A novel approach to the cuts of Feynman integrals

Claude Duhr

in collaboration with S. Abreu, R. Britto & E. Gardi

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Challenges for loop integrals

- Algebraic structure of polylogarithms & differential equations.
  - ➡ How does this generalise to elliptic functions?
  - ➡ Coproducts, symbols, etc. for elliptic functions?

- Construct integrands from unitarity approaches at two loops.
  - ➡ Use cuts to 'project out' master integrals from amplitudes.
  - ➡ Technically, need to find a 'master contour' for each integral.
  - ➡ Many open questions:
    - – Are there enough master contours? Uniqueness?
    - – Why do integrals over master contours satisfy IBPs (but leading singularities do not)?

Challenges for loop integrals

- Aim of this talk:
  - ➡ Discuss some possible avenues to address these issues.
  - ➡ Argue that the 2 questions (special functions & unitarity) may be connected!
  - ➡ Take first steps towards a better understanding of the analytic & algebraic structure of Feynman integrals.
  - ➡ Maybe physics intuition may help to clarify some open questions in pure mathematics..?
- Disclaimer: Many of the ideas are new and under development!
  - ➡ Will discuss mostly one-loop integrals.
  - ➡ General picture emerges, but still a lot to do to go to two loops!

Outline

- Quick review of polylogarithms and their coproduct.
- Cut integrals & homology theory.
- The coproduct of one-loop integrals.
- Outlook & Conjectures.

Quick review of polylogarithms and their coproduct

Polylogarithms

- Large classes of loop integrals can be expressed in terms of polylogarithms.

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1}\, G(a_2,\ldots,a_n;t)$$

$$G(a_1;z) = \log\left(1-\frac{z}{a_1}\right)$$

$$G(0,1;z) = -\text{Li}_2(z)$$

- Polylogarithms form a Hopf algebra. [Goncharov; Brown]

➡ Allows one to 'break' polylogarithms into smaller pieces:

$$\Delta(\log z) = 1\otimes \log z + \log z \otimes 1$$

$$\Delta(\text{Li}_n(z)) = 1\otimes \text{Li}_n(z) + \sum_{k=0}^{n-1} \text{Li}_{n-k}(z)\otimes \frac{\log^k z}{k!}$$

- The two factors encode discontinuities & differential equations:

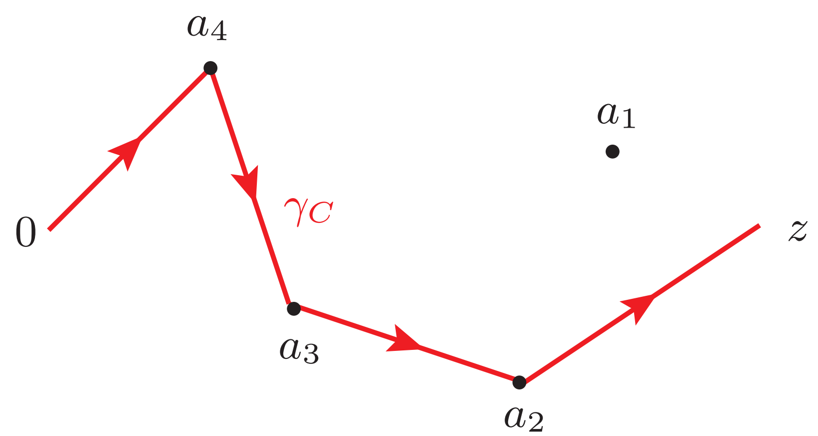
$$\Delta\text{Disc} = (\text{Disc}\otimes \text{id})\Delta \qquad \Delta\partial_z = (\text{id}\otimes\partial_z)\Delta$$

The coproduct

- General formula:

$$\Delta(G(\vec{a};z)) = \sum_{\vec{b}\subset\vec{a}} G(\vec{b};z) \otimes G_{\vec{b}}(\vec{a};z)$$

Integral over a contour that encircles the singularities in $\vec{b}$ = 'Cut Integral'



$\vec{a} = a_1a_2a_3a_4$ $\vec{b} = a_2a_3a_4$

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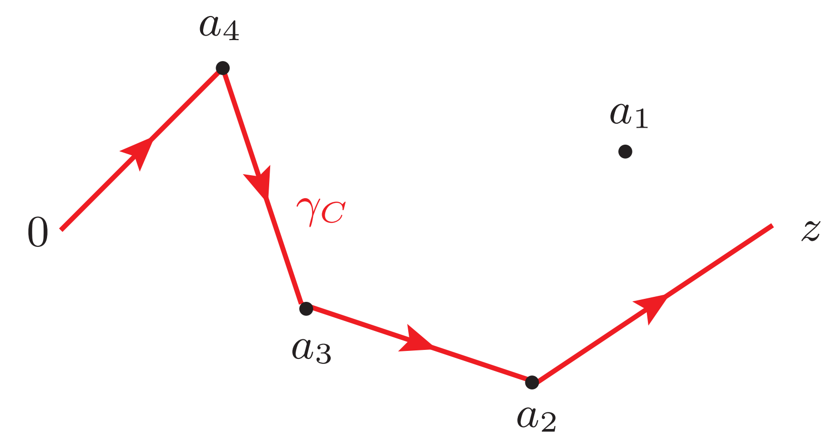

$\vec{a} = a_1a_2a_3a_4$ $\quad$ $\vec{b} = a_2a_3a_4$

- Does this picture generalise to other functions?
  - ➡ Answer is 'Yes' [Brown]:

$$\Delta([\gamma,\omega]^{\mathfrak{m}}) = \sum_i [\gamma,\omega_i]^{\mathfrak{m}} \otimes [\omega_i^\dagger,\omega]^{\mathfrak{dr}}$$

Sum over MIs $\quad$ Master integrals $\quad$ Cut

$$[\gamma,\omega]^{\mathfrak{m}} \sim \int_\gamma \omega$$

- Goal: Make this formula precise!
  - ➡ First step towards understanding mathematical structure of functions that appear in loops and are more polylogarithms.

Cuts integrals & homology theory

Cut integrals

- Traditional definition: replace propagators by delta functions:

$$\frac{1}{p^2-m^2+i\varepsilon} \longrightarrow 2\pi i\,\delta_+(p^2-m^2)$$

[Cutkosky; 't Hooft, Veltmann]

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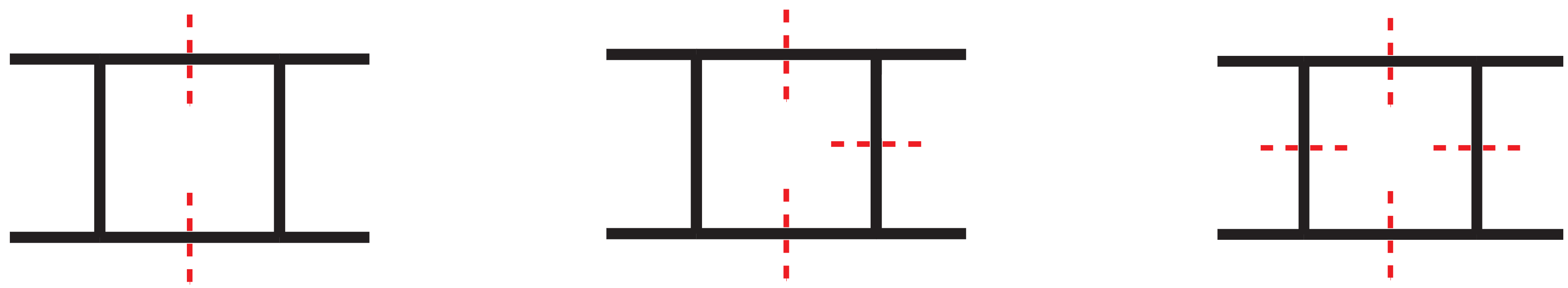
- Folklore:
  - ➡ 'Cuts compute discontinuities' - Which ones..?



