Scattering Amplitudes in Twistor Space

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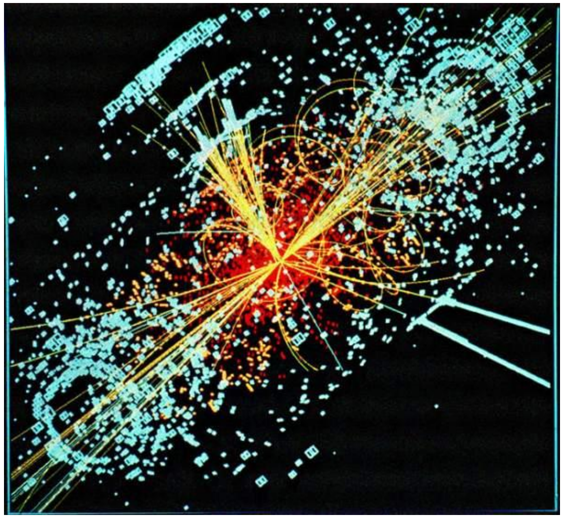
Introduction

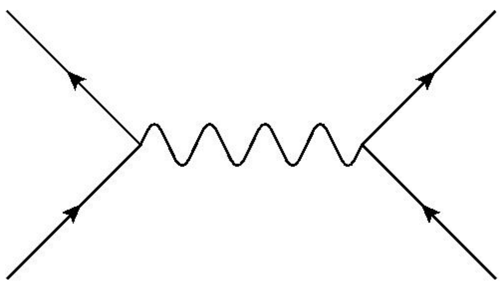
- Scattering amplitudes are the basic quantities used to compare particle theory with experiment.



- They also have a rich mathematical structure which is interesting in its own right.

Feynman Diagrams

- The traditional method for computing scattering amplitudes uses Feynman diagrams:



- As the number of particles increases, the number of Feynman diagrams quickly gets out of hand, even though the final answer is often surprisingly simple.

Spinor-Helicity

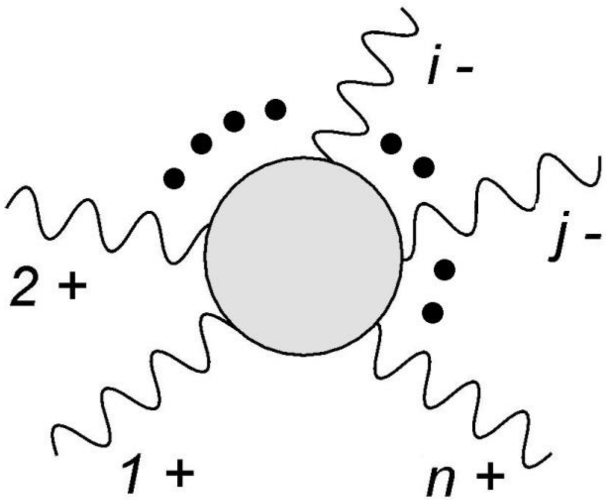
- 4d null momentum:

$$p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}$$

- Expressing amplitudes in terms of these spinors leads to dramatic simplifications.

MHV Amplitudes

At tree-level:



$$\mathcal{A}_n = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$$

(Parke,Taylor)

where $\langle ij \rangle = \epsilon_{\alpha\beta} \lambda_i^{\alpha} \lambda_j^{\beta}$

Twistor String Theory

- The simplicity of MHV amplitudes suggests a deeper mathematical structure.
- Is there a way to reformulate Yang-Mills theory to make this structure manifest?
- Nair,Witten,Berkovits: N=4 super Yang-Mills (SYM) is equivalent to string theory with target space $CP^{3|4}$

Twistors

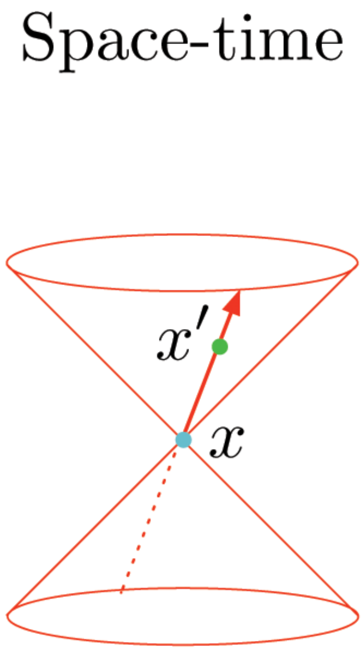
- Twistors: (Penrose)

