4.6(1)	3.04(2)	3.62(3)	142(15)	1.25(9)
24	1.27	1.50	10.3	10.4(2)	7.38(4)	9.21(8)	69(5)	1.05(5)
20	1.52	1.50	8.57	25.7(3)	19.7(1)	27.2(2)	30(2)	0.89(3)
16	1.90	1.50	6.86	61.9(6)	60.1(3)	95.8(8)	15.7(3)	0.85(1)
16	1.90	2.00	9.14	57.6(6)	55.7(2)	86.4(8)	16.8(4)	0.83(1)
16	1.90	2.50	11.4	62.5(7)	76.0(2)	113(1)	19.5(5)	0.81(2)

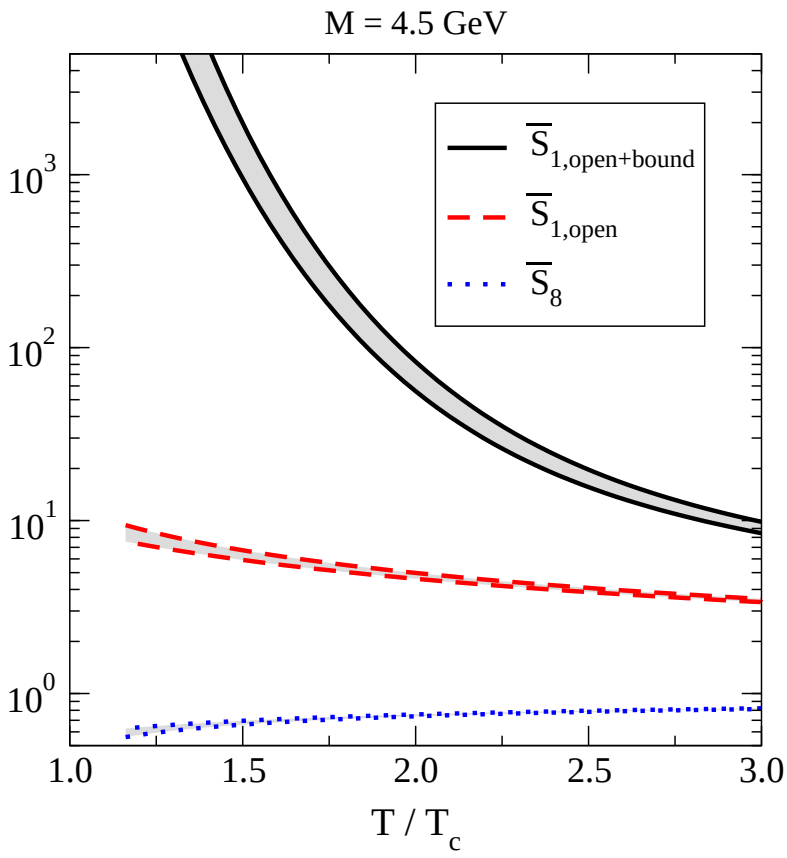
Lattice result of thermal Sommerfeld factor



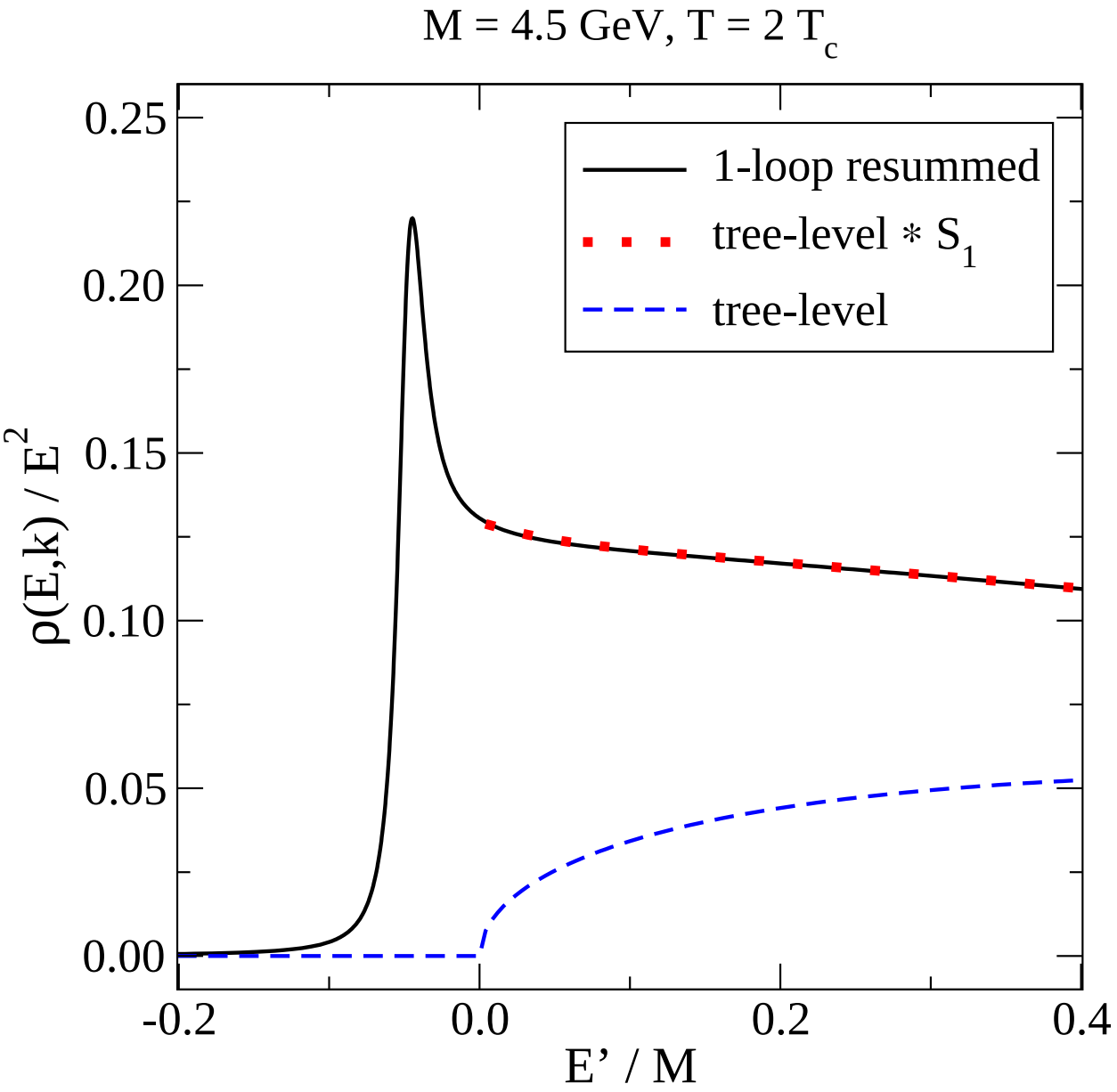


Analytic estimate of thermal Sommerfeld factor





Analytic estimate of thermal Sommerfeld factor



Conclusion

- a real time quantity, chemical equilibriation rate, is calculated non-perturbatively using Euclidean lattice without analytic continuation

- in the current relativistic heavy ion collision experiments, charm quark and bottom quarks do not chemically equilibriate

- thermal Sommerfeld effect for bottomonium co-annihilation is calculated using lattice NRQCD and is found to be two orders of magnitude larger than perturbative estimate

- bound state effect is the cause of large enhancement

- similarly, in “strongly interacting” dark matter scenario, the bound state effect is expected to enhance co-annihilation of dark matter far beyond a naive perturbative estimate.