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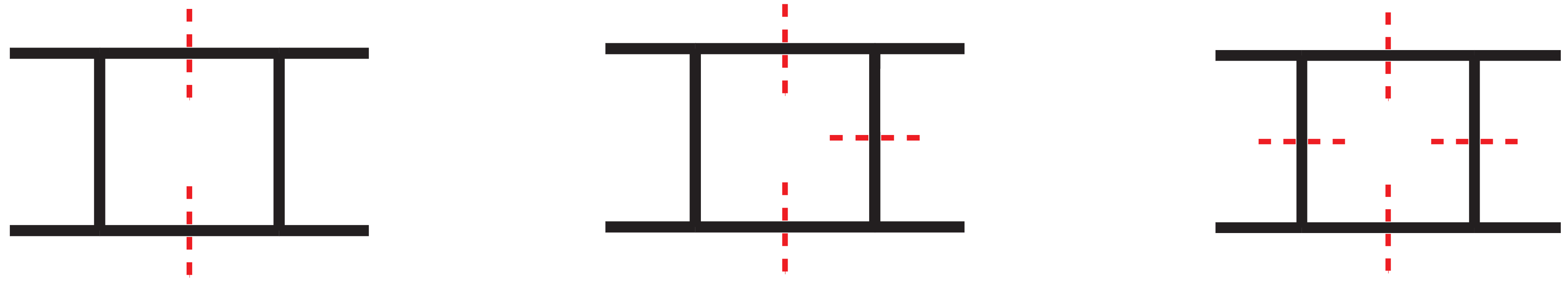

  - ➡ 'Cuts are computed by integrating over a contour that encircles the poles of some propagators' - Which contour..?

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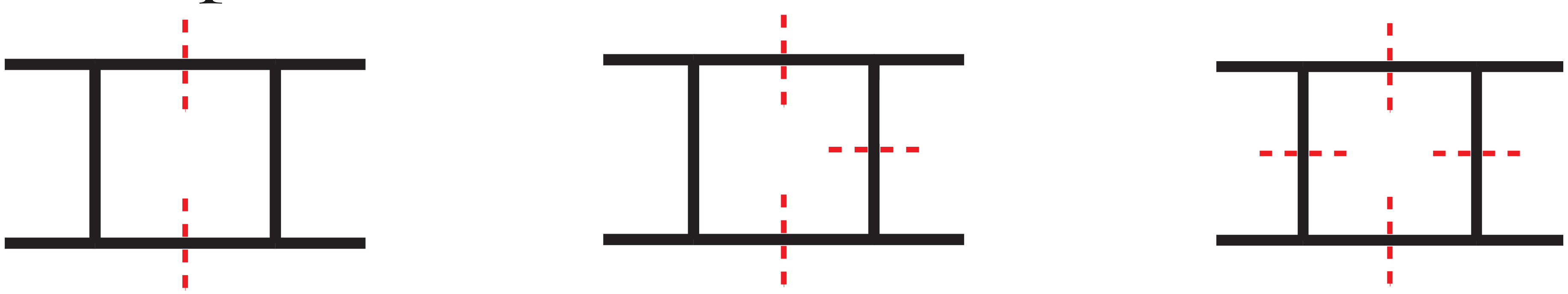
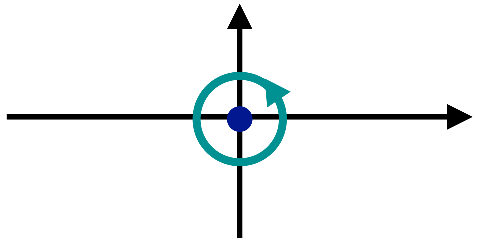


  - ➡ 'Cuts are computed by integrating over a contour that encircles the poles of some propagators' - Which contour..?
  - ➡ 'Leading singularities do not satisfy IBPs… but some linear combinations do!' - What about reverse-unitarity..?

Cut integrals

- Which contours..?
  - ➡ Turns the problem into a problem in homology theory!
- Homology groups: ~ all inequivalent integration contours we can define in our space.
- Example: The plane minus the origin: $\mathbb{C} \setminus \{0\}$



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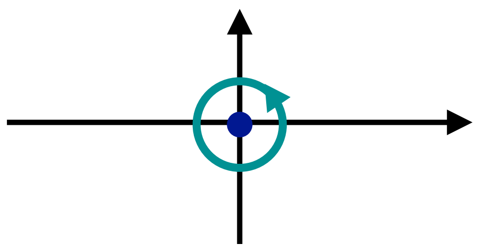

- Homology groups associated to Feynman integrals have been studied in the 60s. [Fotiadi, Pham; Teplitz, Hwa; Federbusch; Landshof, Polkinghorne, ...]
  - ➡ Contours for cuts can be unambiguously defined.
  - ➡ Every cut integrals computes a discontinuity, associated to some Landau singularity (1st & 2nd kind)
  - ➡ Cut integrals always satisfy the same IBP relations and differential equations as uncut integrals.

Homology groups

- At one-loop: interesting contours 'encircle' propagator poles and/or pinch singularity at infinity:

$$\Gamma_{\emptyset}, \qquad \Gamma_{\infty}, \ \Gamma_{1}, \ \Gamma_{2}, \ldots \qquad \Gamma_{12}, \ldots \qquad \Gamma_{\infty 12}, \ldots$$

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- Homology theory: Contours that do not encircle ∞ form a basis:

$$\Gamma_{\infty C} = -2\,\Gamma_C + \sum_X (-1)^{\lfloor |C|/2 \rfloor + \lceil |X|/2 \rceil} \Gamma_X$$

$C \subseteq \{1, 2 \ldots\}$

$|C|$ odd

[Fotiadi, Pham]

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[Fotiadi, Pham]

- Alternate basis: $\Gamma_{\emptyset} \ldots \Gamma_{\infty 123}\,, \Gamma_{1234} \ldots$

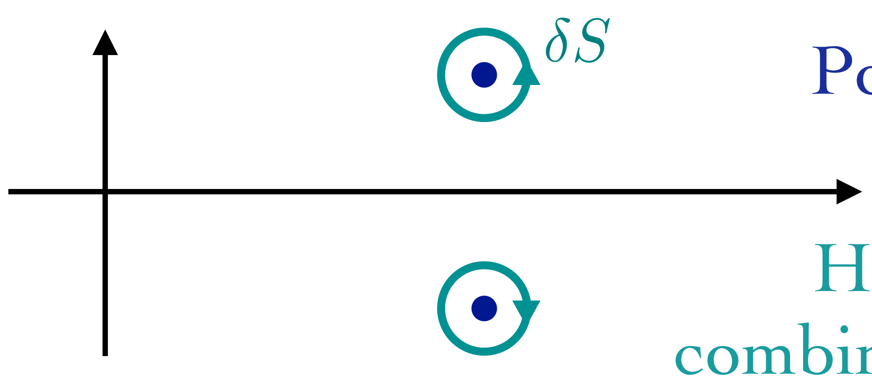
  ➡ Master contours at one loop! [Britto, Cachazo, Feng; Forde; …]

- There is two-loop literature on the homology groups of the double box! [Federbusch]

  ➡ Does this provide two-loop master contours?!

