

A novel approach to the cuts of Feynman integrals

Claude Duhr

in collaboration with S. Abreu, R. Britto & E. Gardi

Future Challenges for Precision QCD
Durham

25 - 28 October 2016



Challenges for loop integrals



- Algebraic structure of polylogarithms & differential equations.
 - ➔ How does this generalise to elliptic functions?
 - ➔ Coproducts, symbols, etc. for elliptic functions?
- Construct integrands from unitarity approaches at two loops.
 - ➔ Use cuts to ‘project out’ master integrals from amplitudes.
 - ➔ Technically, need to find a ‘master contour’ for each integral.
 - ➔ Many open questions:
 - Are there enough master contours? Uniqueness?
 - Why do integrals over master contours satisfy IBPs (but leading singularities do not)?



Challenges for loop integrals



- Aim of this talk:
 - ➔ Discuss some possible avenues to address these issues.
 - ➔ Argue that the 2 questions (special functions & unitarity) may be connected!
 - ➔ Take first steps towards a better understanding of the analytic & algebraic structure of Feynman integrals.
 - ➔ Maybe physics intuition may help to clarify some open questions in pure mathematics..?
- **Disclaimer:** Many of the ideas are new and under development!
 - ➔ Will discuss mostly one-loop integrals.
 - ➔ General picture emerges, but still a lot to do to go to two loops!



Outline



- Quick review of polylogarithms and their coproduct.
- Cut integrals & homology theory.
- The coproduct of one-loop integrals.
- Outlook & Conjectures.

Quick review of polylogarithms and their coproduct



Polylogarithms



- Large classes of loop integrals can be expressed in terms of polylogarithms.

$$G(a_1, \dots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t)$$

$$G(a_1; z) = \log \left(1 - \frac{z}{a_1} \right)$$

$$G(0, 1; z) = -\text{Li}_2(z)$$

- Polylogarithms form a Hopf algebra. [Goncharov; Brown]

➔ Allows one to ‘break’ polylogarithms into smaller pieces:

$$\Delta(\log z) = 1 \otimes \log z + \log z \otimes 1$$

$$\Delta(\text{Li}_n(z)) = 1 \otimes \text{Li}_n(z) + \sum_{k=0}^{n-1} \text{Li}_{n-k}(z) \otimes \frac{\log^k z}{k!}$$

- The two factors encode discontinuities & differential equations:

$$\Delta \text{Disc} = (\text{Disc} \otimes \text{id}) \Delta$$

$$\Delta \partial_z = (\text{id} \otimes \partial_z) \Delta$$



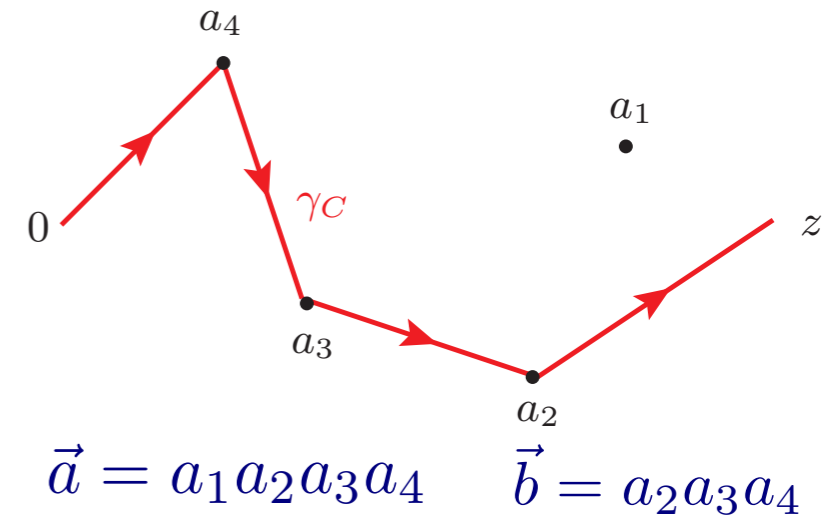
The coproduct



- General formula:

$$\Delta(G(\vec{a}; z)) = \sum_{\vec{b} \subset \vec{a}} G(\vec{b}; z) \otimes G_{\vec{b}}(\vec{a}; z)$$

Integral over a contour that encircles the singularities in \vec{b} = 'Cut Integral'



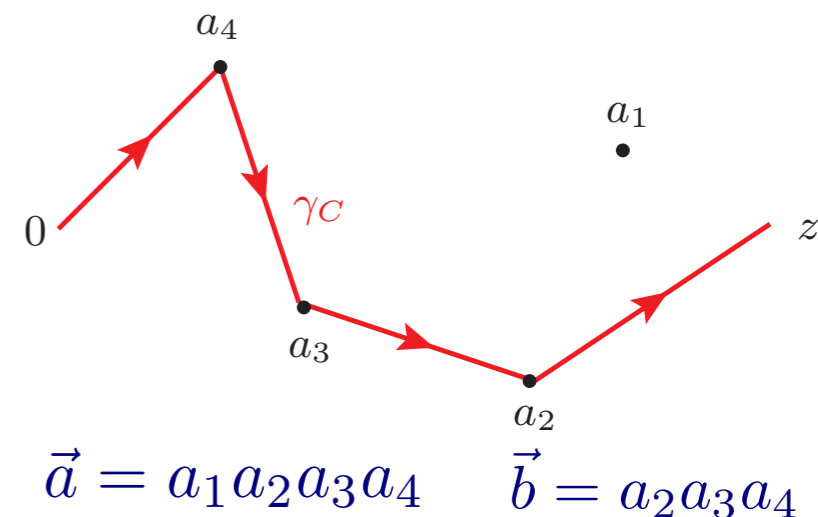


The coproduct

- General formula:

$$\Delta(G(\vec{a}; z)) = \sum_{\vec{b} \subset \vec{a}} G(\vec{b}; z) \otimes G_{\vec{b}}(\vec{a}; z)$$

Integral over a contour that encircles the singularities in \vec{b} = 'Cut Integral'



- Does this picture generalise to other functions?

