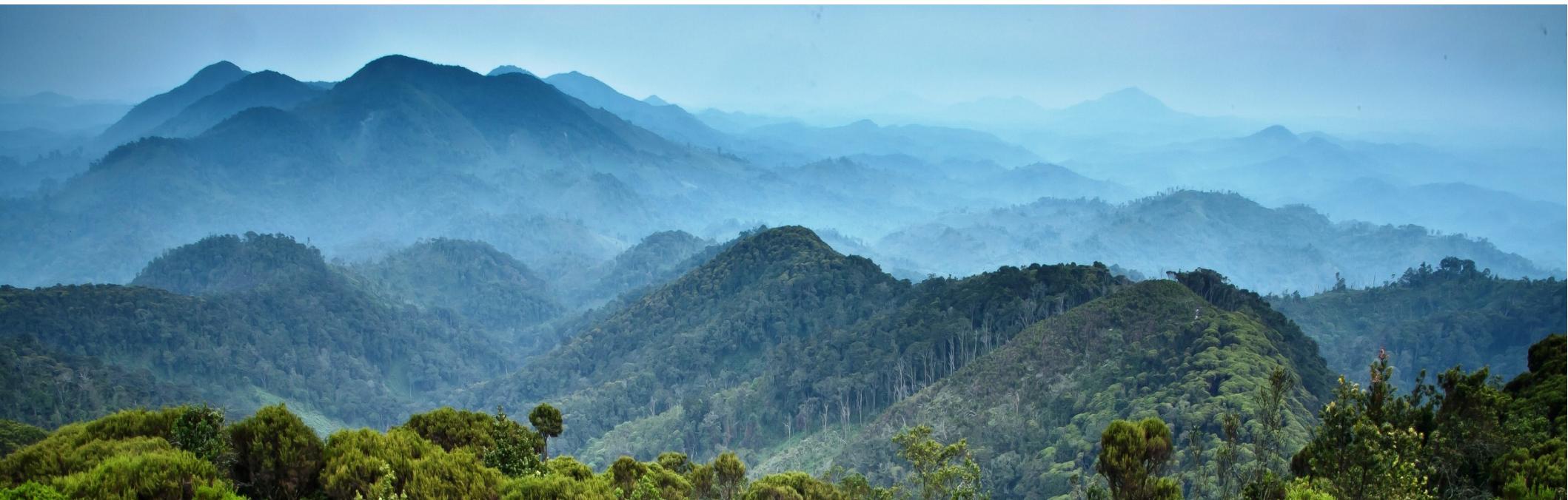


# Forest Formula, IR Divergences & 2pt-functions

Franz Herzog (Nikhef)

Future Challenges for Precision QCD workshop (Durham)  
27.10.2016

Collaborators: Ben Ruijl; Jos Vermaseren, Takahiro Ueda, Andreas Vogt; Konstantin Chetyrkin



A view of the Congo's primary forests from Nyungwe National Park, Rwanda. Photo © Dr Liana Joseph

# Overview

- Introduction

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- Crash course on R, R\*-operations

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- Automation and application to 5-loop beta function (in general gauge group)

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- Automation and application to 5-loop beta function (in general gauge group)
- Outlook

# Prehistory

On the Multiplication of the causal function in the quantum theory of fields  
**ÜBER DIE MULTIPLIKATION DER KAUSALFUNKTIONEN IN  
DER QUANTENTHEORIE DER FELDER**

VON

1956

N. N. BOGOLIUBOW und O. S. PARASIUK

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The Institute for Advanced Study, Princeton, New Jersey

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K.G. CHETYRKIN and F.V. TKACHOV

*Institute for Nuclear Research of the Academy of Sciences of the USSR,  
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6 September 1984

**R\*-OPERATION CORRECTED**

K.G. CHETYRKIN

*Institute for Nuclear Research of the Academy of Sciences of the USSR, Moscow, USSR*

and

V.A. SMIRNOV

*Nuclear Physics Institute of Moscow State University, Moscow, USSR*

# Some Applications of R & R\*

- $Z,\tau \rightarrow \text{Hadrons} / Z\text{-decay}$ : QCD at N4LO  
**(2012 Baykov, Chetyrkin, Kühn) Global R\***
- Anomalous Dimensions:
  - QCD
    - 5-loop Beta function  
**(2016 Baykov, Chetyrkin, Kühn) Global R\***
    - 5-loop quark mass and field anomalous dimension  
**(2014 Baykov, Chetyrkin, Kühn) Global R\***
  - Phi4
    - 6-loop (Batkovich, Chetyrkin, Kompaniets)  
**(2016 Batkovich, Chetyrkin, Kompaniets) Global & Local R\***
- Decay Rates: QCD corrections
  - $H \rightarrow bb$  N4LO **(2005 Baykov, Chetyrkin, Kühn) Global & Local R\***
- g-2: QED corrections
  - 5 loop **(2014 Kinoshita,Aoyama, Hayakawa, Kinoshita, Nio)**  
Local R\*(?) subtraction, even numerical