Master contours

- Consequence: Cut integrals always satisfy IBPs!
  ➡ Contradiction with literature…?
- Let's look at the quadruple cut at one-loop:



Poles of quadruple cut

Homology groups contains linear combination that gives the box coefficient!

Master contours

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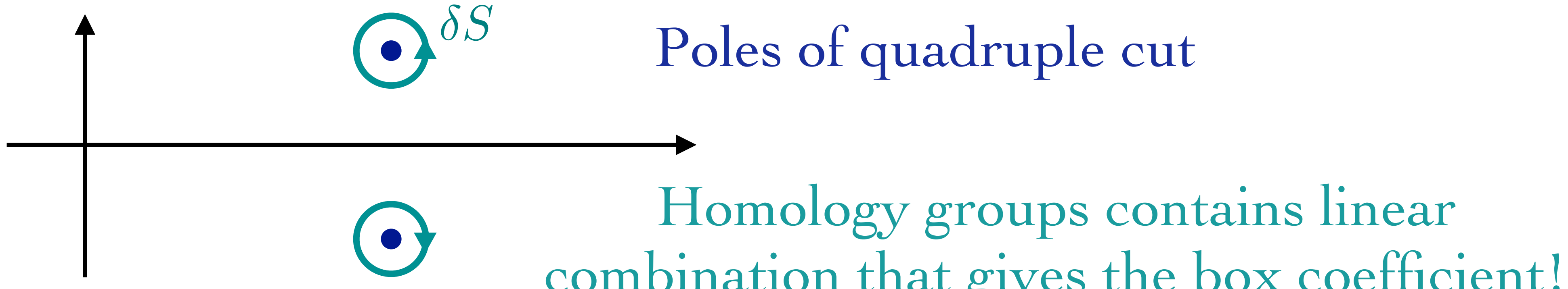


  - ➡ Individual residues do not satisfies IBPs, but the integral over δS does!
- Conclusion: Master contours should not be seen as leading singularities, but as discontinuities!
  - ➡ These contours are dictated by homology theory.

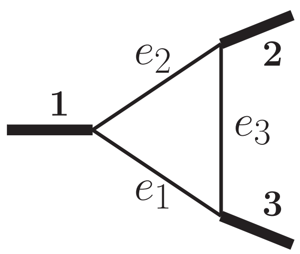
The coproduct
of one-loop integrals

The diagrammatic coaction

$$\Delta([\gamma,\omega]^{\mathfrak{m}}) = \sum_i [\gamma,\omega_i]^{\mathfrak{m}} \otimes [\omega_i^{\dagger},\omega]^{\mathfrak{dr}}$$

Sum over MIs   MI   Cut

- Let us analyse the triangle with massless propagators:

$\Delta \Big[$  $\Big]$

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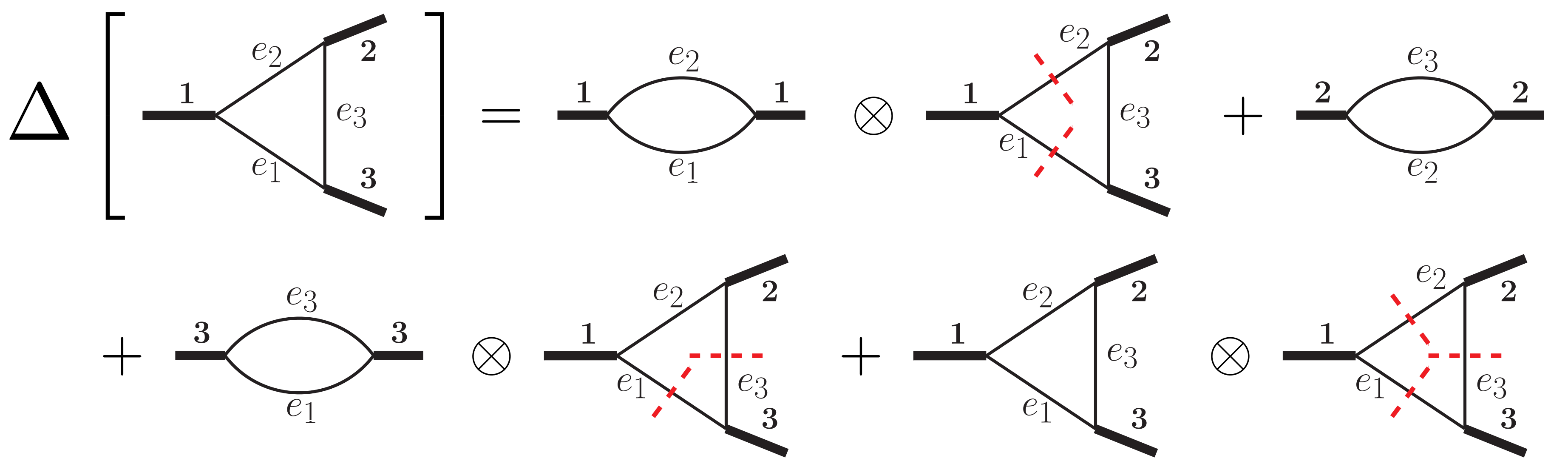
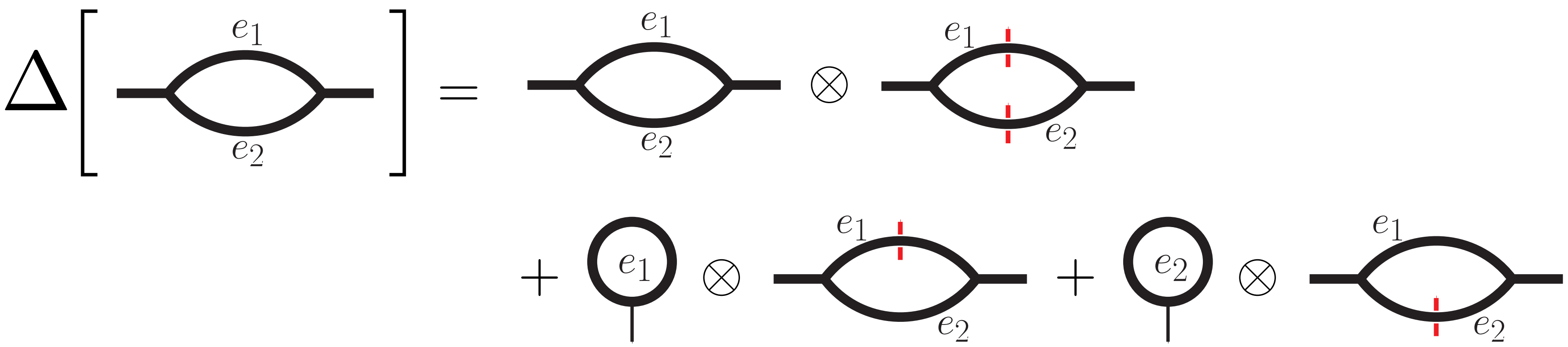


➡ Checked up to terms of weight 4.