➔ Answer is 'Yes' [Brown]:

$$\Delta([\gamma, \omega]^m) = \sum_i [\gamma, \omega_i]^m \otimes [\omega_i^\dagger, \omega]^{\text{dr}}$$

Sum over MIs Master integrals Cut

$$[\gamma, \omega]^m \sim \int_\gamma \omega$$

- Goal: Make this formula precise!

➔ First step towards understanding mathematical structure of functions that appear in loops and are more polylogarithms.

Cuts integrals & homology theory



Cut integrals



- **Traditional definition:** replace propagators by delta functions:

$$\frac{1}{p^2 - m^2 + i\varepsilon} \longrightarrow 2\pi i \delta_+(p^2 - m^2) \quad [\text{Cutkosky; 't Hooft, Veltmann}]$$



Cut integrals



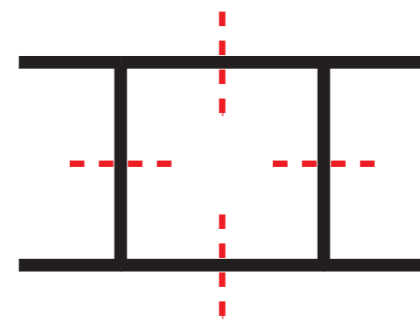
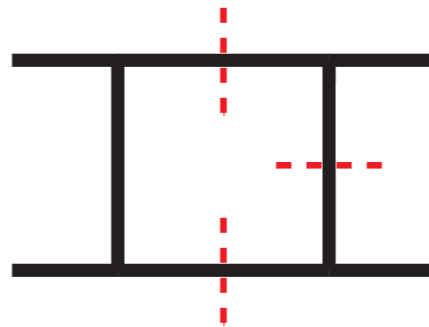
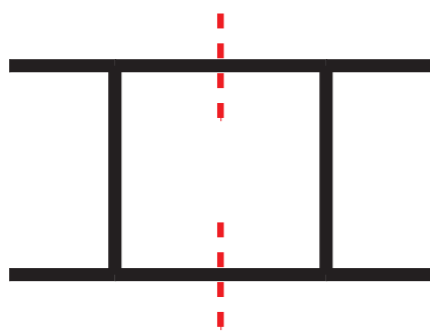
- **Traditional definition:** replace propagators by delta functions:

$$\frac{1}{p^2 - m^2 + i\varepsilon} \longrightarrow 2\pi i \delta_+(p^2 - m^2) \quad [\text{Cutkosky; 't Hooft, Veltmann}]$$

- **Folklore:**

➔ ‘Cuts compute discontinuities’

- Which ones..?





Cut integrals

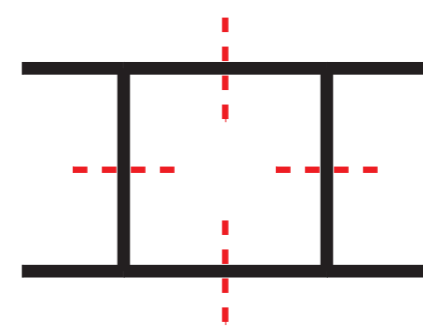
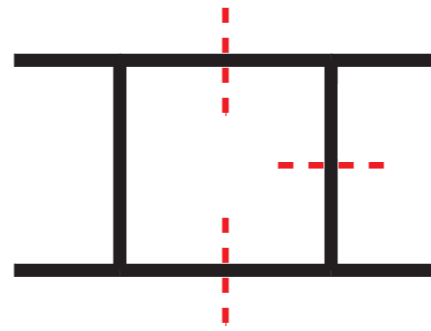
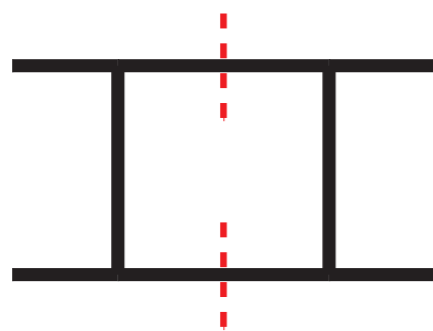


- **Traditional definition:** replace propagators by delta functions:

$$\frac{1}{p^2 - m^2 + i\varepsilon} \longrightarrow 2\pi i \delta_+(p^2 - m^2) \quad [\text{Cutkosky; 't Hooft, Veltmann}]$$

- **Folklore:**

➔ ‘Cuts compute discontinuities’



- Which ones..?

➔ ‘Cuts are computed by integrating over a contour that encircles the poles of some propagators’ - Which contour..?



Cut integrals

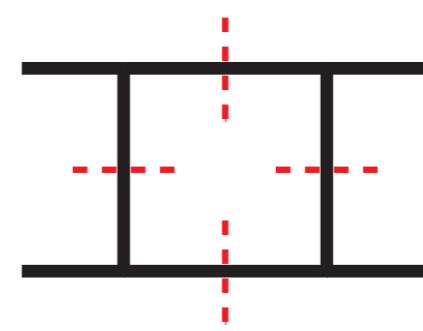
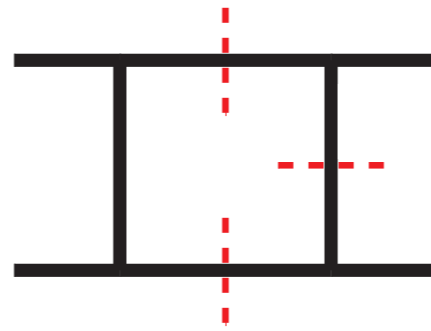
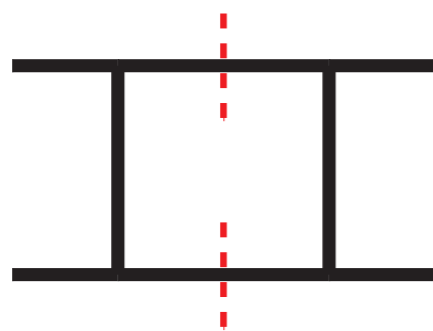


- **Traditional definition:** replace propagators by delta functions:

$$\frac{1}{p^2 - m^2 + i\varepsilon} \longrightarrow 2\pi i \delta_+(p^2 - m^2) \quad [\text{Cutkosky; 't Hooft, Veltmann}]$$

- **Folklore:**

➔ ‘Cuts compute discontinuities’



- Which ones..?