# BPHZ R-operation in the MS scheme

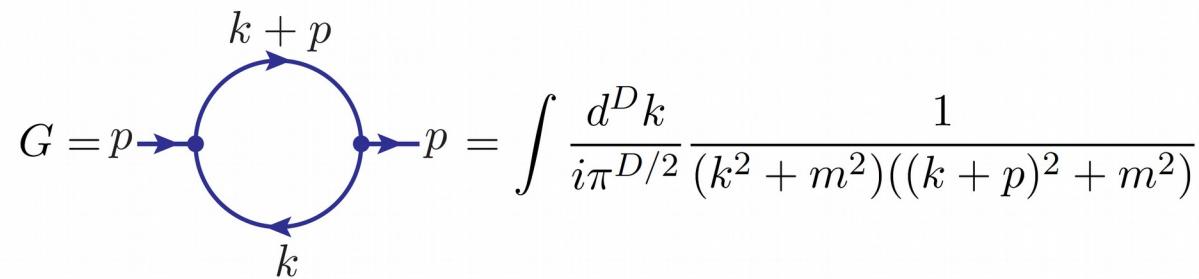
Example of a massive Feynman Graph:

$$G = p \rightarrow \text{Feynman Graph} \rightarrow p = \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2 + m^2)((k + p)^2 + m^2)}$$

The Feynman graph consists of a circular loop with two external lines. The left external line is labeled  $p$  and the right external line is labeled  $p$ . The top arc of the loop is labeled  $k + p$  and the bottom arc is labeled  $k$ . Arrows on the lines indicate the direction of momentum flow.

# BPHZ R-operation in the MS scheme

Example of a massive Feynman Graph:

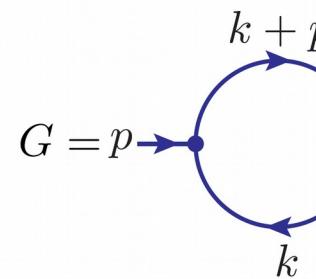
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The R-operation allows to separate finite and divergent parts

$$G = RG + KG$$

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$$KG = \sum_{n=-L}^{-1} G^{(n)} \epsilon^n$$

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$$RG = \sum_{n=0}^{\infty} G^{(n)} \epsilon^n \quad \begin{matrix} / \\ KG = \sum_{n=-L}^{-1} G^{(n)} \epsilon^n \end{matrix}$$

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- R renormalizes a Feynman diagram

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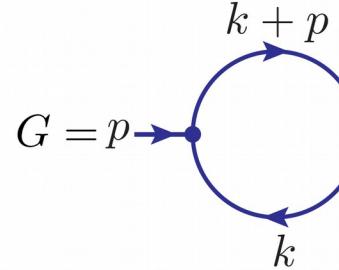
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- R renormalizes a Feynman diagram
- R is scheme dependent
- $R=1-K$  in the MS scheme
- R can be also be expressed as a recursive subtraction operator
- The solution to this recursive subtraction operator is Zimmermann's Forest Formula

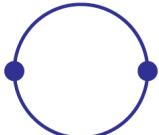
# UV Divergences at 1 loop

$$G = p \rightarrow \text{circle} \rightarrow p = \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2 + m^2)((k + p)^2 + m^2)}$$


# UV Divergences at 1 loop

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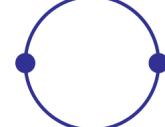

$$RG = G + \Delta_G^{\text{mom}}$$

$$\Delta_G^{\text{mom}} = -G|_{p=0} = - \bullet \text{---} \circlearrowleft \text{---} \bullet$$