➡ Requires highly non-trivial conspiracy of terms!

The diagrammatic coaction

- Bubble with massive propagators:



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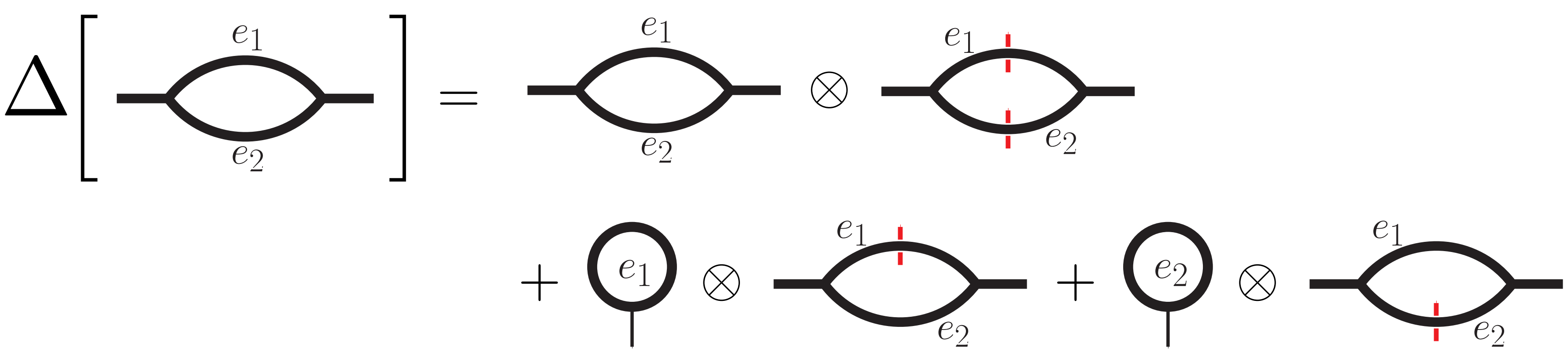

➡ This relation is incorrect…

The diagrammatic coaction

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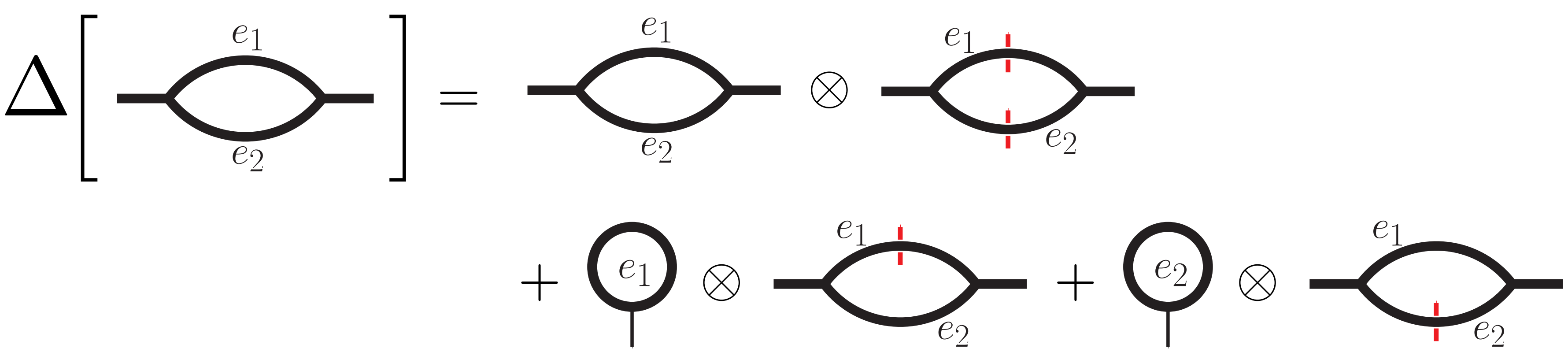
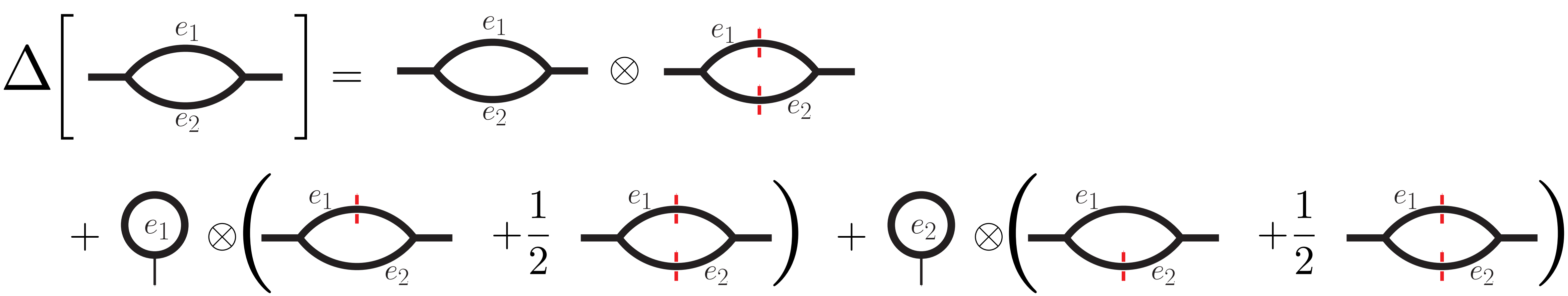
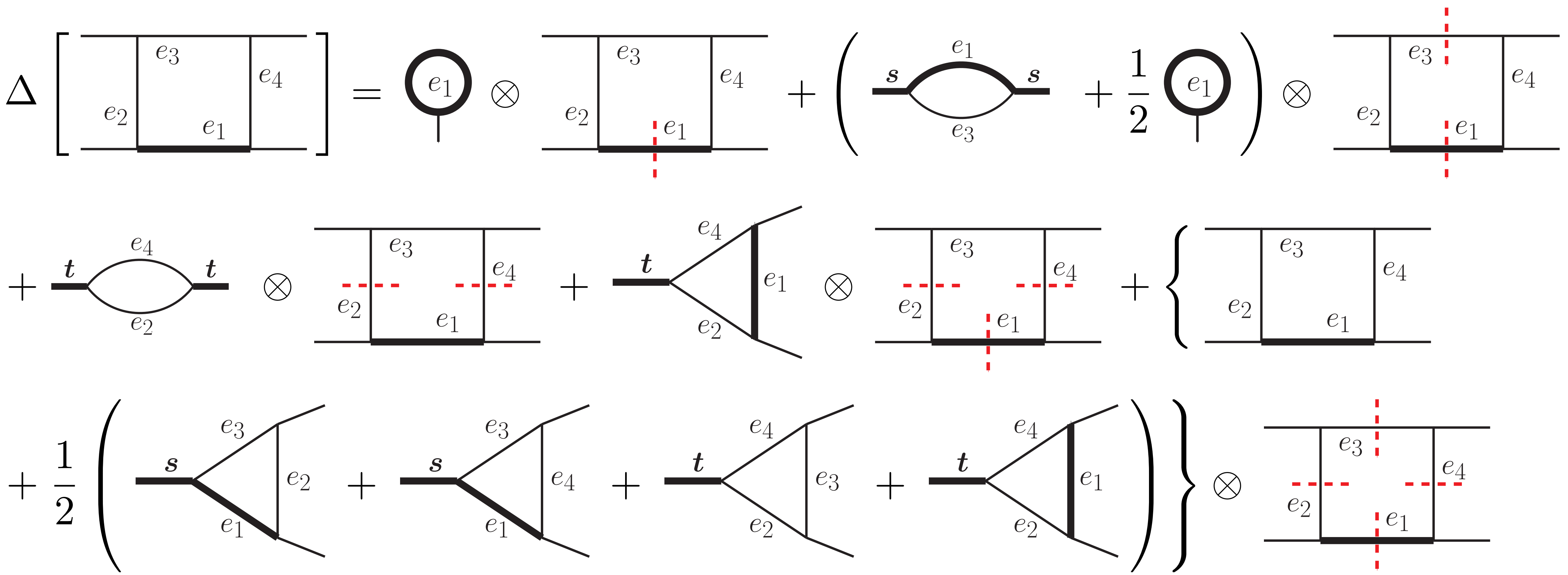