➔ ‘Cuts are computed by integrating over a contour that encircles the poles of some propagators’ - Which contour..?

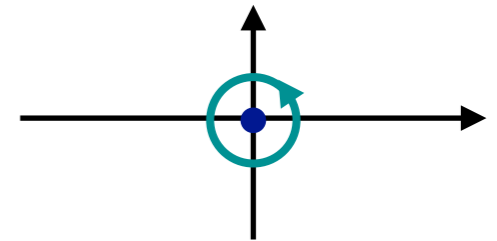
➔ ‘Leading singularities do not satisfy IBPs... but some linear combinations do!’ - What about reverse-unitarity..?



Cut integrals



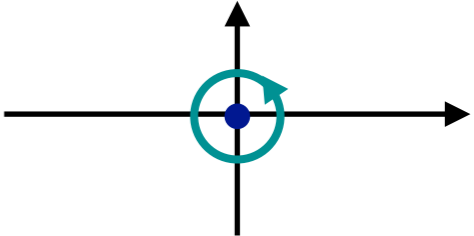
- Which contours..?
 - ➔ Turns the problem into a problem in homology theory!
- **Homology groups:** \sim all inequivalent integration contours we can define in our space.
- **Example:** The plane minus the origin: $\mathbb{C} \setminus \{0\}$





Cut integrals



- Which contours..?
 - ➔ Turns the problem into a problem in homology theory!
- **Homology groups:** ~ all inequivalent integration contours we can define in our space.
- **Example:** The plane minus the origin: $\mathbb{C} \setminus \{0\}$ 
- Homology groups associated to Feynman integrals have been studied in the 60s.

[Fotiadi, Pham; Teplitz, Hwa; Federbusch; Landshof, Polkinghorne, ...]

 - ➔ Contours for cuts can be unambiguously defined.
 - ➔ Every cut integrals computes a discontinuity, associated to some Landau singularity (1st & 2nd kind)
 - ➔ Cut integrals always satisfy the same IBP relations and differential equations as uncut integrals.



Homology groups



- *At one-loop*: interesting contours ‘encircle’ propagator poles and/or pinch singularity at infinity:

$$\Gamma_{\emptyset}, \quad \Gamma_{\infty}, \Gamma_1, \Gamma_2, \dots \quad \Gamma_{12}, \dots \quad \Gamma_{\infty 12}, \dots$$



Homology groups



- **At one-loop:** interesting contours ‘encircle’ propagator poles and/or pinch singularity at infinity:

$$\Gamma_{\emptyset}, \quad \Gamma_{\infty}, \Gamma_1, \Gamma_2, \dots \quad \Gamma_{12}, \dots \quad \Gamma_{\infty 12}, \dots$$

- **Homology theory:** Contours that do not encircle ∞ form a basis:

$$\Gamma_{\infty C} = -2\Gamma_C + \sum_X (-1)^{\lfloor |C|/2 \rfloor + \lceil |X|/2 \rceil} \Gamma_X \quad \begin{array}{l} C \subseteq \{1, 2, \dots\} \\ |C| \text{ odd} \end{array} \quad [\text{Fotiadi, Pham}]$$



Homology groups



- **At one-loop:** interesting contours ‘encircle’ propagator poles and/or pinch singularity at infinity:

$$\Gamma_{\emptyset}, \quad \Gamma_{\infty}, \Gamma_1, \Gamma_2, \dots \quad \Gamma_{12}, \dots \quad \Gamma_{\infty 12}, \dots$$

- **Homology theory:** Contours that do not encircle ∞ form a basis:

$$\Gamma_{\infty C} = -2\Gamma_C + \sum_X (-1)^{\lfloor |C|/2 \rfloor + \lceil |X|/2 \rceil} \Gamma_X \quad \begin{array}{l} C \subseteq \{1, 2, \dots\} \\ |C| \text{ odd} \end{array} \quad [\text{Fotiadi, Pham}]$$

- **Alternate basis:** $\Gamma_{\emptyset} \dots \Gamma_{\infty 123}, \Gamma_{1234} \dots$

➡ Master contours at one loop! [Britto, Cachazo, Feng; Forde; ...]

- There is two-loop literature on the homology groups of the double box! [Federbusch]

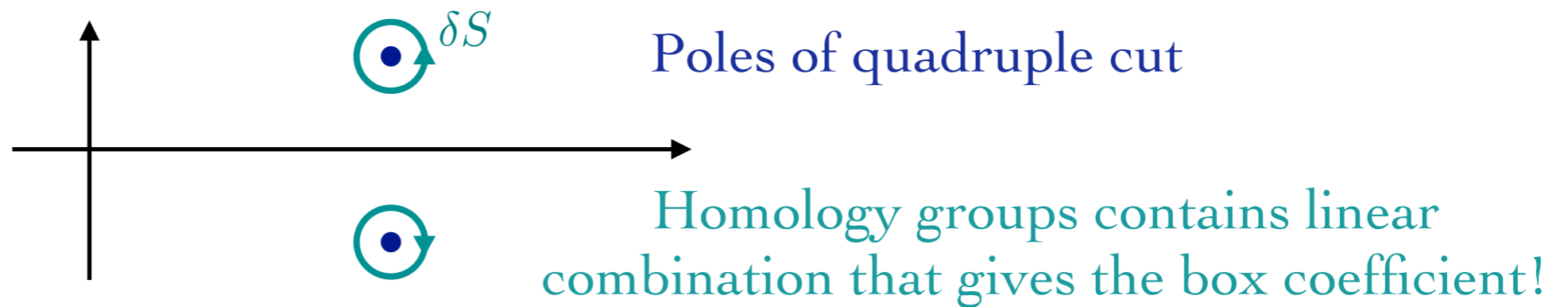
➡ Does this provide two-loop master contours?!



