New Methods in Supersymmetric Quantum Field Theory

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- We are used to the idea that a scalar particle can be composite, for example the pion π has constituents which can be resolved if one does experiments at $\sim 1GeV$.
- We are even used to the idea that a massive spin 1 particle can be composite, e.g. the rho meson ρ_μ.

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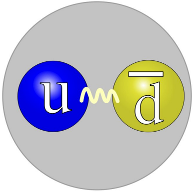
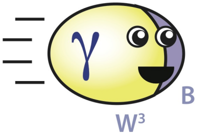


It is clearly an outstanding question to understand whether the W, Z bosons and the Higgs field are fundamental or composite.

But what about the photon?



I am not trying to suggest seriously that this is a phenomenological question that needs an urgent solution. It is just a theoretical question that I will use as an excuse to survey some recent developments in QFT.

Traditionally, one would argue that a massless gauge field A_μ cannot be composite.

- $A_\mu \simeq A_\mu + \partial_\mu \Lambda$. Necessary for unitarity. For example, if two fermions have a massless composite vector bound state, where would this gauge symmetry come from?
- Weinberg-Witten theorem: The gauge symmetry cannot be the manifestation of any conserved current in the system:

$$\langle VAC|j_\mu|A_\mu\rangle = 0 \quad \text{for all} \quad \partial^\mu j_\mu = 0$$

- A_μ couples to some conserved current (electron minus positron number). Where would this come from if, according to Witten-Witten, A_μ cannot "talk" to any current?

Where would the gauge symmetry come from if it did not exist in the fundamental theory?

Take a $U(1)$ Goldstone scalar in $2+1$-dimensional quantum field theory, $\pi \simeq \pi + f$. It can be transformed to a gauge potential

$$\partial_\mu \pi = \epsilon_{\mu\nu\rho}\partial^\nu A^\rho \ .$$

This transformation has the inherent ambiguity

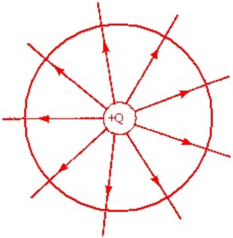
$$A_\mu \to A_\mu + \partial_\mu \Lambda \ .$$

The transformation from the fundamental degrees of freedom to the low energy degrees of freedom could have an inherent ambiguity.

Where does the current to which the gauge field couples come from?

By the Weinberg-Witten theorem, it cannot be one of the conserved currents in the theory.

But the current to which a gauge field couples is not a real current in the theory anyway: If we quantize the theory on $\mathcal{M} \times \mathbb{R}$ with compact $\mathcal{M}$, then the Hilbert space consists of gauge singlets (Gauss' law).



There is thus no theoretical obstruction for the compositeness of massless spin 1 particles. In nature, it could actually pertain to the photon and also the W, Z bosons.

The simplest existence proof is provided by $d = 4$, $\mathcal{N} = 1$ theories [Seiberg].

The simplest possible case is that we start with $SU(4)$ gauge theory with 6 fundamental fields Q_i^A and 6 anti-fundamental fields $\tilde{Q}_A^i$. This theory has a negative beta function

$$\beta \sim -3 \times 3 + 4 = -5 \ .$$

It therefore develops strong coupling at some scale Λ_{QCD} and becomes intractable.

What happens in the infrared?

Seiberg has made a guess that the infrared theory is actually weakly coupled but in terms of different variables:

It is an $SU(2)$ gauge theory with 12 fundamentals and 36 neutral scalar fields.

The beta function is

$$\beta = -6 + 6 = 0$$

at one-loop, but it is positive at two loops. So the theory is free in the infrared.

The $SU(2)$ gauge fields have nothing to do with the original $SU(4)$ gauge theory. The $SU(2)$ gauge fields are new, emergent, weakly-coupled massless composite spin 1 particles.

We often refer to this guess of the low energy degrees of freedom as "duality." It is a duality in the sense that the original strongly-coupled $SU(4)$ description has a more useful, weakly-coupled, description as an $SU(2)$ gauge theory.