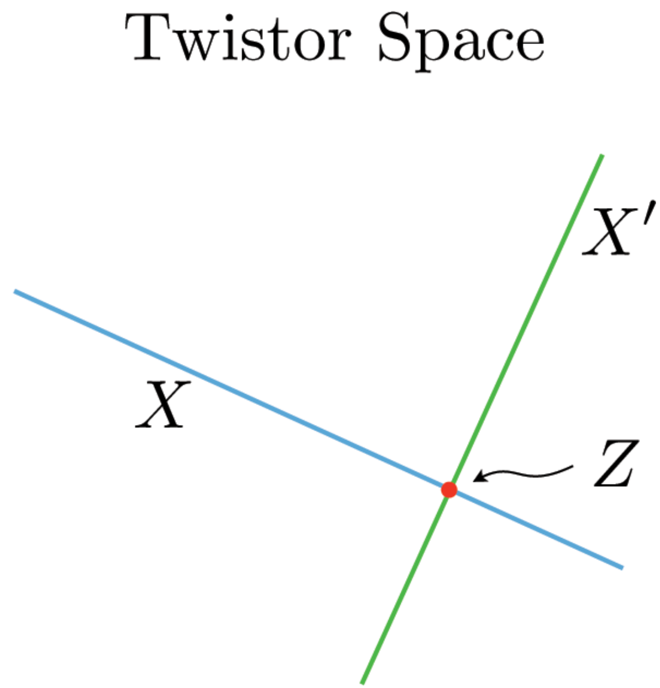
$$\begin{pmatrix} Z^A \\ \chi^a \end{pmatrix}, \quad Z^A = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot\alpha} \end{pmatrix}$$

- Incidence relations:

$$\mu^{\dot\alpha} = -i x^{\dot\alpha\alpha}\lambda_\alpha, \quad \chi^a = -i\theta^{a\alpha}\lambda_\alpha$$

Spacetime vs Twistor Space



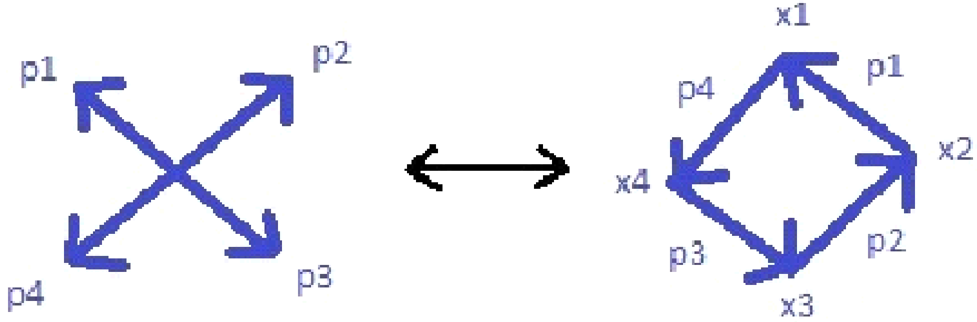


Point in spacetime ⟷ CP^1 in twistor space

Point in twistor space ⟷ null ray in spacetime

Dual Conformal Symmetry

- Dual variables: $x_i - x_{i+1} = p_i$



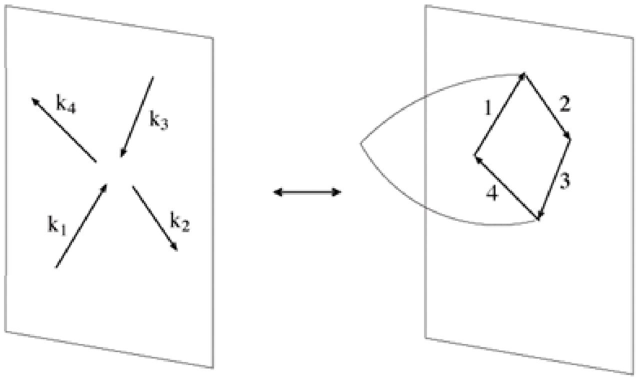
- Tree-level amplitudes and loop integrands transform covariantly when

$$x_i \to x_i^{-1}$$

(Drummond,Henn,Korchemsky,Smirnov,Sokatchev)