Master contours



- **Consequence:** Cut integrals always satisfy IBPs!
➔ Contradiction with literature...?
- Let's look at the quadruple cut at one-loop:

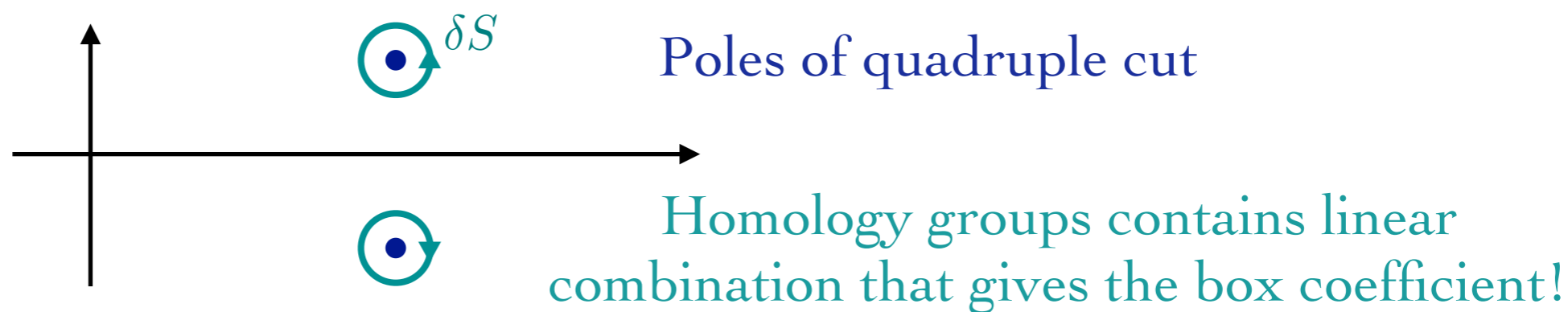




Master contours



- **Consequence:** Cut integrals always satisfy IBPs!
➔ Contradiction with literature...?
- Let's look at the quadruple cut at one-loop:



- ➔ Individual residues do not satisfy IBPs, but the integral over δS does!
- **Conclusion:** Master contours should not be seen as leading singularities, but as discontinuities!
➔ These contours are dictated by homology theory.

The coproduct of one-loop integrals

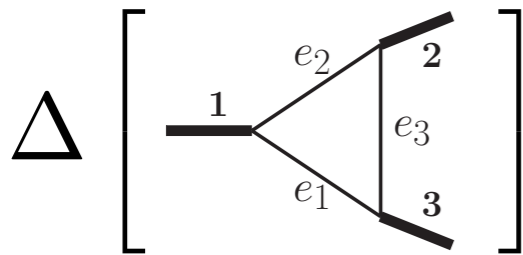


The diagrammatic coaction



$$\Delta([\gamma, \omega]^{\mathfrak{m}}) = \underbrace{\sum_i}_{\text{Sum over MIs}} \underbrace{[\gamma, \omega_i]^{\mathfrak{m}}}_{\text{MI}} \otimes \underbrace{[\omega_i^{\dagger}, \omega]^{\mathfrak{dr}}}_{\text{Cut}}$$

- Let us analyse the triangle with massless propagators:





The diagrammatic coaction



$$\Delta([\gamma, \omega]^m) = \sum_i [\gamma, \omega_i]^m \otimes [\omega_i^\dagger, \omega]^{\text{dr}}$$

Sum over MIs
MI
Cut

- Let us analyse the triangle with massless propagators:

$$\Delta \left[\text{Triangle}(e_1, e_2, e_3) \right] = \text{Bubble}(e_1, e_2) \otimes \text{CutTriangle}(e_1, e_2, e_3) + \text{Bubble}(e_2, e_3) \otimes \text{CutTriangle}(e_1, e_2, e_3) + \text{Bubble}(e_1, e_3) \otimes \text{CutTriangle}(e_1, e_2, e_3) + \dots$$

The diagrammatic equation shows the coaction on a triangle diagram with massless propagators. The left side is the coaction Δ applied to a triangle with external lines 1, 2, 3 and internal lines e_1, e_2, e_3 . The right side is a sum of four terms, each representing a bubble diagram (MI) tensored with a cut triangle diagram (Cut). The bubbles are formed by two internal lines, and the cuts are indicated by red dashed lines on the triangle diagrams.

- ➔ Checked up to terms of weight 4.
- ➔ Requires highly non-trivial conspiracy of terms!



The diagrammatic coaction



- Bubble with massive propagators:

$$\Delta \left[\text{bubble}(e_1, e_2) \right] = \text{bubble}(e_1, e_2) \otimes \text{cut_bubble}(e_1, e_2) + \text{cut_e1_bubble}(e_1, e_2) + \text{cut_e2_bubble}(e_1, e_2)$$

The diagrammatic equation shows the coaction Δ applied to a bubble diagram with two external lines and two internal propagators labeled e_1 and e_2 . The result is a sum of four terms:

- The first term is the original bubble diagram $\text{bubble}(e_1, e_2)$ tensored with a cut bubble diagram $\text{cut_bubble}(e_1, e_2)$, where the internal propagators are cut (indicated by red dashed lines).
- The second term is a cut on the e_1 propagator, represented by a circle with a vertical line and label e_1 tensored with a bubble diagram where the top propagator is cut.
- The third term is a cut on the e_2 propagator, represented by a circle with a vertical line and label e_2 tensored with a bubble diagram where the bottom propagator is cut.



The diagrammatic coaction



- Bubble with massive propagators:

$$\Delta \left[\text{Bubble}(e_1, e_2) \right] = \text{Bubble}(e_1, e_2) \otimes \text{Bubble}(e_1, e_2) + \text{Bubble}(e_1, e_2) \otimes \text{Bubble}(e_1, e_2) + \text{Bubble}(e_1, e_2) \otimes \text{Bubble}(e_1, e_2)$$

The diagrammatic equation shows the coaction Δ applied to a bubble diagram with two external lines and two internal propagators labeled e_1 and e_2 . The result is a sum of three terms, each representing a tensor product of two diagrams. The first term is the tensor product of two identical bubble diagrams. The second and third terms are tensor products of a bubble diagram with a bubble diagram that has a red dashed line on one of its internal propagators, indicating a cut or a specific type of singularity.