Seiberg's guess is essentially based on 't Hooft anomaly matching and comparing the vacua of the two theories.

Recent developments allow to make extremely detailed tests of this proposal.

The duality is essentially non-perturbative because it involves a strongly coupled description and a weakly coupled description.

Suppose we have a theory with a conserved fermionic charge Q such that $Q^2 = H$ and $Q|boson\rangle = |fermion\rangle$ and vice versa.

Suppose $H|\Psi\rangle = E|\Psi\rangle$, $E \neq 0$. Then, $Q|\Psi\rangle = |\Psi'\rangle \neq 0$. Thus, $Tr(-1)^F = 0$ for all the states with $E \neq 0$.

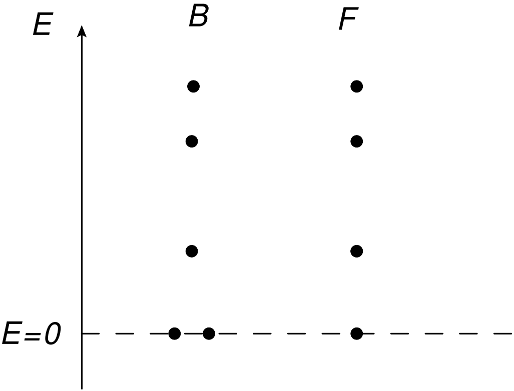
Define

$$I = Tr_{\mathcal{H}}(-1)^F$$

over the whole Hilbert space $\mathcal{H}$. The contributions only come from $H = 0$ states (vacua).

$$I = n_B - n_F$$

This is the Witten Index.



$I = 2 - 1 = 1$

Since the index does no depend on the renormalization group scale, we would like to compare the Index I of our $SU(4)$ and $SU(2)$ gauge theories.

The trouble is that it diverges. This is due to the infinitely many SUSY vacua on both sides (flat directions).

We now know how to overcome this longstanding problem! The first steps were done by Römelsberger. We can study the theory on $\mathcal{M}_3 \times \mathbb{R}$ rather than on $\mathbb{R}^4$ and consider

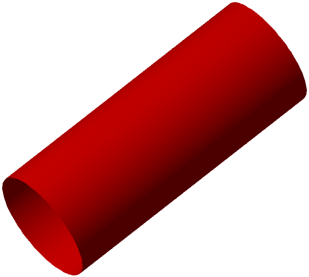
$$I(\mu_i) = Tr_{\mathcal{H}}((-1)^F e^{\mu_i q_i})$$

for various charges q_i that commute with the supercharge.

This is well defined for interesting theories such as our $SU(4)$ and $SU(2)$ theories.

In order to preserve sufficient supersymmetry, $\mathcal{M}_3$ must look locally as $S^1 \times \Sigma^{(2)}$ for some Riemann surface $\Sigma^{(2)}$ [Dumitrescu-Festuccia-Seiberg, Closset-Dumitrescu-Festuccia-ZK].

A particularly nice example is $\mathcal{M}_3 \sim S^3$ (topologically). This is what Römelsberger and subsequently many others studied.



We can imagine that we take $g_{YM} \to 0$ and remove the gauge fields. Then, the corresponding gauge chemical potentials r_i are observables.

The effect of gauging with infinitesimal g_{YM} is to integrate over r_i. Hence,

$$I_{gauged}[\mu_i] \sim \int [dr_i] \, Tr_{\mathcal{H}_{ungauged}} \left((-1)^F e^{\mu_i q_i} e^{r_i Q_i} \right)$$

$[dr_i]$ is the Haar measure.