Amplitude/Wilson Loop Duality

- Alday,Maldacena: Amplitudes mapped into null polygonal Wilson loops by T-duality:

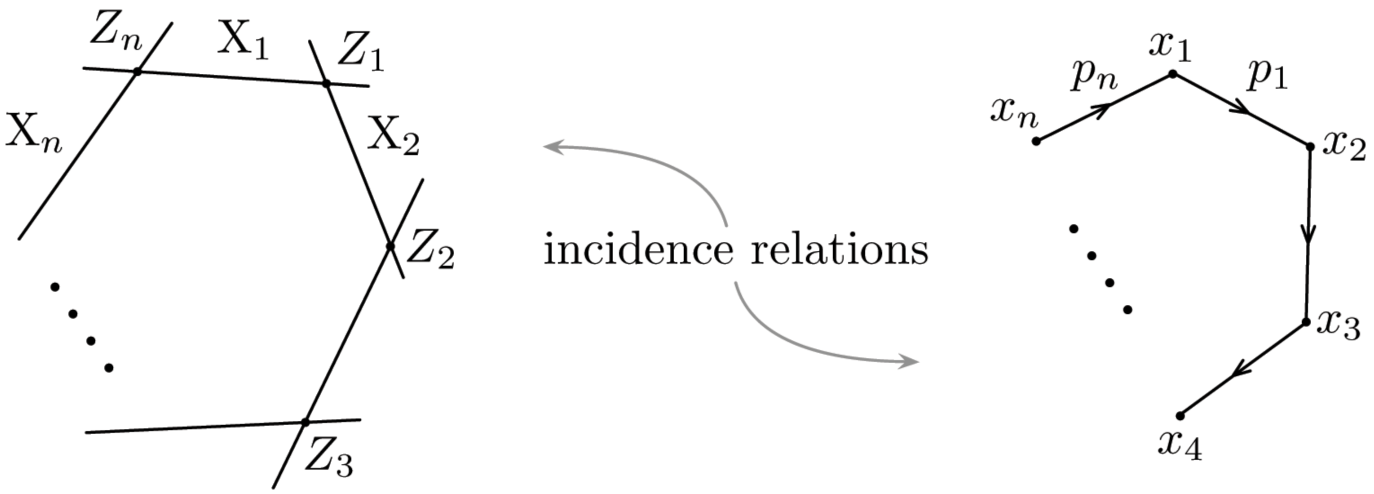


- To compute an amplitude at strong coupling, compute the area of a “soap bubble” in string theory.

- Remarkably, this duality extends to weak coupling! (Brandhuber,Heslop,Travaglini)

Twistor Wilson Loop

- Null polygon in spacetime corresponds to polygon in twistor space:



- Expectation value of the twistor Wilson loop computes planar S-matrix! (Mason,Skinner,Caron-Huot)

Dlog Form

- Lipstein,Mason: Twistor Wilson loop reveals new mathematical simplicity of loop amplitudes.