➡ This relation is incorrect...



The diagrammatic coaction



- Bubble with massive propagators:

$$\Delta \left[\text{bubble}(e_1, e_2) \right] = \text{bubble}(e_1, e_2) \otimes \text{cut_bubble}(e_1, e_2) + \text{cut_e1}(e_1) \otimes \text{cut_bubble}(e_1, e_2) + \text{cut_e2}(e_2) \otimes \text{cut_bubble}(e_1, e_2)$$

➔ This relation is incorrect...

- ... but the following relation holds!

$$\Delta \left[\text{bubble}(e_1, e_2) \right] = \text{bubble}(e_1, e_2) \otimes \text{cut_bubble}(e_1, e_2) + \text{cut_e1}(e_1) \otimes \left(\text{cut_bubble}(e_1, e_2) + \frac{1}{2} \text{cut_cut_bubble}(e_1, e_2) \right) + \text{cut_e2}(e_2) \otimes \left(\text{cut_bubble}(e_1, e_2) + \frac{1}{2} \text{cut_cut_bubble}(e_1, e_2) \right)$$



● Example:

$$\begin{aligned}
 \Delta \left[\begin{array}{c|c|c} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} \right] &= \text{circle}(e_1) \otimes \begin{array}{c|c|c} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} + \left(\text{bubble}(e_1, e_3, s, s) + \frac{1}{2} \text{circle}(e_1) \right) \otimes \begin{array}{c|c|c} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} \\
 &+ \text{bubble}(e_4, e_2, t, t) \otimes \begin{array}{c|c|c} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} + \text{triangle}(e_4, e_1, e_2, t) \otimes \begin{array}{c|c|c} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} + \left\{ \begin{array}{c|c|c} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} \right. \\
 &+ \frac{1}{2} \left(\text{triangle}(e_3, e_2, e_1, s) + \text{triangle}(e_3, e_4, e_1, s) + \text{triangle}(e_4, e_2, e_3, t) + \text{triangle}(e_4, e_1, e_2, t) \right) \otimes \begin{array}{c|c|c} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array}
 \end{aligned}$$



The diagrammatic coaction



- What is the meaning of the $1/2$ term..?

$$\begin{aligned}
 \Delta \left[\text{Diagram with edges } e_1, e_2 \right] = & \text{Diagram with edges } e_1, e_2 \otimes \text{Diagram with edges } e_1, e_2 \text{ (with red dashed lines)} \\
 & + \text{Diagram with edge } e_1 \otimes \left(\text{Diagram with edges } e_1, e_2 \text{ (with red dashed lines)} + \frac{1}{2} \text{Diagram with edges } e_1, e_2 \text{ (with red dashed lines)} \right) \\
 & + \text{Diagram with edge } e_2 \otimes \left(\text{Diagram with edges } e_1, e_2 \text{ (with red dashed lines)} + \frac{1}{2} \text{Diagram with edges } e_1, e_2 \text{ (with red dashed lines)} \right)
 \end{aligned}$$



The diagrammatic coaction



- What is the meaning of the $1/2$ term..?

$$\Delta \left[\text{Diagram with two vertices } e_1, e_2 \text{ and two internal lines} \right] = \text{Diagram with two vertices } e_1, e_2 \text{ and two internal lines} \otimes \text{Diagram with two vertices } e_1, e_2 \text{ and two internal lines, with red dashed lines on the internal lines}$$

$$+ \text{Diagram with one vertex } e_1 \text{ and one internal line} \otimes \left(\text{Diagram with two vertices } e_1, e_2 \text{ and two internal lines, with red dashed lines on the internal lines} + \frac{1}{2} \text{Diagram with two vertices } e_1, e_2 \text{ and two internal lines, with red dashed lines on the internal lines} \right) + \text{Diagram with one vertex } e_2 \text{ and one internal line} \otimes \left(\text{Diagram with two vertices } e_1, e_2 \text{ and two internal lines, with red dashed lines on the internal lines} + \frac{1}{2} \text{Diagram with two vertices } e_1, e_2 \text{ and two internal lines, with red dashed lines on the internal lines} \right)$$

- Using the ‘homological relation’ $\Gamma_{\infty 1} = -2\Gamma_1 - \Gamma_{12}$, we find

$$\text{Diagram with one vertex } e_1 \text{ and one internal line} \otimes \left(\text{Diagram with two vertices } e_1, e_2 \text{ and two internal lines, with red dashed lines on the internal lines} + \frac{1}{2} \text{Diagram with two vertices } e_1, e_2 \text{ and two internal lines, with red dashed lines on the internal lines} \right) = \int \omega_1 \otimes \left(\int_{\Gamma_1} \omega_2 + \frac{1}{2} \int_{\Gamma_{12}} \omega_2 \right)$$

$$= \int \omega_1 \otimes \left(-\frac{1}{2} \int_{\Gamma_{\infty 1}} \omega_2 \right)$$

$$\text{Diagram with two vertices } e_1, e_2 \text{ and two internal lines} \otimes \text{Diagram with two vertices } e_1, e_2 \text{ and two internal lines, with red dashed lines on the internal lines} = \int \omega_2 \otimes \int_{\Gamma_{12}} \omega_2$$



The diagrammatic coaction



- What is the meaning of the $1/2$ term..?

$$\Delta \left[\text{Diagram with edges } e_1, e_2 \right] = \text{Diagram with edges } e_1, e_2 \otimes \text{Diagram with edges } e_1, e_2 \text{ (cut)} + \text{Diagram with edge } e_1 \otimes \left(\text{Diagram with edges } e_1, e_2 \text{ (cut)} + \frac{1}{2} \text{Diagram with edges } e_1, e_2 \text{ (cut)} \right) + \text{Diagram with edge } e_2 \otimes \left(\text{Diagram with edges } e_1, e_2 \text{ (cut)} + \frac{1}{2} \text{Diagram with edges } e_1, e_2 \text{ (cut)} \right)$$