For $\mathcal{M}_3 = S^3$ the computation was done and one finds that if $SU(4)$ indeed flows to $SU(2)$ with massless emergent gauge fields, then the following identity should hold true (schematically):

$$(p,p)^2(q,q)^2 \int \prod_{i=1,..,4} [dr_i] \frac{\prod_{i,j\leq 4} \Gamma(\mu_i r_j, 1/(\tilde{\mu}_i r_j), p, q)}{\prod_{i,j\leq 4} \Gamma(r_i/r_j, r_j/r_i, p, q)}$$

$$= \left[\prod_{i,j\leq 2} \Gamma(\mu_i/\tilde{\mu}_j, p, q)\right] \int \prod_{i=1,2} [dr_i] \frac{\prod_{i,j\leq 2} \Gamma(\mu_i r_j, 1/(\tilde{\mu}_i r_j), p, q)}{\prod_{i,j\leq 2} \Gamma(r_i/r_j, r_j/r_i, p, q)}$$

$\Gamma(\cdot)$ is the elliptic hypergeometric gamma function. $(\cdot,\cdot)$ is the q-Pochhammer symbol.

It appears that mathematicians have independently proved such identities quite recently [Spiridonov, Rains, Rahman, van de Bult...]. In particular, the identity above holds true!

The elliptic hypergeometric function Γ that appeared above is a "higher version" of the Jacobi theta function $\Theta_n(z; q)$. The latter are central in complex analysis in *two dimensions*.

It appears that the computations of

$$I(\mu_i) = Tr_{\mathcal{H}}((-1)^F e^{\mu_i q_i})$$

are closely linked with the theory of complex geometry in *four dimensions*:

- In order to preserve SUSY on some four-fold $\mathcal{M}_4$ the four-fold needs to be complex. [Klare-Tomasiello-Zaffaroni, Dumitrescu-Festuccia-Seiberg]
- If $\mathcal{M}_4 = \mathcal{M}_3 \times S^1$ then if locally $\mathcal{M}_3 = S^1 \times \Sigma^{(2)}$ we have complex structure and a holomorphic Killing vector, guaranteeing 2 supercharges.
- The Index is independent of the Hermitian metric on $\mathcal{M}_4$, and it depends only on the complex structure. [Closset-Dumitrescu-Festuccia-ZK]
- The Index I is thus a function of the complex structure parameters τ_i and of the moduli of holomorphic vector bundles. This is the geometric meaning of μ_i.
- The complex structure moduli space of $S^3 \times S^1$ is two-complex dimensional [Kodaira-Spencer], accounting for the parameters p, q that appeared above.

The index only depends on non-metric data, i.e. the complex structure moduli.

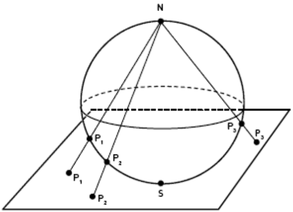
This is somewhat reminiscent of twisting in $\mathcal{N} = 2$, where the theory becomes completely topological.

We see that one can also “twist” $\mathcal{N} = 1$ theories, but one gets a theory of the complex structure moduli space, rather than a topological theory.

The study of four-dimensional theories on spaces like $\mathcal{M}_3 \times S^1$ opened many new directions but at the end it is a combinatorial object that counts the states in the Hilbert space of the theory on $\mathcal{M}_3$.

Now I would like to briefly describe recent developments of an altogether different sort, where one has access to objects which are not combinatorial in nature.

Suppose we have a Conformal Field Theory (CFT). Then we can also place it on the space $\mathbb{S}^4$ via the stereographic map.



This is an angle-preserving transformation,

$$ds^2 = dx^i dx^i \longrightarrow \frac{1}{(x^2 + r^2)^2} dx^i dx^i \ .$$

Now the theory is free of infrared divergences because space is compact. The UV divergences are the same as in flat space. One can therefore try to compute

$$Z_{S^d} \equiv \int [DX] e^{-S[X; g_{ij}]} \; , \qquad g_{ij} = \delta_{ij} \frac{1}{(x^2 + r^2)^2}$$

Assume the theory has various **exactly** marginal coupling constants λ^i. Then we can compute

$$Z_{S^4}(\lambda^i)$$

Is this well defined?