- **Example:** 1-loop MHV amplitude

$$K_{ij} = -\frac{1}{4\pi^2} \int \mathrm{d}\ln s_0 \, \mathrm{d}\ln t_0 \, \mathrm{d}\ln s \, \mathrm{d}\ln t$$

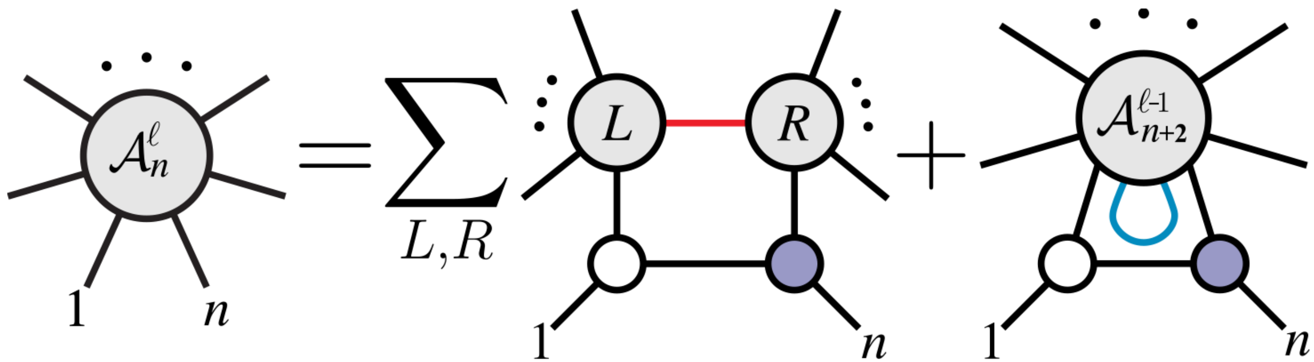
where

$$s_0 = \bar{s}_0 \qquad t_0 = \bar{t}_0$$

$$s = -\frac{\bar{t}\,(a_{i-1j} - v) + a_{i-1j-1} - v}{\bar{t}\,(a_{i\,j} - v) + a_{ij-1} - v}$$

On-Shell Diagrams

- Alternative dlog form follows from "on-shell diagrams"



(Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka)

- No virtual particles!
- Implement loop-level Britto-Cachazo-Feng-Witten (BCFW) recursion relations.
- Reveal new connections to combinatorics, algebraic geometry, and cluster algebras

Summary

- Combining insights from AdS/CFT and twistor string theory has lead to powerful new techniques for computing amplitudes of planar N=4 SYM.

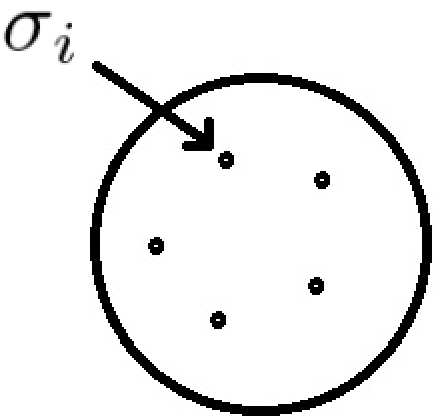
- Question: Can these ideas be extended to other theories?

Scattering Equations

external momentum

$$\sum_{i \neq j} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0$$

point on 2-sphere



- First discovered by Fairlie and Roberts.
- Gross,Mende: arise from the tensionless limit of string amplitudes
- Cachazo,He,Yuan (CHY): These equations underlie the scattering amplitudes of massless particles in any dimension!

Ambitwistor Strings

- Mason,Skinner: Amplitudes of massless point particles can be computed using a chiral, infinite tension limit of string theory:

$$S = \int_{\Sigma} P_\mu \bar{\partial} X^\mu + \frac{e}{2} P_\mu P^\mu + \ldots$$

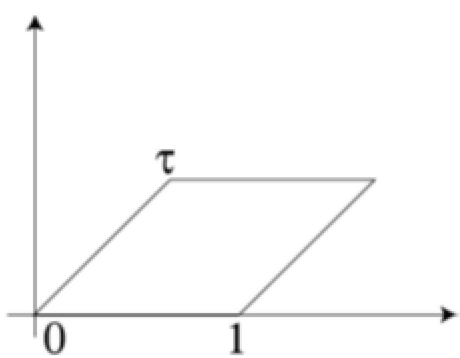
- Tree-level correlators reproduce the CHY formulae!