- Using the ‘homological relation’ $\Gamma_{\infty 1} = -2\Gamma_1 - \Gamma_{12}$, we find

$$\begin{aligned} \text{Diagram with edge } e_1 \otimes \left(\text{Diagram with edges } e_1, e_2 \text{ (cut)} + \frac{1}{2} \text{Diagram with edges } e_1, e_2 \text{ (cut)} \right) &= \int \omega_1 \otimes \left(\int_{\Gamma_1} \omega_2 + \frac{1}{2} \int_{\Gamma_{12}} \omega_2 \right) \\ &= \int \omega_1 \otimes \left(-\frac{1}{2} \int_{\Gamma_{\infty 1}} \omega_2 \right) \\ &\stackrel{\text{Master integrands}}{=} \text{Diagram with edges } e_1, e_2 \otimes \text{Diagram with edges } e_1, e_2 \text{ (cut)} \\ &= \int \omega_2 \otimes \int_{\Gamma_{12}} \omega_2 \quad \text{Master contours} \end{aligned}$$

Outlook & Conjectures



The Master formula



- Brown's motivic coaction:

$$\Delta([\gamma, \omega]^{\mathfrak{m}}) = \underbrace{\sum_i}_{\text{Sum over MIs}} \underbrace{[\gamma, \omega_i]^{\mathfrak{m}}}_{\text{Master integrals}} \otimes \underbrace{[\omega_i^{\dagger}, \omega]^{\mathfrak{dr}}}_{\text{Cut}} \quad [\gamma, \omega]^{\mathfrak{m}} \sim \int_{\gamma} \omega$$

- Conjecture:

$$\Delta \left(\int_{\gamma} \omega \right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega \quad [\text{Abreu, Britto, CD, Gardi}]$$

➔ γ_i is the master contour of the master integrand ω_i .

- Works for multiple polylogs and one-loop integrals.

➔ Does it have any predictive power?

➔ Does predict the correct results?



Hypergeometric functions



- Consider the integrals

$$T(a_1, a_2, a_3; z) = \int_0^1 \underbrace{dx x^{a_1} (1-x)^{a_2} (1-zx)^{a_3}}_{=\omega(a_1, a_2, a_2)} \quad a_i = n_i + \alpha_i \epsilon \quad \begin{array}{l} n_i, \alpha_i \in \mathbb{Z} \\ \alpha_i \neq 0 \end{array}$$

➔ Closely connected to hypergeometric ${}_2F_1$.



Hypergeometric functions



- Consider the integrals

$$T(a_1, a_2, a_3; z) = \int_0^1 \underbrace{dx x^{a_1} (1-x)^{a_2} (1-zx)^{a_3}}_{=\omega(a_1, a_2, a_3)} \quad a_i = n_i + \alpha_i \epsilon \quad \begin{array}{l} n_i, \alpha_i \in \mathbb{Z} \\ \alpha_i \neq 0 \end{array}$$

➔ Closely connected to hypergeometric ${}_2F_1$.

- Using IBPs, we find two master integrands.

$$\omega_1 = \omega(\alpha_1 \epsilon, -1 + \alpha_2 \epsilon, \alpha_3 \epsilon) \quad \omega_2 = \omega(\alpha_1 \epsilon, \alpha_2 \epsilon, -1 + \alpha_3 \epsilon)$$



Hypergeometric functions



- Consider the integrals

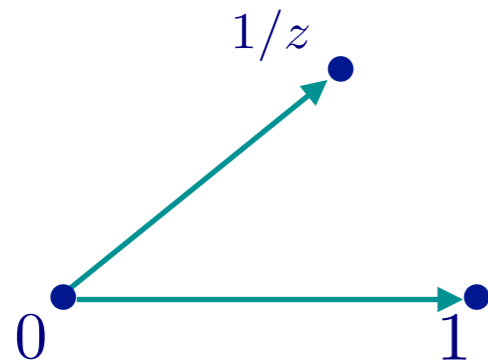
$$T(a_1, a_2, a_3; z) = \int_0^1 \underbrace{dx x^{a_1} (1-x)^{a_2} (1-zx)^{a_3}}_{=\omega(a_1, a_2, a_3)} \quad a_i = n_i + \alpha_i \epsilon \quad \begin{array}{l} n_i, \alpha_i \in \mathbb{Z} \\ \alpha_i \neq 0 \end{array}$$

➔ Closely connected to hypergeometric ${}_2F_1$.

- Using IBPs, we find two master integrands.

$$\omega_1 = \omega(\alpha_1 \epsilon, -1 + \alpha_2 \epsilon, \alpha_3 \epsilon) \quad \omega_2 = \omega(\alpha_1 \epsilon, \alpha_2 \epsilon, -1 + \alpha_3 \epsilon)$$

- Associated Geometry:** straight lines connecting $0, 1, 1/z, \infty$:



∞

➔ **Homology theory:** Only two of these segments are independent!

[Vassiliev]



Hypergeometric functions



- Master contours:

$$\int_0^1 \omega_1 = \frac{1}{a_2 \epsilon} + \dots \quad \int_0^1 \omega_2 = 0 + \dots \quad \int_0^{1/z} \omega_1 = 0 + \dots \quad \int_0^{1/z} \omega_2 = \frac{1}{a_3 z \epsilon} + \dots$$

➡ Dots indicate higher-weights terms.

$$a_i = n_i + \alpha_i \epsilon \quad \begin{array}{l} n_i, \alpha_i \in \mathbb{Z} \\ \alpha_i \neq 0 \end{array}$$



Hypergeometric functions



- Master contours:

$$\int_0^1 \omega_1 = \frac{1}{a_2 \epsilon} + \dots \quad \int_0^1 \omega_2 = 0 + \dots \quad \int_0^{1/z} \omega_1 = 0 + \dots \quad \int_0^{1/z} \omega_2 = \frac{1}{a_3 z \epsilon} + \dots$$

➔ Dots indicate higher-weights terms.

- Master formula: $\Delta \left(\int_{\gamma} \omega \right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$

$$a_i = n_i + \alpha_i \epsilon \quad \begin{array}{l} n_i, \alpha_i \in \mathbb{Z} \\ \alpha_i \neq 0 \end{array}$$



Hypergeometric functions



- Master contours:

$$\int_0^1 \omega_1 = \frac{1}{a_2 \epsilon} + \dots \quad \int_0^1 \omega_2 = 0 + \dots \quad \int_0^{1/z} \omega_1 = 0 + \dots \quad \int_0^{1/z} \omega_2 = \frac{1}{a_3 z \epsilon} + \dots$$

➔ Dots indicate higher-weights terms.