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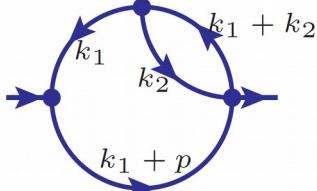

$$RG = G + \Delta_G^{\text{mom}}$$

$$\Delta_G^{\text{mom}} = -G|_{p=0} = - \text{circle}$$


Works because  $G$  and  $\Delta_G^{\text{mom}}$  have identical singular limits:

$$\lim_{k \rightarrow \infty} -\Delta_G \sim \lim_{k \rightarrow \infty} G \sim \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{[k^2]^2}$$

# UV Divergences at 2 loops

$$G = \text{Diagram} = \int \frac{d^D k_1}{i\pi^{D/2}} \frac{d^D k_2}{i\pi^{D/2}} \frac{1}{(k_1^2 + m^2)((k_1 + k_2)^2 + m^2)(k_2^2 + m^2)((k_1 + p)^2 + m^2)}$$


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The diagram shows a two-loop Feynman graph. It consists of two external lines meeting at a central vertex. A loop is attached to this vertex. The top arc of the loop is labeled  $k_1$ , the bottom arc is labeled  $k_2$ , and the rightmost line of the loop is labeled  $k_1 + k_2$ . The leftmost line of the loop is labeled  $k_1 + p$ .

UV divergent (sub) graphs:

- $\gamma = \text{Diagram}$  :  $\lim_{k_2 \rightarrow \infty} G \sim \text{Diagram} \cdot \text{Diagram}$
- $\gamma = G$  :  $\lim_{k_1, k_2 \rightarrow \infty} G \sim \text{Diagram}$

# UV Divergences at 2 loops

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The diagram shows a two-loop Feynman graph. It consists of two nested loops. The outer loop has a clockwise arrow and vertices labeled  $k_1 + k_2$  and  $k_1 + p$ . The inner loop also has a clockwise arrow and vertex labels  $k_1$  and  $k_2$ .

UV divergent (sub) graphs:

- $\gamma = \text{Diagram}$  :  $\lim_{k_2 \rightarrow \infty} G \sim \text{Diagram} \cdot \text{Diagram}$
- $\gamma = G$  :  $\lim_{k_1, k_2 \rightarrow \infty} G \sim \text{Diagram}$

The R-operation will construct a counter term from the above limits.  
Overlapping divergences are avoided by recursive subtraction of sub-divergences

$$R \text{Diagram} = \text{Diagram} - \left( \text{Diagram} - \text{Diagram} \cdot \text{Diagram} \right) - \text{Diagram} \cdot \text{Diagram} = \Delta_G = \Delta_\gamma * G / \gamma$$

The R-operation is defined as the difference between the full two-loop graph and the sum of its two UV divergent subgraphs. This results in a finite counter term  $\Delta_G$ , which is proportional to the full graph  $G$  divided by the divergence factor  $\gamma$ .

# UV Divergences at L loops in the MS scheme

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# UV Divergences at L loops in the MS scheme

$$RG = \sum_{\Gamma} \prod_{\gamma \in \Gamma} \Delta_{\gamma} * G / \Gamma$$

- Each  $\Gamma$  is called a spinney, a set of disjoint 1PI UV divergent subgraphs, it contains also the empty and the full graph.
- The set  $\{\Gamma(G)\}$  is called a wood, e.g.  $\left\{ \Gamma(\text{-}\bullet^1 \text{-} \bullet^2 \text{-} \bullet^3 \text{-} \bullet^4) \right\} = \left\{ \emptyset, \left\{ \text{-}\bullet^2 \text{-} \bullet^3 \text{-} \bullet^4 \right\}, \left\{ \text{-}\bullet^1 \text{-} \bullet^2 \text{-} \bullet^3 \text{-} \bullet^4 \right\} \right\}$

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- $\Delta_{\gamma}$  Is a counter term operation. It isolates the singularity due to the divergent subgraph.

$$\Delta_{\gamma} = - \sum_{n=0}^{\omega(\gamma)} K \bar{R} \mathcal{T}_p^{(n)} \gamma \quad \bar{R}G = \sum_{\bar{\Gamma}} \prod_{\gamma \in \bar{\Gamma}} \Delta_{\gamma} * G / \Gamma$$

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- $\bar{\Gamma}$  Does not include the full graph  $G$ .
- $\mathcal{T}_p^{(n)}$  is a Taylor expansion operator.

$$\left\{ \bar{\Gamma}(\text{---}\bullet^1_2\bullet^3_4\text{---}) \right\} = \left\{ \emptyset, \{ \text{---}\bullet^2_3\text{---} \} \right\}$$



<http://www.wallpapermania.eu/>

What happens when  $m \rightarrow 0$  ?