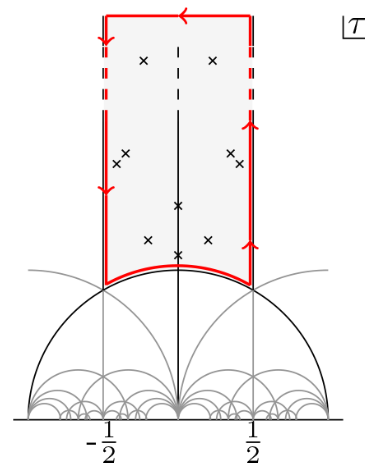
- Critical in d=26 (bosonic) and d=10 (superstring)

Genus 1 Amplitudes

- Adamo,Casali,Skinner proposed a formula for genus-1 amplitudes of the critical 10d ambitwistor string.

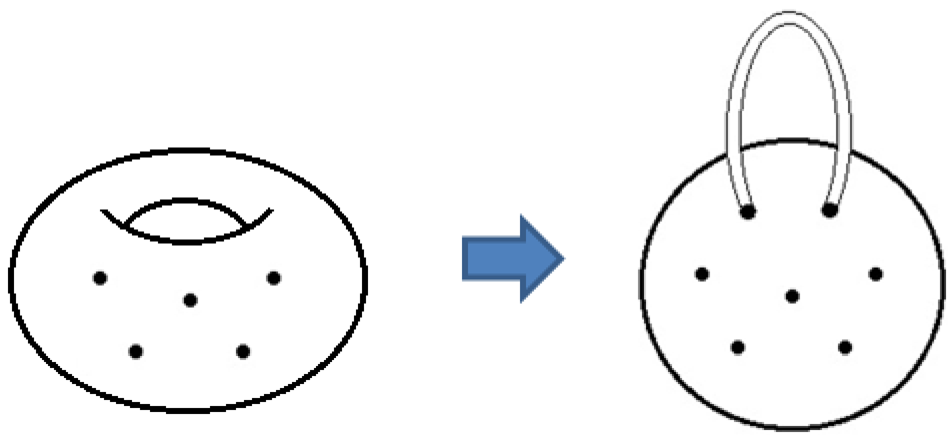
- This formula involves an integral over the complex structure of the worldsheet, τ. In this form, it's difficult to see how it can be a supergravity amplitude.





Loops from Riemann Sphere

- When Im(τ) = ∞, the worldsheet degenerates to a sphere with two additional punctures and the integrand becomes rational.



- Remarkably, the integral over τ localizes to Im(τ) = ∞, giving rise to one-loop supergravity amplitudes! (Geyer,Mason,Monteiro,Tourkine)

4d Ambitwistor Strings

- In four dimensions, the CHY formulae take a particularly elegant form and can be derived from a 2d model known as 4d ambitwistor string theory (Geyer, Lipstein, Mason).

$$\mathcal{L} = W_A \bar{\partial} Z^B + \tilde{\rho}_A \bar{\partial} \rho^A$$

$$Z^A = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot{\alpha}} \\ \chi^a \end{pmatrix}, \quad W_A = \begin{pmatrix} \tilde{\mu}^\alpha \\ \tilde{\lambda}_{\dot{\alpha}} \\ \tilde{\chi}_a \end{pmatrix}, \quad \rho^A = \begin{pmatrix} \rho_\alpha \\ \rho^{\dot{\alpha}} \\ \omega^a \end{pmatrix}, \quad \tilde{\rho}_A = \begin{pmatrix} \tilde{\rho}^\alpha \\ \tilde{\rho}_{\dot{\alpha}} \\ \tilde{\omega}_a \end{pmatrix}$$

- Describe tree-level super Yang-Mills and supergravity amplitudes with any amount of supersymmetry.