- Master formula: $\Delta \left(\int_{\gamma} \omega \right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$

$$\Delta \left(\int_0^1 \omega(a_1, a_2, a_3) \right) = \int_0^1 \omega_1 \otimes \epsilon a_2 \int_0^1 \omega(a_1, a_2, a_3) + \int_0^1 \omega_2 \otimes \epsilon a_3 z \int_0^{1/z} \omega(a_1, a_2, a_3)$$

➔ Checked explicitly up to weight 5 in ϵ expansion!

$$a_i = n_i + \alpha_i \epsilon \quad \begin{array}{l} n_i, \alpha_i \in \mathbb{Z} \\ \alpha_i \neq 0 \end{array}$$



Hypergeometric functions



- Master contours:

$$\int_0^1 \omega_1 = \frac{1}{a_2 \epsilon} + \dots \quad \int_0^1 \omega_2 = 0 + \dots \quad \int_0^{1/z} \omega_1 = 0 + \dots \quad \int_0^{1/z} \omega_2 = \frac{1}{a_3 z \epsilon} + \dots$$

➔ Dots indicate higher-weights terms.

- Master formula: $\Delta \left(\int_{\gamma} \omega \right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$

$$\Delta \left(\int_0^1 \omega(a_1, a_2, a_3) \right) = \int_0^1 \omega_1 \otimes \epsilon a_2 \int_0^1 \omega(a_1, a_2, a_3) + \int_0^1 \omega_2 \otimes \epsilon a_3 z \int_0^{1/z} \omega(a_1, a_2, a_3)$$

➔ Checked explicitly up to weight 5 in ϵ expansion!

- Can do the same for Appell F_1 function:

$$\int_0^1 dx x^{a_1} (1-x)^{a_2} (1-yx)^{a_3} (1-zx)^{a_4} \quad a_i = n_i + \alpha_i \epsilon \quad \begin{array}{l} n_i, \alpha_i \in \mathbb{Z} \\ \alpha_i \neq 0 \end{array}$$

➔ Master formula was checked up to weight 5!



Conclusion



- New mathematical ideas and homology theory may be able to tell us something about multi-loop integrals.
 - ➔ Rigorous way to define and investigate cuts!
 - ➔ Two-loop master contours from homology groups?
 - ➔ New way to look at unitarity techniques?
- Conjectured ‘master formula’ for coproduct.
 - ➔ Shown to work for polylogarithms, one-loop integrals, (some classes of) hypergeometric and Appell functions.
 - ➔ Hidden algebraic structure of loop integrals?
- Expansion of some hypergeometric functions cannot be expressed in terms of polylogarithms.
 - ➔ Gives hints of how mathematics of polylogarithms extends elliptic functions?



Multi-variate residues



- If S is a surface given by $s(z) = 0$, a differential form ω (integrand) has a pole on S , then

$$\omega = \frac{ds}{s} \wedge \psi + \theta \quad \psi, \theta \text{ regular on } S$$

- The residue of ω is $\text{Res}_S[\omega] = \psi|_S$.
- Generalisation to several singular surfaces is straightforward.



Multi-variate residues



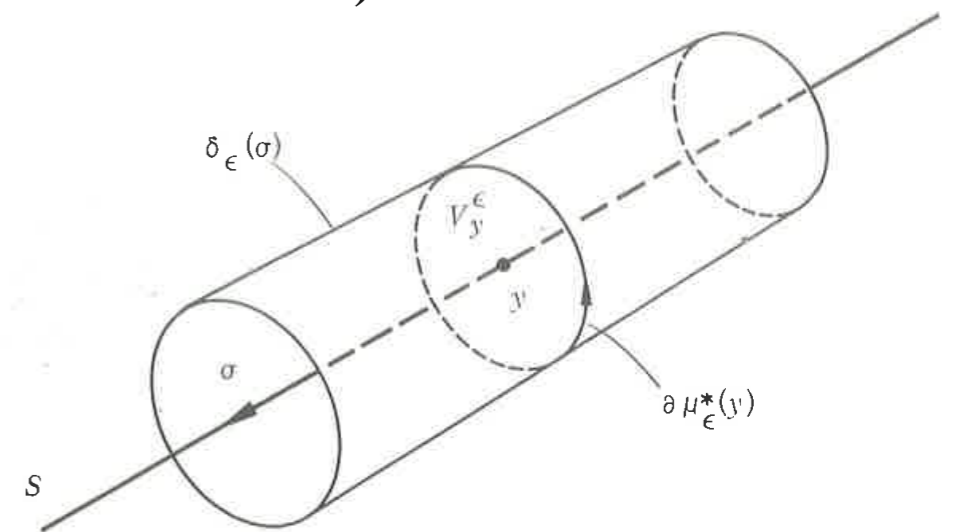
- If S is a surface given by $s(z) = 0$, a differential form ω (integrand) has a pole on S , then

$$\omega = \frac{ds}{s} \wedge \psi + \theta \quad \psi, \theta \text{ regular on } S$$

- The residue of ω is $\text{Res}_S[\omega] = \psi|_S$.
- Generalisation to several singular surfaces is straightforward.
- **Residue Theorem:** If γ is a contour contained in S , then

$$\int_{\delta\gamma} \omega = 2\pi i \int_{\gamma} \text{Res}_S[\omega]$$

➔ δ is the Leray coboundary operator.