# Introducing $R^*$

$$G = R^* G + K G$$

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- $R^*$  subtracts only those IR Divergences associated to massless internal lines
- In MS scheme  $R^*=1-K$

Why are scaleless integrals zero in dimension regularisation?

$$\text{Diagram: A circle with two dots on its circumference.} = \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{[k^2]^2}$$

Why are scaleless integrals zero in dimension regularisation?

$$\text{Diagram} = \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{[k^2]^2} = \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} = 0$$

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$$k \rightarrow \infty \quad k \rightarrow 0$$

To see this just insert  $1 = \frac{k^2}{k^2 + m^2} + \frac{m^2}{k^2 + m^2}$

# Extracting IR poles from UV poles

$$R^* \circ = 0$$

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$$R^* \circlearrowleft = 0$$

$$R^* \circlearrowleft = \circlearrowleft - K \bar{R}^* \circlearrowleft - K \underline{R}^* \circlearrowleft = 0$$

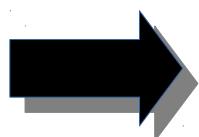
UV counter term                    IR counter term

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UV counter term                    IR counter term



$$K \underline{R}^* \text{ } \circlearrowleft = -K \bar{R}^* \text{ } \circlearrowleft = -K \text{ } \circlearrowleft$$

Infrared Rearrangement (Vladimirov 1980)

# Subtraction of IR and UV at 2 loops

$$R^* - \text{Diagram} = .$$

The diagram consists of a circle with three external legs. The top leg has a dot at its vertex with the number 1 above it. The bottom-left leg has a dot at its vertex with the number 3 below it. The bottom-right leg has a dot at its vertex with the number 2 between 1 and 3.

# Subtraction of IR and UV at 2 loops

UV divergences:  $\{\Gamma(G)\} = \{\emptyset, \{ \text{---} \circlearrowleft \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \text{---} \}, \{ \text{---} \circlearrowright \begin{array}{c} 2 \\ 3 \end{array} \text{---} \}\}$

$$R^* \text{---} \circlearrowleft \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \text{---} = .$$

# Subtraction of IR and UV at 2 loops

UV divergences:

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IR divergences:

$$\{\Gamma'(G)\} = \{\emptyset, \{ \text{---} \begin{array}{c} \circ \\[-1ex] \bullet \\[-1ex] \circ \end{array} \text{---} \}\}$$

# Subtraction of IR and UV at 2 loops

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$$R^* \text{---} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} = \text{---} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} - K \bar{R}^* \text{---} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} - K \bar{R}^* \text{---} \begin{array}{c} 2 \\ 3 \end{array} * \text{---} \begin{array}{c} 1 \end{array} - K \underline{R}^* \text{---} \begin{array}{c} 1 \\ 1* \end{array} \text{---} \begin{array}{c} 2 \\ 3 \end{array} - K \underline{R}^* \text{---} \begin{array}{c} 1 \\ 1* \end{array} K \bar{R}^* \text{---} \begin{array}{c} 2 \\ 3 \end{array}$$

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$$\{\Gamma'(G)\} = \{\emptyset, \text{---} \begin{array}{c} 1 \end{array}\}$$

# Subtraction of IR and UV at 2 loops

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IR divergences:

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"UV Rearrangement"  $\underline{R}^* \text{---} \begin{array}{c} \circ \\ \bullet \\ \circ \end{array} = \underline{R}^* \text{---} \begin{array}{c} \bullet \\ \circlearrowleft \end{array}$

# Subtraction of IR and UV at L loops

(for massive external legs)

$$R^*G = \sum_{\Gamma \cap \Gamma' = \emptyset} \prod_{\gamma' \in \Gamma'} \Delta'_{\gamma'} * \prod_{\gamma \in \Gamma} \Delta_\gamma * G / \Gamma \backslash \Gamma'$$

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IR counter term:

$$\Delta'_{\gamma'} = -K \underline{R}^* \tilde{\gamma}' \mathcal{T}_p^{(\omega)}$$

UV counter term:

$$\Delta_\gamma = -K \bar{R}^* \mathcal{T}_p^{(\omega)} \gamma$$

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- $\tilde{\gamma}'$  is the contracted vacuum IR subgraph

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(for massive external legs)

