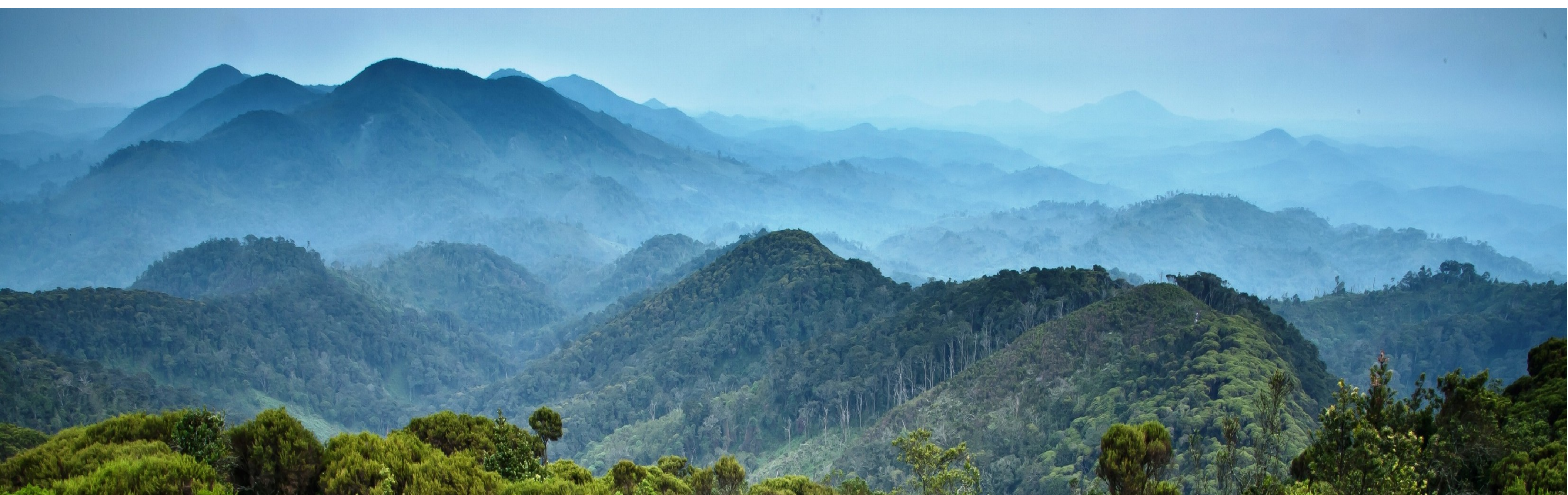
Forest Formula, IR Divergences & 2pt-functions

Franz Herzog (Nikhef)

Future Challenges for Precision QCD workshop (Durham)
27.10.2016

Collaborators: Ben Ruijl; Jos Vermaseren, Takahiro Ueda, Andreas Vogt; Konstantin Chetyrkin



A view of the Congo's primary forests from Nyungwe National Park, Rwanda. Photo © Dr Liana Joseph

Overview

- Introduction

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Prehistory

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R^*-OPERATION CORRECTED

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and

V.A. SMIRNOV

Nuclear Physics Institute of Moscow State University, Moscow, USSR

Some Applications of R & R*

- Z,tau → Hadrons / Z-decay: QCD at N4LO
  (2012 Baykov, Chetyrkin, Kühn) Global R*

- Anomalous Dimensions:
  - QCD
    - 5-loop Beta function
      (2016 Baykov, Chetyrkin, Kühn) Global R*
    - 5-loop quark mass and field anomalous dimension
      (2014 Baykov, Chetyrkin, Kühn) Global R*
  - Phi4
    - 6-loop (Batkovich, Chetyrkin, Kompaniets)
      (2016 Batkovich, Chetyrkin, Kompaniets) Global & Local R*

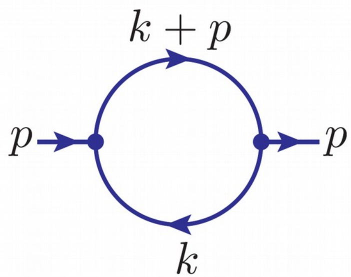
- Decay Rates: QCD corrections
  - H→bb N4LO (2005 Baykov, Chetyrkin, Kühn) Global & Local R*

- g-2: QED corrections
  - 5 loop (2014 Kinoshita,Aoyama, Hayakawa, Kinoshita, Nio)
    Local R*(?) subtraction, even numerical