➡ This relation is incorrect…

- … but the following relation holds!

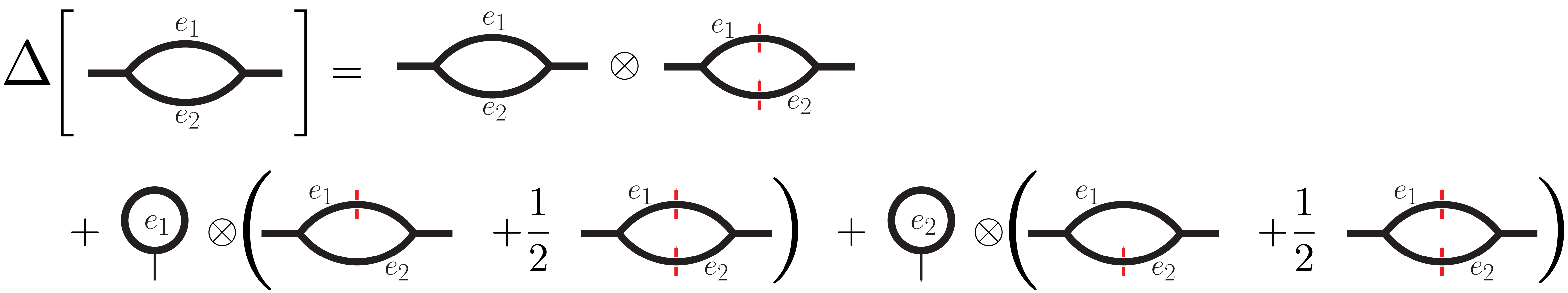


- # Example:



The diagrammatic coaction

- What is the meaning of the 1/2 term..?



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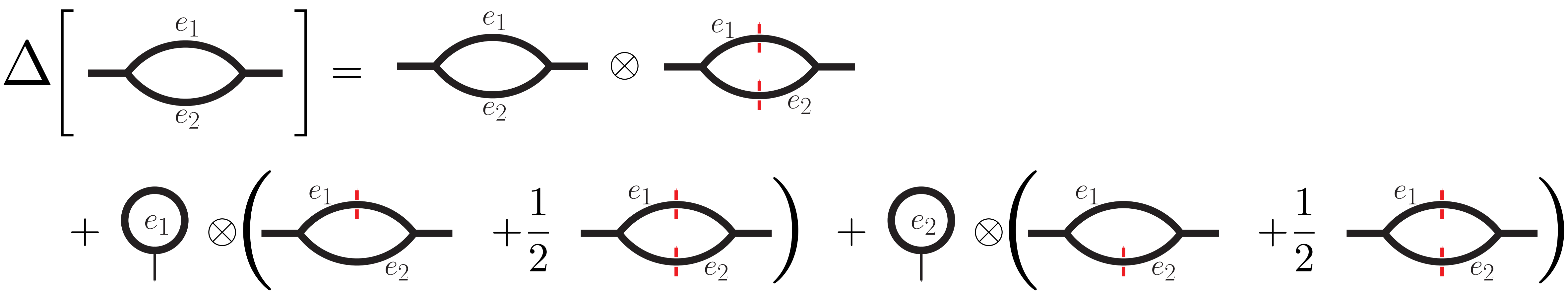
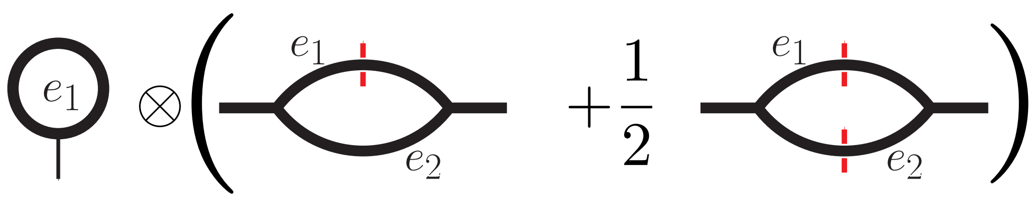
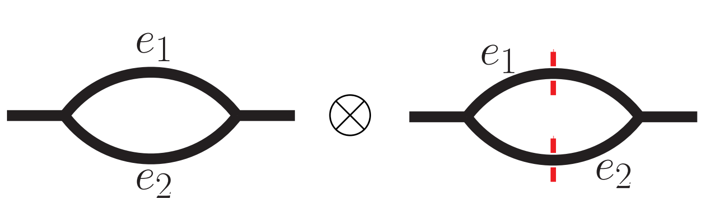


- Using the 'homological relation' $\Gamma_{\infty 1} = -2\Gamma_1 - \Gamma_{12}$, we find



$$= \int \omega_1 \otimes \left(\int_{\Gamma_1} \omega_2 + \frac{1}{2} \int_{\Gamma_{12}} \omega_2 \right)$$

$$= \int \omega_1 \otimes \left(-\frac{1}{2} \int_{\Gamma_{\infty 1}} \omega_2 \right)$$



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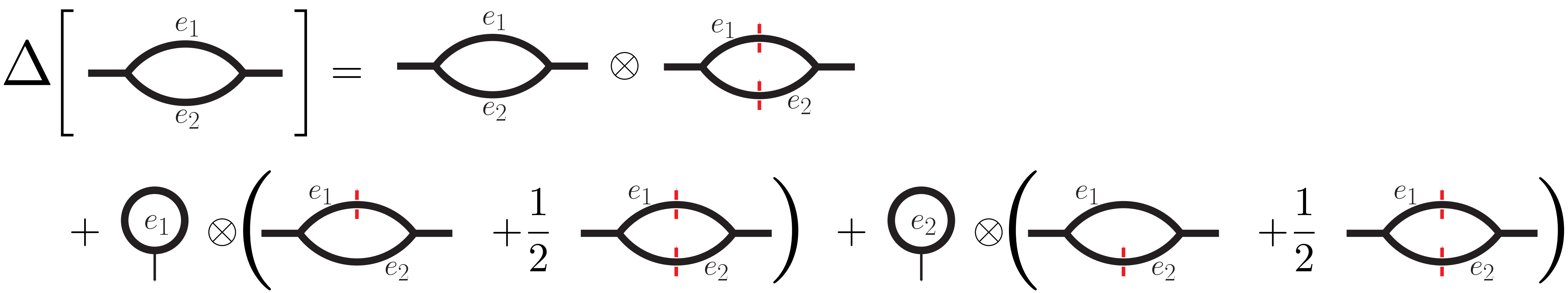