Correlators

- Vertex operators:

$$\mathcal{V}_h = \int \left[W, \frac{\partial h}{\partial Z}\right] + \left[\tilde{\rho}, \frac{\partial}{\partial Z}\right] \rho \cdot \frac{\partial h}{\partial Z}$$

$$\widetilde{\mathcal{V}}_{\tilde{h}} = \int \left\langle Z, \frac{\partial \tilde{h}}{\partial W}\right\rangle + \left\langle \rho, \frac{\partial}{\partial W}\right\rangle \tilde{\rho} \cdot \frac{\partial \tilde{h}}{\partial W}$$

- N^{k-2}MHV amplitude: $\mathcal{A} = \left\langle \widetilde{\mathcal{V}}_1 \ldots \widetilde{\mathcal{V}}_k \mathcal{V}_{k+1} \ldots \mathcal{V}_n \right\rangle$

- Localize onto solutions of the 4d scattering equations:

$$\tilde{\lambda}_l^{\dot{\alpha}} - \sum_{r=k+1}^{n} \frac{\tilde{\lambda}_r^{\dot{\alpha}}}{(\sigma\sigma_r)} = 0, \;\; l \in 1, ..., k$$

$$\lambda_r^{\alpha} - \sum_{l=1}^{k} \frac{\lambda_l^{\alpha}}{(\sigma\sigma_l)} = 0, \;\; r \in k+1, ..., n$$

Tree Amplitudes

- For N=8 supergravity:

$$\mathcal{M}_n^k = \int \frac{\Pi_{i=1}^n \mathrm{d}^2\sigma_i}{\mathrm{VolGL}(2,\mathbb{C})} \det H \det \tilde{H} \Pi_{l=1}^k \delta^2\left(\tilde{\lambda}_l - \tilde{\lambda}(\sigma_l)\right)$$

$$\Pi_{r=k+1}^n \delta^{2|8}\left(\lambda_r - \lambda(\sigma_r)\,|\eta_r - \chi(\sigma_r)\right)$$

where

$$\mathbb{H}_{ij} = \frac{\langle i\, j\rangle}{(i\, j)}, \quad i \neq j, \qquad \mathbb{H}_{ii} = -\sum_{j=1, j\neq i}^{k} \mathbb{H}_{ij}$$

$$\widetilde{\mathbb{H}}_{pq} = \frac{[p\, q]}{(p\, q)}, \quad p \neq q, \qquad \widetilde{\mathbb{H}}_{pp} = -\sum_{q=k+1, q\neq p}^{n} \widetilde{\mathbb{H}}_{pq}.$$

Soft Theorems

- Soft Graviton Theorem:

$$\lim_{k_n \to 0} \mathcal{A}_n = \left(S^{(-1)} + S^{(0)} + S^{(1)} \right) \mathcal{A}_{n-1}$$

$$S^{(-1)} = \sum_{i=1}^{n-1} \frac{(\epsilon \cdot k_i)^2}{k_n \cdot k_i}, \quad S^{(0)} = \sum_{i=1}^{n-1} \frac{\epsilon \cdot k_i k_{n,\mu} \epsilon_\nu J_i^{\mu\nu}}{k_n \cdot k_i}, \quad S^{(1)} = \sum_{i=1}^{n-1} \frac{(k_{n,\mu} \epsilon_\nu J_i^{\mu\nu})^2}{k_n \cdot k_i}$$

(Weinberg, White, Cachazo, Strominger)

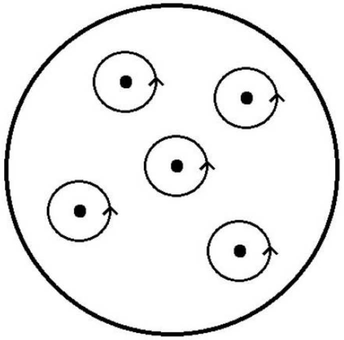
- Can be understood as Ward identities of infinite dimensional asymptotic symmetries discovered by Bondi, van der Burg, Metzner, and Sachs (BMS).

- Analogous symmetries at the black hole horizon give rise to black hole hair! (Hawking, Perry, Strominger)

Soft Theorems from 2d CFT

- Using ambitwistor string theory, the soft theorems can be proven from the point of view of 2d CFT (Geyer,Lipstein,Mason).

- Key idea: Take a vertex operator in correlator to be soft, integrate it around the hard ones, and add up the residues.



- Furthermore, the algebra of soft limits can be encoded in the OPE of the vertex operators (Lipstein).