BPHZ R-operation in the MS scheme

Example of a massive Feynman Graph:

$G =$



$$= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2+m^2)((k+p)^2+m^2)}$$

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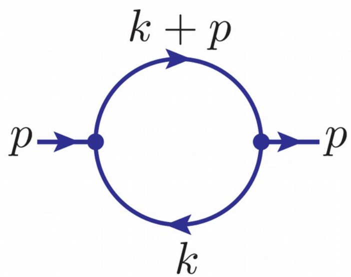
$G =$ 

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$$G = RG + KG$$

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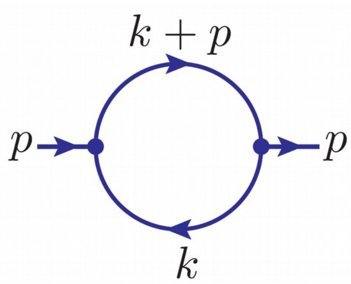
$G =$  $= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2+m^2)((k+p)^2+m^2)}$

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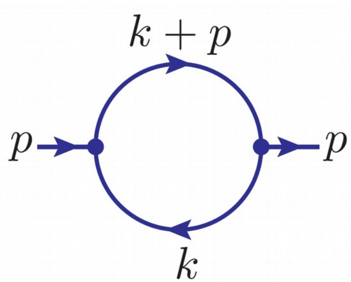

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What is R?

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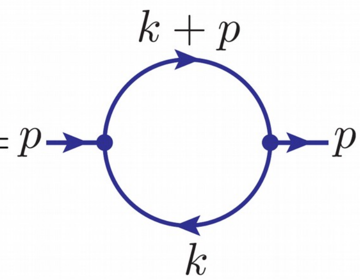
- R can be also be expressed as a recursive subtraction operator
- The solution to this recursive subtraction operator is Zimmermann's Forest Formula

UV Divergences at 1 loop

$G =$ 

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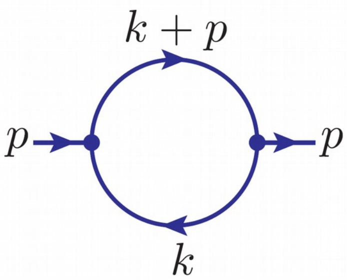
$G =$ 

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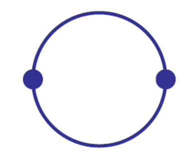
$$RG = G + \Delta_G^{\mathrm{mom}} \qquad \Delta_G^{\mathrm{mom}} = -G\Big|_{p=0} = -$$ 

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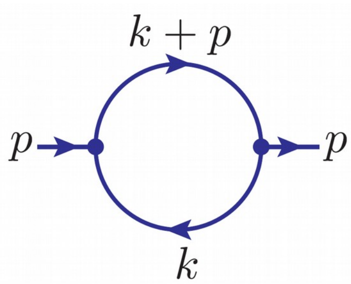
$G =$ 

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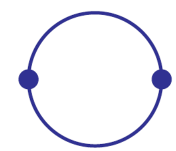
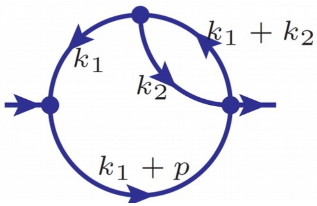
$$RG = G + \Delta_G^{\mathrm{mom}} \qquad \Delta_G^{\mathrm{mom}} = -G\big|_{p=0} = -$$ 

Works because G and Δ_G^{mom} have identical singular limits:

$$\lim_{k\to\infty} -\Delta_G \sim \lim_{k\to\infty} G \sim \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{[k^2]^2}$$

UV Divergences at 2 loops

$G =$ 

$$= \int \frac{d^D k_1}{i\pi^{D/2}} \frac{d^D k_2}{i\pi^{D/2}} \frac{1}{(k_1^2 + m^2)((k_1 + k_2)^2 + m^2)(k_2^2 + m^2)((k_1 + p)^2 + m^2)}$$

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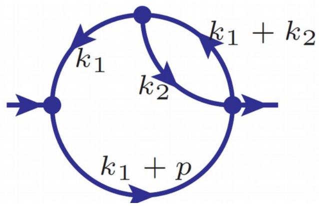
$G =$ 

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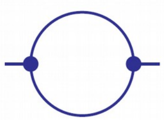
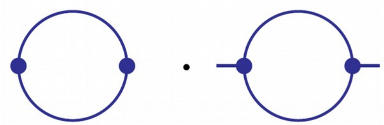
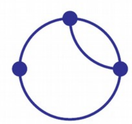
UV divergent (sub) graphs:

- $\gamma =$  : $\lim_{k_2 \to \infty} G \sim$ 
- $\gamma = G$: $\lim_{k_1, k_2 \to \infty} G \sim$ 

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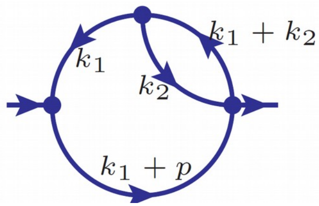
$G =$ 

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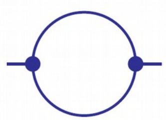
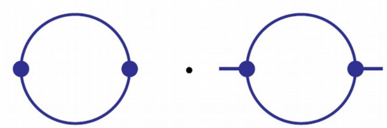
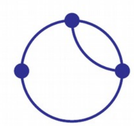
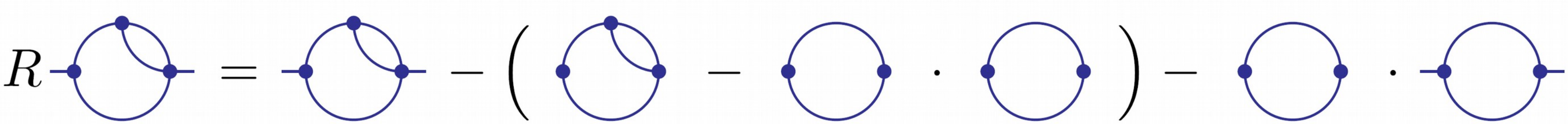
- $\gamma =$  : $\lim_{k_2 \to \infty} G \sim$ 
- $\gamma = G$: $\lim_{k_1, k_2 \to \infty} G \sim$ 

The R-operation will construct a counter term from the above limits.
Overlapping divergences are avoided by recursive subtraction of sub-divergences



G $\quad = \Delta_G$ $\quad = \Delta_\gamma * G/\gamma$

UV Divergences at L loops in the MS scheme

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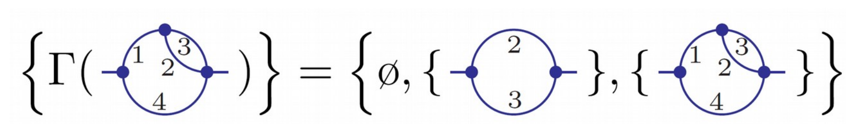
- The set $\{\Gamma(G)\}$ is a called a wood, .e.g.



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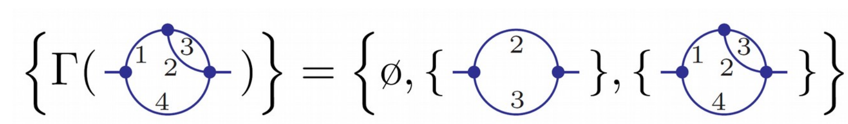
- The set $\{\Gamma(G)\}$ is a called a wood, .e.g. 

- Δ_γ Is a counter term operation. It isolates the singuarity due to the divergent subgraph.

$$\Delta_\gamma = -\sum_{n=0}^{\omega(\gamma)} K\bar{R}\mathcal{T}_p^{(n)}\gamma \qquad \bar{R}G = \sum_{\bar{\Gamma}} \prod_{\gamma \in \Gamma} \Delta_\gamma * G/\Gamma$$

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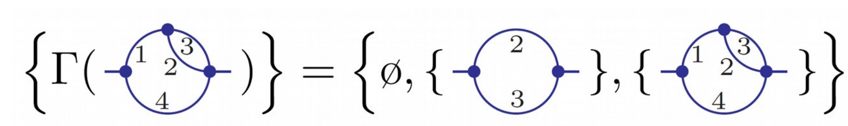

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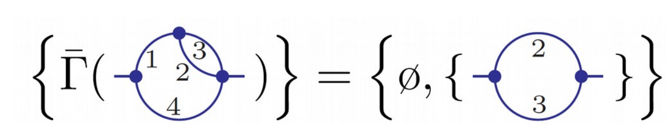
- $\bar{\Gamma}$ Does not include the full graph G.
- $\mathcal{T}_p^{(n)}$ is a Taylor expansion operator.



What happens when $m \to 0$?