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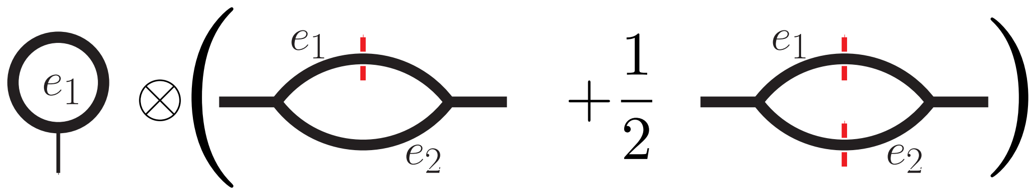

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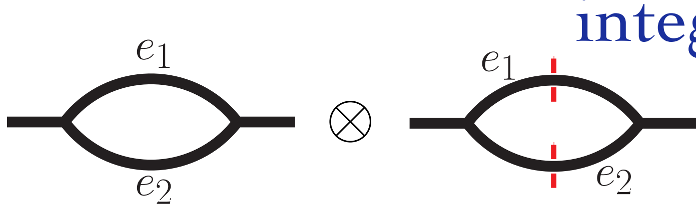
Master integrands



$$= \int \omega_2 \otimes \int_{\Gamma_{12}} \omega_2$$

Master contours

Outlook
&
Conjectures

The Master formula

- Brown's motivic coaction:

$$\Delta([\gamma,\omega]^{\mathfrak{m}}) = \sum_i [\gamma,\omega_i]^{\mathfrak{m}} \otimes [\omega_i^\dagger,\omega]^{\mathfrak{dr}} \qquad [\gamma,\omega]^{\mathfrak{m}} \sim \int_\gamma \omega$$

Sum over MIs — Master integrals — Cut

- Conjecture:

$$\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$$

[Abreu, Britto, CD, Gardi]

➡ γ_i is the master contour of the master integrand ω_i.

- Works for multiple polylogs and one-loop integrals.

➡ Does it have any predictive power?

➡ Does predict the correct results?

Hypergeometric functions

- Consider the integrals

$$T(a_1, a_2, a_3; z) = \int_0^1 \underbrace{dx\, x^{a_1}\,(1-x)^{a_2}\,(1-zx)^{a_3}}_{=\omega(a_1,a_2,a_2)} \qquad a_i = n_i + \alpha_i \epsilon \qquad \begin{array}{r} n_i, \alpha_i \in \mathbb{Z} \\ \alpha_i \neq 0 \end{array}$$

➡ Closely connected to hypergeometric ${}_2F_1$.

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- Using IBPs, we find two master integrands.

$$\omega_1 = \omega(\alpha_1\epsilon, -1+\alpha_2\epsilon, \alpha_3\epsilon) \qquad \omega_2 = \omega(\alpha_1\epsilon, \alpha_2\epsilon, -1+\alpha_3\epsilon)$$

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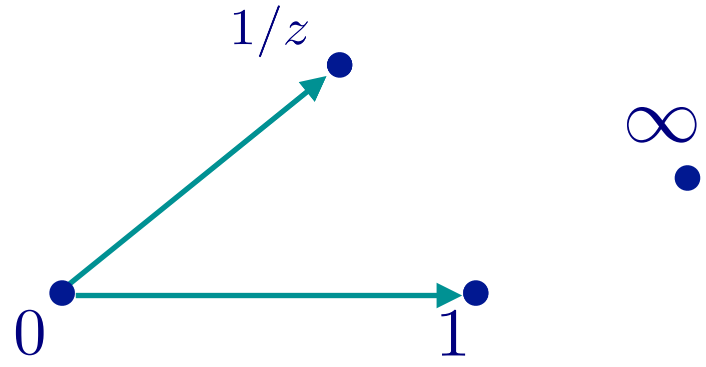
- Associated Geometry: straight lines connecting $0, 1, 1/z, \infty$:



➡ Homology theory: Only two of these segments are independent!

[Vassiliev]

Hypergeometric functions

- Master contours:

$$\int_0^1 \omega_1 = \frac{1}{a_2\epsilon} + \dots \qquad \int_0^1 \omega_2 = 0 + \dots \qquad \int_0^{1/z} \omega_1 = 0 + \dots \qquad \int_0^{1/z} \omega_2 = \frac{1}{a_3 z\epsilon} + \dots$$

➡ Dots indicate higher-weights terms.

$$a_i = n_i + \alpha_i\epsilon \qquad n_i, \alpha_i \in \mathbb{Z} \qquad \alpha_i \neq 0$$

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$$\Delta\left(\int_0^1 \omega(a_1,a_2,a_3)\right) = \int_0^1 \omega_1 \otimes \epsilon a_2 \int_0^1 \omega(a_1,a_2,a_3) + \int_0^1 \omega_2 \otimes \epsilon a_3 z \int_0^{1/z} \omega(a_1,a_2,a_3)$$

➡ Checked explicitly up to weight 5 in ϵ expansion!

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Hypergeometric functions

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➡ Checked explicitly up to weight 5 in ϵ expansion!

- Can do the same for Appell F_1 function:

$$\int_0^1 dx\, x^{a_1} (1-x)^{a_2} (1-yx)^{a_3} (1-zx)^{a_4} \qquad a_i = n_i + \alpha_i \epsilon \qquad \begin{array}{r} n_i, \alpha_i \in \mathbb{Z} \\ \alpha_i \neq 0 \end{array}$$

➡ Master formula was checked up to weight 5!

Conclusion

- New mathematical ideas and homology theory may be able to tell us something about multi-loop integrals.
  - ➡ Rigorous way to define and investigate cuts!
  - ➡ Two-loop master contours from homology groups?
  - ➡ New way to look at unitarity techniques?
- Conjectured 'master formula' for coproduct.
  - ➡ Shown to work for polylogarithms, one-loop integrals, (some classes of) hypergeometric and Appell functions.
  - ➡ Hidden algebraic structure of loop integrals?
- Expansion of some hypergeometric functions cannot be expressed in terms of polylogarithms.
  - ➡ Gives hints of how mathematics of polylogarithms extends elliptic functions?

Multi-variate residues

- If S is a surface given by $s(z) = 0$, a differential form ω (integrand) has a pole on S, then

$$\omega = \frac{ds}{s} \wedge \psi + \theta \qquad\qquad \psi, \theta \text{ regular on } S$$

- The residue of ω is $\mathrm{Res}_S[\omega] = \psi_{|S}$.

- Generalisation to several singular surfaces is straightforward.