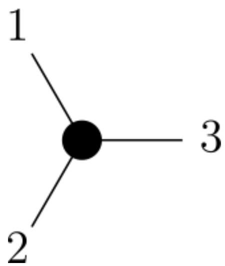
N=8 SUGRA

- We have seen that the ideas of twistor string theory can be generalized beyond N=4 SYM. What about on-shell diagrams?

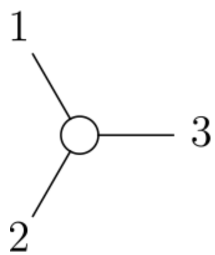
- With Paul Heslop, I generalized on-shell diagrams to N=8 supergravity, which has been argued to be the simplest QFT in four dimensions (Arkani-Hamed, Cachazo, Kaplan). Related work by Herrmann and Trnka.

- Perturbative finiteness of N=8 SUGRA is an important open question. (Green, Russo, Vanhove, Bern, Carrasco, Dixon, Johansson, Kosower, Roiban)

Building Blocks



$$= \frac{\delta^{16}\left(\lambda_1\tilde{\eta}_1 + \lambda_2\tilde{\eta}_2 + \lambda_3\tilde{\eta}_3\right)\delta^4\left(\lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3\right)}{\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2}$$

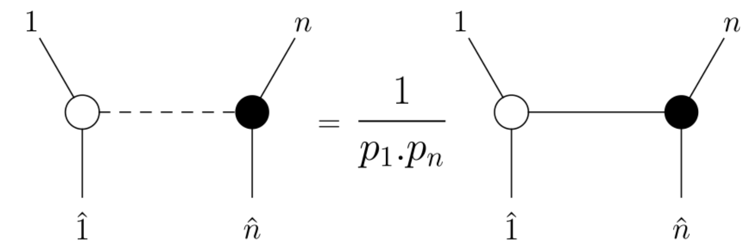


$$= \frac{\delta^{8}\left([12]\,\tilde{\eta}_3 + [23]\tilde{\eta}_1 + [31]\tilde{\eta}_2\right)\delta^4\left(\lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3\right)}{[12]^2[23]^2[31]^2}$$

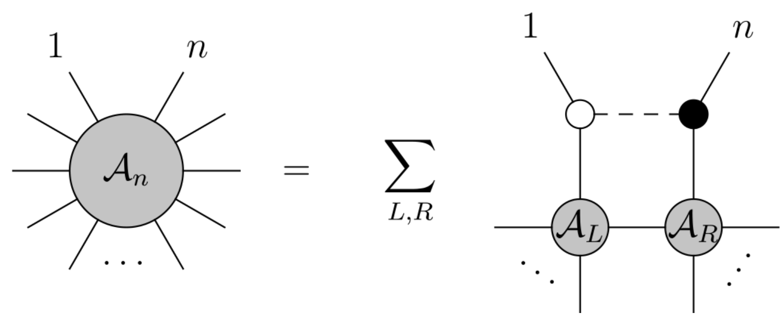
$$\int \frac{\mathrm{d}^8\tilde{\eta}\,\mathrm{d}^2\lambda\,\mathrm{d}^2\tilde{\lambda}}{\mathrm{VolGL}(1)}$$

Tree-Level Recursion

- Naïve BCFW bridge doesn't work; need to decorate it!



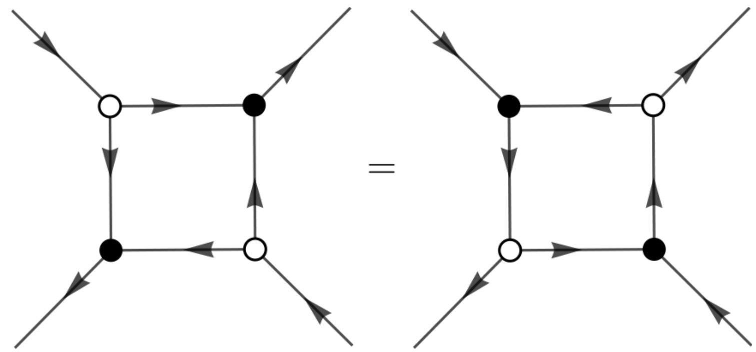
- In terms of the decorated bridge, BCFW is given by