Introducing R*

$$G = R^* G + KG$$

- G may have massless lines, but massive external lines

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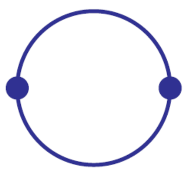
- In MS scheme R*=1-K

Why are scaleless integrals zero in dimension regularisation?



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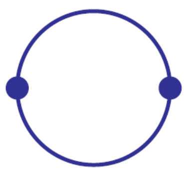

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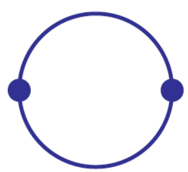

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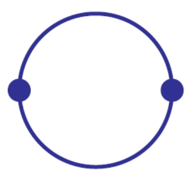

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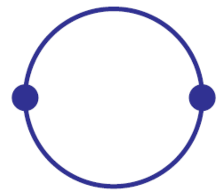
$k \to \infty \qquad k \to 0$

To see this just insert $\quad 1 = \frac{k^2}{k^2 + m^2} + \frac{m^2}{k^2 + m^2}$

Extracting IR poles from UV poles

R^*  $= 0$

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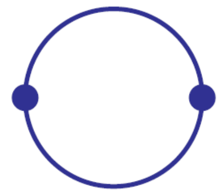
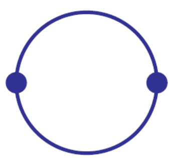
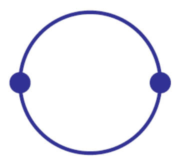
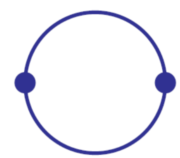
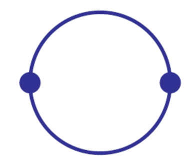
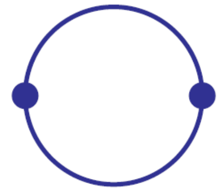
R^*  $= 0$

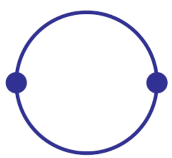
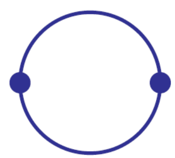
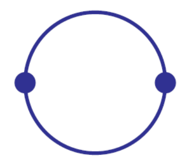
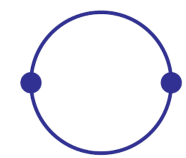
R^*  $=$  $- K\bar{R}^*$  $- K\underline{R}^*$  $= 0$

UV counter term

IR counter term

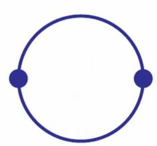
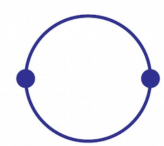
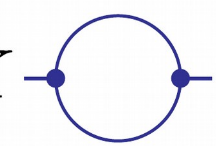
Extracting IR poles from UV poles

R^*  $= 0$

R^*  $=$  $- K\bar{R}^*$  $- K\underline{R}^*$  $= 0$

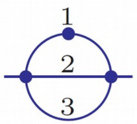
UV counter term

IR counter term

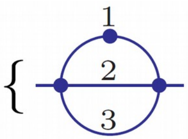
$\Rightarrow$ $K\underline{R}^*$  $= -K\bar{R}^*$  $= -K$ 

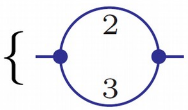
Infrared Rearrangement (Vladimirov 1980)

Subtraction of IR and UV at 2 loops

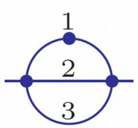
R^*  $=$

Subtraction of IR and UV at 2 loops

UV divergences: $\{\Gamma(G)\} = \{\emptyset, \{$ 

$\}, \{$ 

$\}\}$

R^* 

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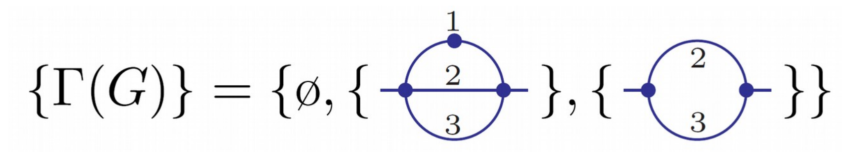

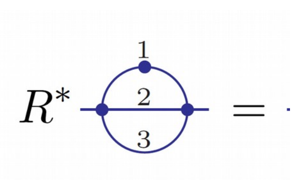

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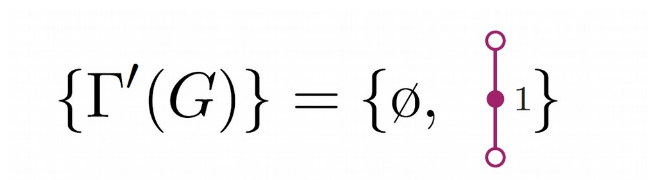