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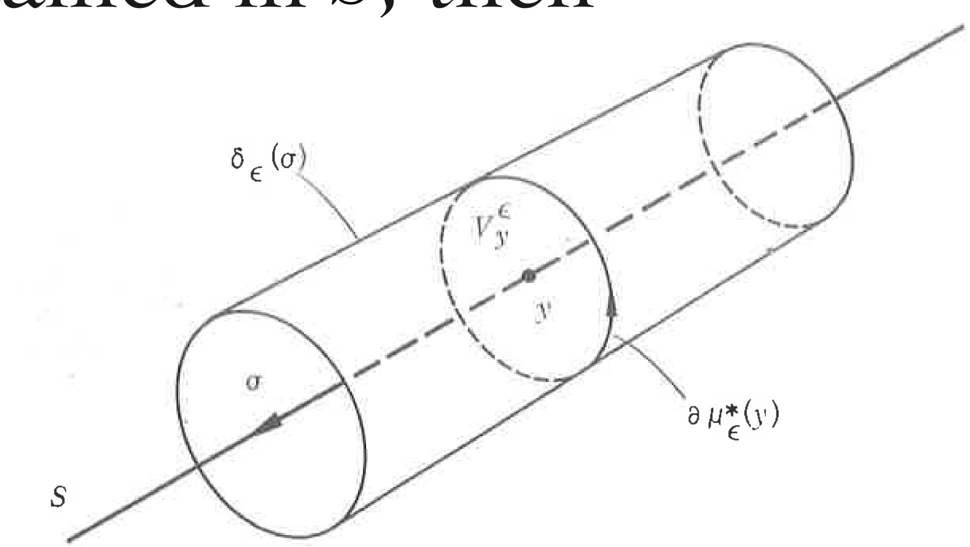
- Generalisation to several singular surfaces is straightforward.

- Residue Theorem: If γ is a contour contained in S, then

$$\int_{\delta\gamma}\omega = 2\pi i\int_\gamma \text{Res}_S[\omega]$$

➡ δ is the Leray coboundary operator.



[Picture from Hwa & Teplitz]

Cut integrals

- Using this language we can make all the cut-folklore precise.
- Let S_i denote the surface where the i-th propagator is on shell.
  - ➡ Each S_i is a sphere, and so is their intersection S.
  - ➡ Cut integral = integrating the residue over the sphere S.

$$I_n = \int \omega_n \longrightarrow \mathcal{C}_{S_1\ldots S_k} I_n = \int_S \mathrm{Res}_{S_1\ldots S_k}[\omega_n] = (2\pi i)^{-k} \int_{\delta S} \omega_n$$

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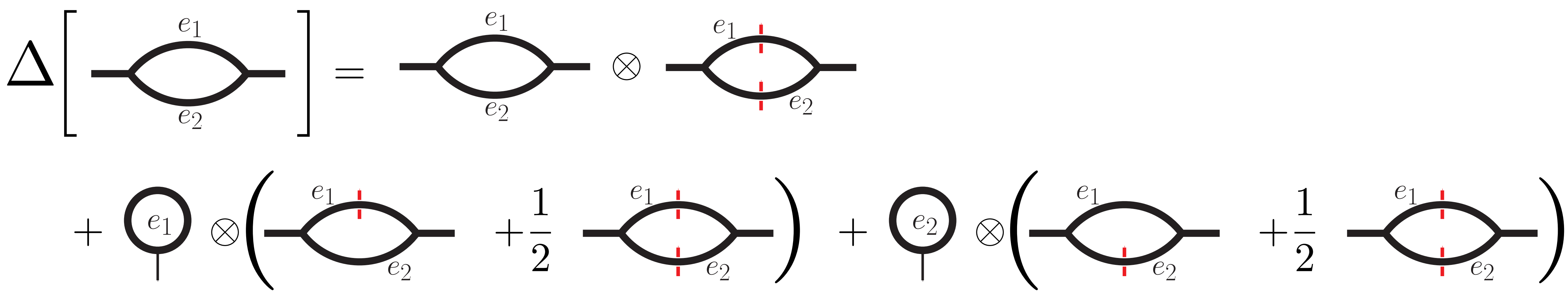
Contour that encircles propagator poles

- Each such integral computes a discontinuity, associated to some pinch singularity (cf. Landau conditions).
  - ➡ Picard-Lefschetz theorem and homology theory.
- Works also for Landau singularities of second type.

The diagrammatic coaction



- The coproduct is consistent with the action of discontinuities:

$$\Delta \mathrm{Disc} = (\mathrm{Disc} \otimes \mathrm{id})\Delta$$

  ➡ (Stronger version of) First entry condition [Gaiotto, Maldacena, Sever, Vieira] built in.

  ➡ Analytic continuation can be read of from coproduct.

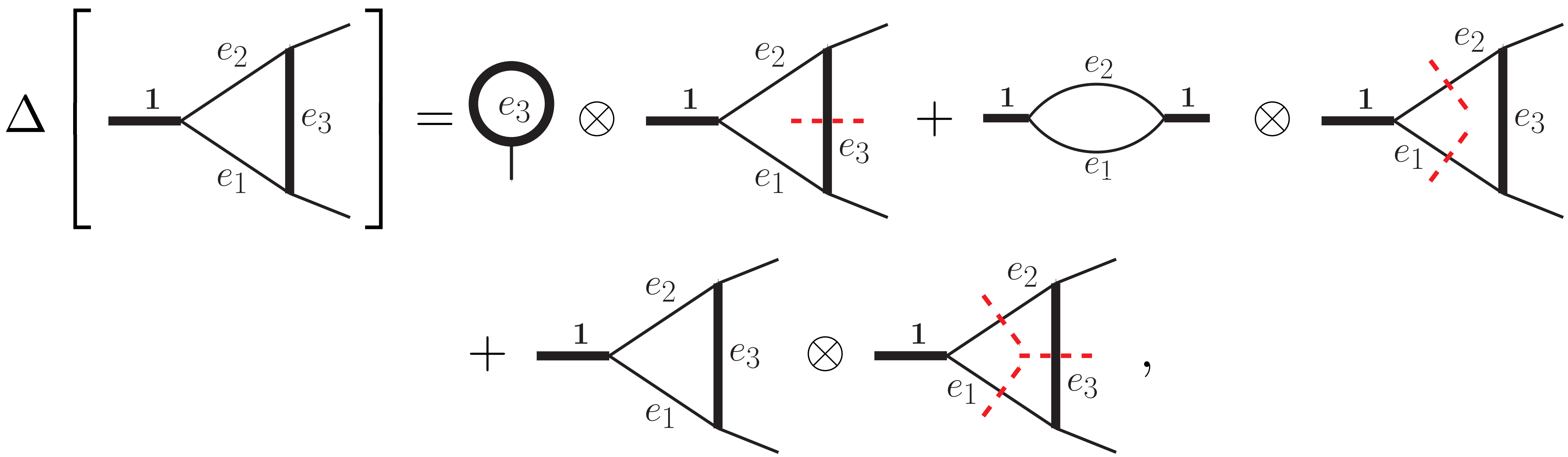
- The coproduct is consistent with the action of derivatives:

$$\Delta \partial_z = (\mathrm{id} \otimes \partial_z)\Delta$$

  ➡ Can read off differential equations from cuts!