The partition function has various power divergences with the UV cutoff Λ_{UV} of the sort

$$\log Z_{S^4} = \Lambda_{UV}^4 r^4 + \Lambda_{UV}^2 r^2 + ...$$

which correspond to the counter-terms

$$\Lambda_{UV}^4 \int d^4x \sqrt{g} + \Lambda_{UV}^2 \int d^4x \sqrt{g} R + \cdots$$

Each of these can be multiplied by an arbitrary function of the λ^i. and we can have both a log and a finite piece

$$\log Z_{S^4} = \cdots + a(\lambda^i) \log(r\Lambda_{UV}) - F(\lambda^i) \ .$$

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easy to prove that $a(\lambda^i) = a$. It counts degrees of freedom and decreases along renormalization group flows

$$a_{UV} > a_{IR} \ .$$

It can also be interpreted as the Entanglement Entropy inside a ball.
The term $F(\lambda^i)$ is unphysical. It can be removed by the counter-term

$$\int d^4x F(\lambda^i) E_4$$

with E_4 the Euler density.

It would therefore seem that $\log Z_{S^4}$ only contains the a-anomaly and no other physical information. However, it turns out that the counter-term

$$\int d^4x F(\lambda^i) E_4$$

is not generally allowed in $\mathcal{N} = 2$ supersymmetric theories.

It is therefore not entirely surprising that for $\mathcal{N} = 2$ theories in $d = 4$ we find

$$Z_{S^4} = r^{-4a} e^{K(\lambda,\bar{\lambda})/12}$$

with λ the coefficients of exactly marginal operators

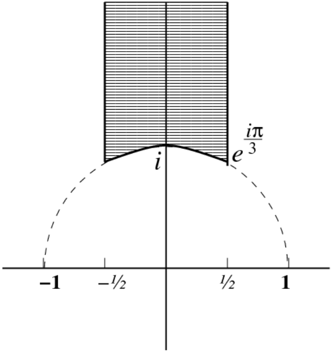
$$\delta S = \sum_i \lambda^i \int d^4x d^4\theta \mathcal{O}_i + c.c.$$

The Kähler potenitial K is not a holomorphic quantity and so is not accessible via standard techniques that have been utilized in the 90s.

With supersymmetric localization [Pestun] we can now compute these sphere partition functions and thus find the metric on the space of theories.
This is a very interesting non-perturbative observable which has various applications.

Consider for instance Seiberg-Witten theory with $SU(2), N_f = 4$.

$$ds^2 = d\tau d\bar{\tau}\left[\frac{3}{8(Im\tau)^2} - \frac{135\zeta(3)}{32\pi^2(Im\tau)^4} - \frac{1575\zeta(5)}{64\pi^3(Im\tau)^5} + \cdots \right.$$
$$\left. - \cos(\theta)e^{-8\pi^2/g^2}\frac{3}{4\pi(Im\tau)^{3/2}} + \cdots\right]$$



These agree with previous perturbative computations. One can also make contact with the general question of whether instantons and perturbation theory are linked in QFT. One can prove that the perturbative series is Borel summable (no poles on the positive Borel plane) [Russo, Honda]. It is therefore unclear how to extract the instantons from the perturbative series.
One can also show that Pade approximations converge exponentially fast (conjectured in the context of QCD by Karliner et al.).

Various other applications of the ideas above

- Many checks of dualities, new dualities...
- Wilson loops expectation values (non-perturbative) [Pestun...]
- Exact computations of the extremal correlators [Gerchkovitz et al, Baggio et al ...]
- Relations between field theories in different dimensions [Alday-Gaiotto-Tachikawa,...]
- Novel tests of AdS/CFT [see e.g. Martelli-Sparks, Cassani-Martelli,...]
- Monotonicity of Renormalization Group Flows in $d = 3$ [Jafferis-Klebanov-Pufu-Safdi...]
- Many new relations to mathematics

We see that the field of non-perturbative aspects of supersymmetric field theories is rapidly evolving and major longstanding problems have been recently solved. This has opened the way for many applications in physics and mathematics.

Thank You For Your Attention