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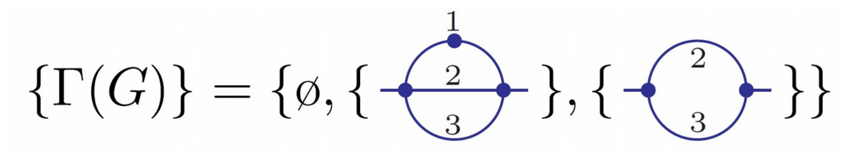

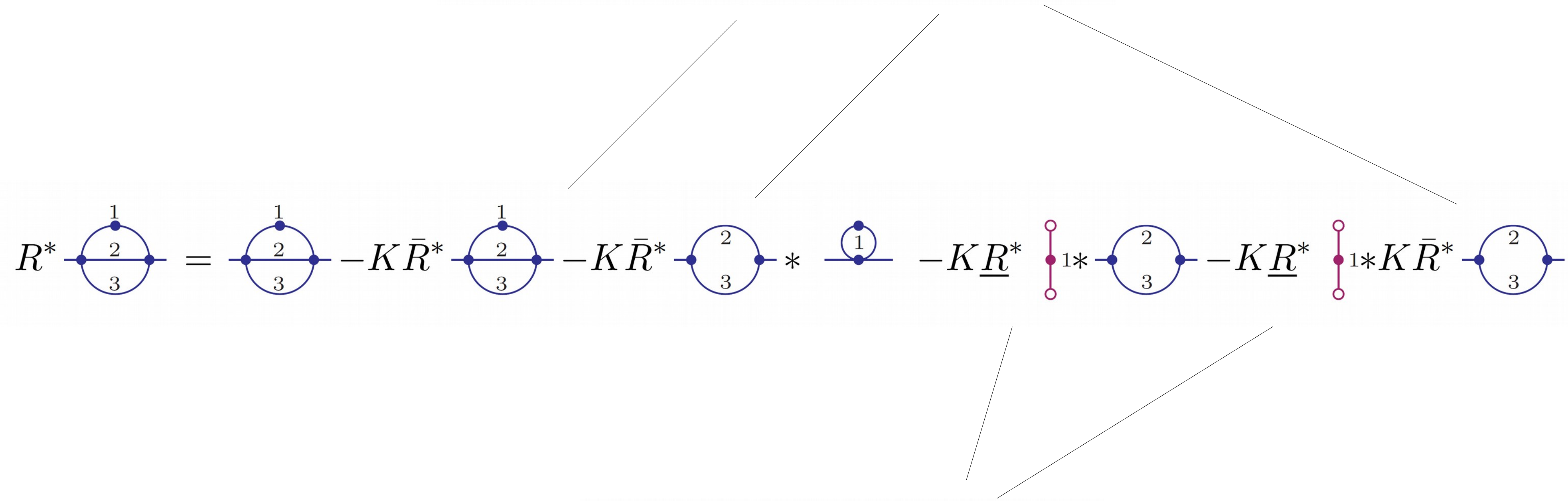

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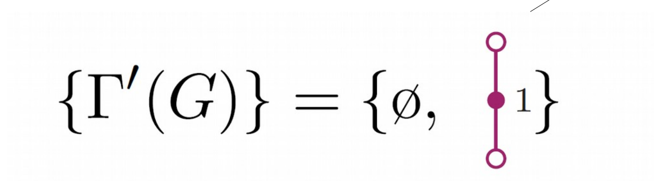

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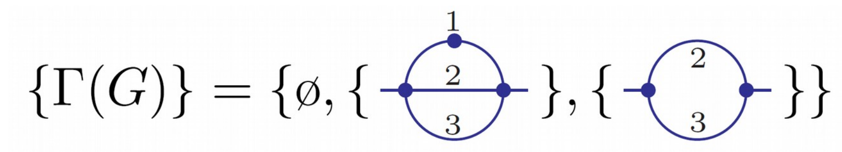