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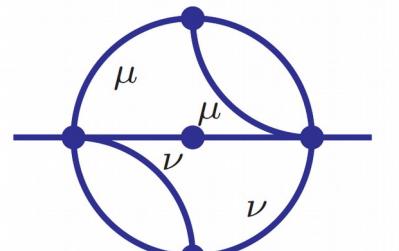
UV counter term:

$$\Delta_\gamma = -K \bar{R}^* \mathcal{T}_p^{(\omega)} \gamma$$

- $\tilde{\gamma}'$  is the contracted vacuum IR subgraph
- $G/\Gamma \setminus \Gamma'$  is constructed by contracting UV subgraphs to points and deleting IR subgraphs

# Automation: Straight forward but tedious!

- Have automated the  $R^*$ -operator for arbitrary Tensor integrals with massive external legs, and massless internal legs, in both FORM and Maple.

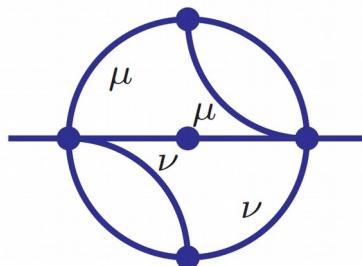


A Feynman diagram representing the  $R^*$ -operator. It consists of a horizontal line with three vertices. The leftmost vertex has a loop attached to it, labeled with indices  $\mu$  and  $\nu$ . The middle vertex also has a loop attached, labeled with index  $\mu$ . The rightmost vertex has a loop attached, labeled with index  $\nu$ .

$$\underline{\mu\nu} = \frac{q^\mu q^\nu}{q^2}$$

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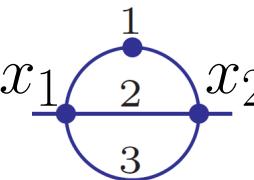
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- Challenges:

- Subgraph finding:

- Use x-representation

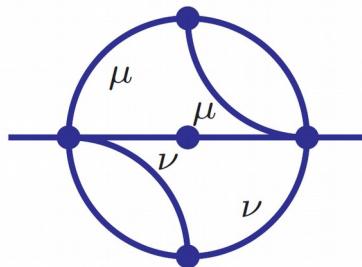


A Feynman diagram showing a central circular loop with three external lines. The top line is labeled with index 1 at its top vertex. The bottom line is labeled with index 3 at its bottom vertex. The right line is labeled with index 2 at its right vertex. The left vertex of the loop is connected to two other lines, which are part of a larger diagram.

$$x_1 \circlearrowleft^1 \underset{2}{\circlearrowright} \underset{3}{\circlearrowright} x_2 = l_1(x_1, x_2) l_2(x_1, x_2) l_3(x_1, x_2)$$

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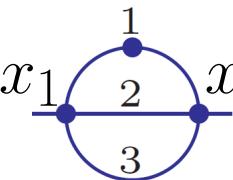


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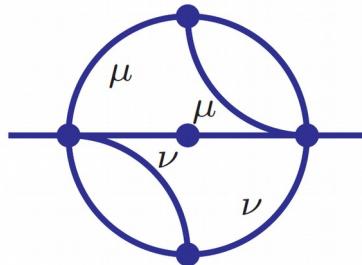
A Feynman diagram representing a subgraph  $l_3$ . It consists of a circle with three external lines. The top line is labeled with index 1 at its top vertex. The bottom line is labeled with index 3 at its bottom vertex. The right line is labeled with index 2 at its right vertex. The internal lines are labeled 1, 2, and 3.

$$x_1 \circlearrowleft^1 \underset{2}{\text{---}} \underset{3}{\text{---}} \circlearrowright^{x_2} = l_1(x_1, x_2) l_2(x_1, x_2) l_3(x_1, x_2)$$

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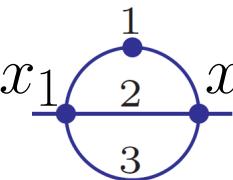


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$$x_1 \circlearrowleft^1 \underset{2}{\circlearrowright} \circlearrowright^3 x_2 = l_1(x_1, x_2) l_2(x_1, x_2) l_3(x_1, x_2)$$

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  - Deletion and contraction of subgraphs
  - Differential Operators
- Evaluation of counter-terms: use Forcer, a highly optimised 4-loop reduction routine in FORM (Ruijl, Ueda, Vermaseren to be published)

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- Long-term Aims: Can we learn something for LHC Physics?
  - Generalise  $R^*$  to massless external states (?)
  - Automate expansions by region using subgraph search