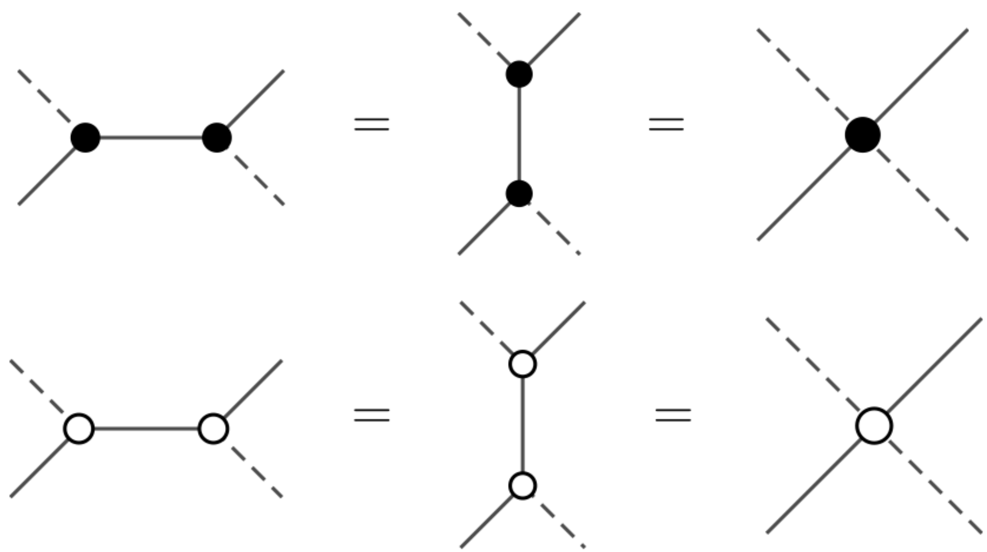
where the sum is over all partitions of particles {2,…,n-1} into two sets L,R.

Equivalence Relations

- Square move:

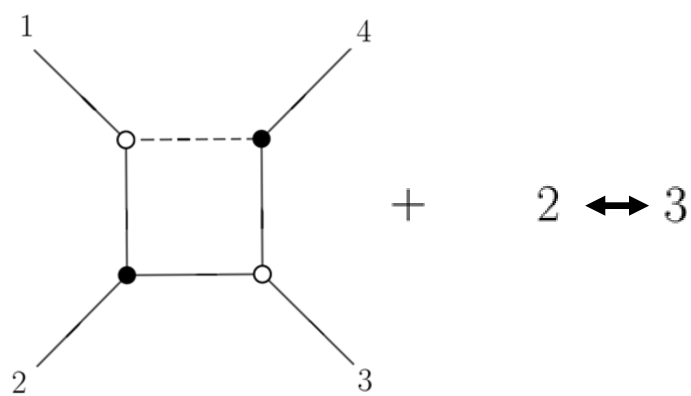


- Mergers:



Grassmannians

- The on-shell diagrams give rise to expressions in the form of integrals over k-dimensional planes in n dimensions, or equivalently the set of kxn matrices modulo the left action of GL(k), which also played a prominent role in N=4 SYM:



$$= \int \frac{d^{2\times 4}C}{GL(2)} \frac{\delta^{4|16}\left(C\cdot\tilde{\lambda}|C\cdot\tilde{\eta}\right)\delta^{4}\left(\lambda\cdot C^{\perp}\right)}{\Pi_{i<j}(ij)} \frac{\langle kl\rangle}{(kl)} \frac{[pq]}{(p^{\perp}q^{\perp})}$$

Summary

- The twistor string description of N=4 SYM can be extended to more general quantum field theories using ambitwistor string theory.
- This framework provides new insight into soft theorems and their relation to asymptotic symmetries.
- On-shell diagrams can also be generalized beyond N=4 SYM and appear to be intimately related to ambitwistor string theory.

Future Directions

- Loop amplitudes of N=8 SUGRA
- Curved backgrounds
- Amplitudes relevant to phenomenology and cosmology.
- Many more…

Thank You