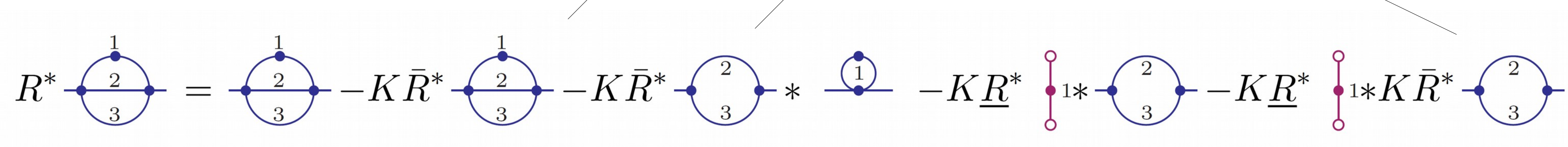

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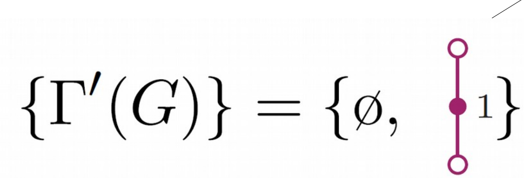
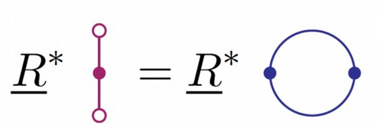


"UV Rearrangement"



Subtraction of IR and UV at L loops

(for massive external legs)

$$R^*G = \sum_{\Gamma \cap \Gamma' = \emptyset} \prod_{\gamma' \in \Gamma'} \Delta'_{\gamma'} * \prod_{\gamma \in \Gamma} \Delta_\gamma * G/\Gamma \backslash \Gamma'$$

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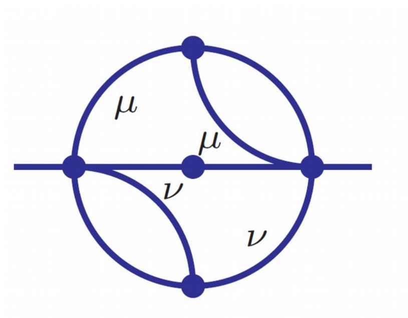
- $G/\Gamma\backslash\Gamma'$ is constructed by contracting UV subgraphs to points and deleting IR subgraphs

Automation:
Straight forward but tedious!

- Have automated the R*-operator for arbitrary Tensor integrals with massive external legs, and massless internal legs, in both FORM and Maple.



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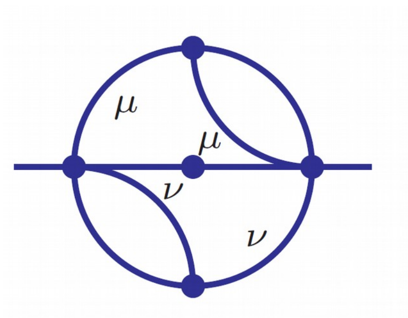

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- Challenges:
  - Subgraph finding:
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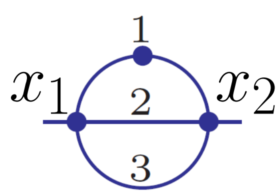

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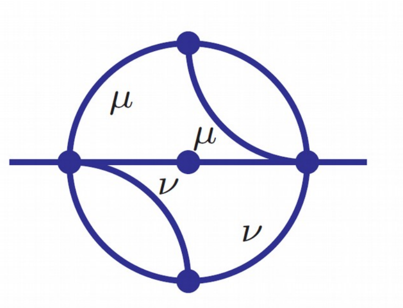

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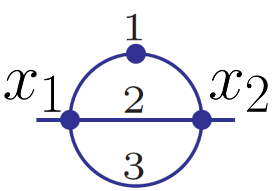

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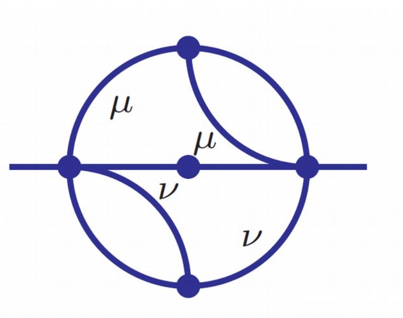

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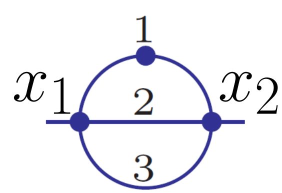

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  - Counter term construction:
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  - Evaluation of counter-terms: use Forcer, a highly optimised 4-loop reduction routine in FORM (Ruijl, Ueda, Vermaseren to be published)

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- Long-term Aims: Can we learn something for LHC Physics?

    - Generalise R* to massless external states (?)

    - Automate expansions by region using subgraph search