The diagrammatic coaction

- Example:



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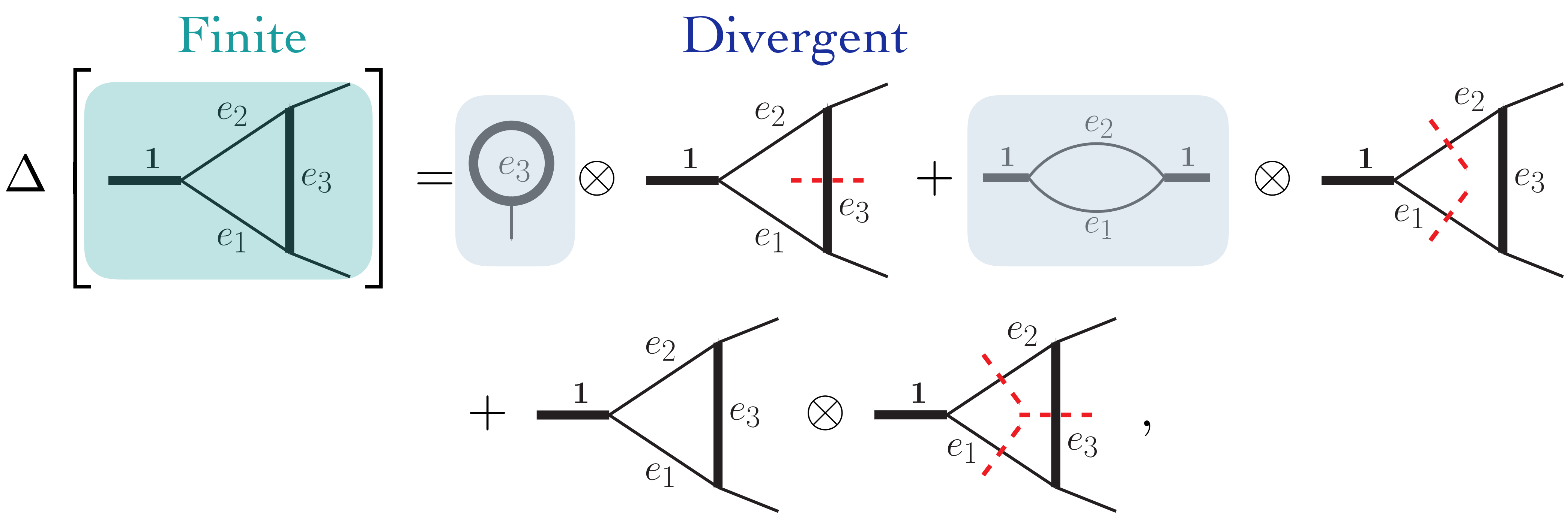

The diagrammatic coaction

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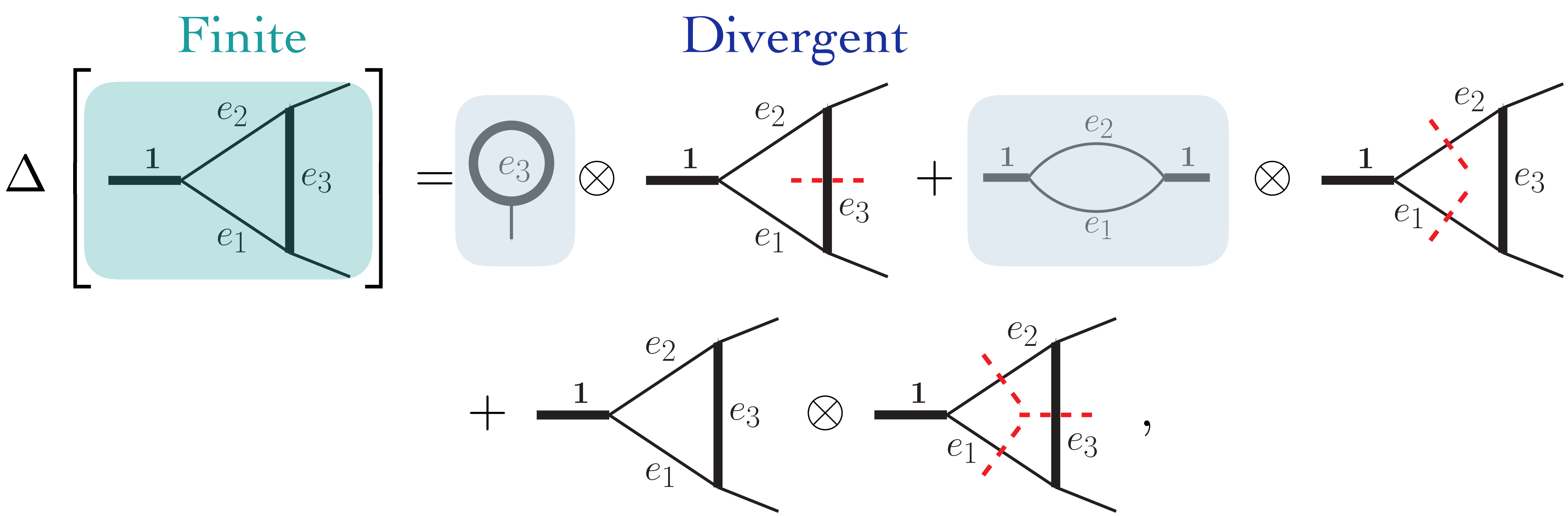
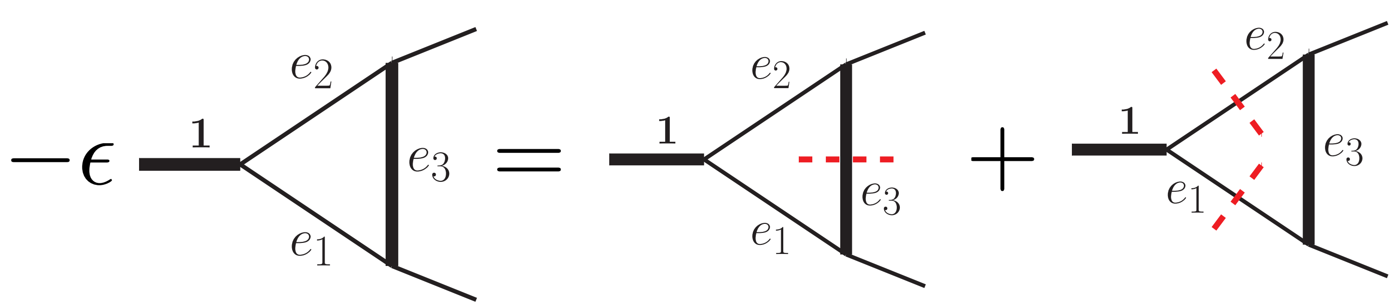


- Poles cancel due to ‘homological identity’: $\Gamma_\infty = \Gamma_3 + \Gamma_{12} + \ldots$



The Master formula

$$\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$$

- Magic formula:
  - ➡ Coproduct = 'insert a complete set of states' $1 = \sum_i |\omega_i\rangle \otimes \langle\gamma_i|$
  - ➡ Knows about analytic continuation: $\Delta\text{Disc} = (\text{Disc} \otimes \text{id})\Delta$
  - ➡ Knows about differential equations: $\Delta\partial_z = (\text{id} \otimes \partial_z)\Delta$
  - ➡ Knows about master integrands and master contours.
  - ➡ Works in DimReg.
- Works in 'all known cases' (MPLs & one-loop integrals).
- Does it make (correct!) predictions?
  - ➡ Beyond one loop? Elliptic functions?