[Picture from Hwa & Teplitz]



Cut integrals



- Using this language we can make all the cut-folklore precise.
- Let S_i denote the surface where the i -th propagator is on shell.
 - ➔ Each S_i is a sphere, and so is their intersection S .
 - ➔ Cut integral = integrating the residue over the sphere S .

$$I_n = \int \omega_n \longrightarrow \mathcal{C}_{S_1 \dots S_k} I_n = \int_S \text{Res}_{S_1 \dots S_k} [\omega_n] = (2\pi i)^{-k} \int_{\delta S} \omega_n$$



Cut integrals



- Using this language we can make all the cut-folklore precise.
- Let S_i denote the surface where the i -th propagator is on shell.
 - ➔ Each S_i is a sphere, and so is their intersection S .
 - ➔ Cut integral = integrating the residue over the sphere S .

$$I_n = \int \omega_n \longrightarrow \mathcal{C}_{S_1 \dots S_k} I_n = \int_S \text{Res}_{S_1 \dots S_k} [\omega_n] = (2\pi i)^{-k} \int_{\delta S} \omega_n$$

Contour that encircles propagator poles



Cut integrals



- Using this language we can make all the cut-folklore precise.
- Let S_i denote the surface where the i -th propagator is on shell.
 - ➔ Each S_i is a sphere, and so is their intersection S .
 - ➔ Cut integral = integrating the residue over the sphere S .

$$I_n = \int \omega_n \longrightarrow \mathcal{C}_{S_1 \dots S_k} I_n = \int_S \text{Res}_{S_1 \dots S_k} [\omega_n] = (2\pi i)^{-k} \int_{\delta S} \omega_n$$

Contour that encircles propagator poles

- Each such integral computes a discontinuity, associated to some pinch singularity (cf. Landau conditions).
 - ➔ Picard-Lefschetz theorem and homology theory.
- Works also for Landau singularities of second type.



The diagrammatic coaction



$$\Delta \left[\text{Diagram with two vertices } e_1, e_2 \text{ connected by two lines} \right] = \text{Diagram with two vertices } e_1, e_2 \text{ connected by two lines} \otimes \text{Diagram with two vertices } e_1, e_2 \text{ connected by two lines with red dashed lines} \\ + \text{Diagram with one vertex } e_1 \text{ and a line} \otimes \left(\text{Diagram with two vertices } e_1, e_2 \text{ connected by two lines with red dashed lines} + \frac{1}{2} \text{Diagram with two vertices } e_1, e_2 \text{ connected by two lines with red dashed lines} \right) \\ + \text{Diagram with one vertex } e_2 \text{ and a line} \otimes \left(\text{Diagram with two vertices } e_1, e_2 \text{ connected by two lines with red dashed lines} + \frac{1}{2} \text{Diagram with two vertices } e_1, e_2 \text{ connected by two lines with red dashed lines} \right)$$

- The coproduct is consistent with the action of discontinuities:

$$\Delta \text{Disc} = (\text{Disc} \otimes \text{id}) \Delta$$

➔ (Stronger version of) First entry condition [Gaiotto, Maldacena, Sever, Vieira] built in.

➔ Analytic continuation can be read off from coproduct.

- The coproduct is consistent with the action of derivatives:

$$\Delta \partial_z = (\text{id} \otimes \partial_z) \Delta$$

➔ Can read off differential equations from cuts!



The diagrammatic coaction



- Example:

$$\Delta \left[\text{Diagram 1} \right] = \text{Diagram 2} \otimes \text{Diagram 3} + \text{Diagram 4} \otimes \text{Diagram 5} + \text{Diagram 6} \otimes \text{Diagram 7},$$

The diagrammatic coaction is illustrated by the following equation:

The left-hand side shows the coaction Δ applied to a diagram (a triangle with edges e_1, e_2, e_3 and a thick vertical line). The right-hand side is the sum of four tensor products of diagrams:

- Diagram 1 (a circle with e_3) \otimes Diagram 2 (triangle with e_1, e_2, e_3 and a thick vertical line, with a dashed red line connecting e_1 and e_2).
- Diagram 3 (a loop with edges e_1, e_2) \otimes Diagram 4 (triangle with e_1, e_2, e_3 and a thick vertical line, with dashed red lines on e_1 and e_2).
- Diagram 5 (triangle with e_1, e_2, e_3 and a thick vertical line) \otimes Diagram 6 (triangle with e_1, e_2, e_3 and a thick vertical line, with dashed red lines on e_1 and e_2).



The diagrammatic coaction



- Example:

$$\Delta \left[\begin{array}{c} \text{Finite} \\ \text{Diagram} \end{array} \right] = \begin{array}{c} \text{Divergent} \\ \text{Diagram 1} \end{array} + \begin{array}{c} \text{Diagram 2} \end{array} + \begin{array}{c} \text{Diagram 3} \end{array} + \begin{array}{c} \text{Diagram 4} \end{array},$$

The diagrammatic coaction Δ is applied to a Feynman diagram labeled "Finite". The result is a sum of four terms, each representing a divergent part of the diagram. The first term is a diagram with a loop labeled e_3 and a vertical line labeled e_3 , with external lines labeled 1 , e_2 , and e_1 . The second term is a diagram with a loop labeled e_2 and a vertical line labeled e_3 , with external lines labeled 1 , e_2 , and e_1 . The third term is a diagram with a loop labeled e_1 and a vertical line labeled e_3 , with external lines labeled 1 , e_2 , and e_1 . The fourth term is a diagram with a loop labeled e_2 and a vertical line labeled e_3 , with external lines labeled 1 , e_2 , and e_1 . The diagrams are connected by plus signs and tensor products (\otimes).



The diagrammatic coaction



- Example:

$$\begin{aligned}
 \Delta \left[\text{Finite} \right] &= \text{Divergent} + \text{Divergent} + \text{Divergent} \\
 &= \left(\text{Diagram 1} \otimes \text{Diagram 2} \right) + \left(\text{Diagram 3} \otimes \text{Diagram 4} \right) + \left(\text{Diagram 5} \otimes \text{Diagram 6} \right),
 \end{aligned}$$

The diagrams are as follows:

- Finite:** A triangle diagram with a thick horizontal line labeled '1' on the left, a thick vertical line labeled 'e₃' on the right, and two diagonal lines labeled 'e₁' (bottom) and 'e₂' (top). The entire diagram is enclosed in a light blue rounded rectangle.
- Divergent (Term 1):** A diagram with a light blue rounded rectangle containing a circle labeled 'e₃' with a vertical line extending downwards from its center. This is tensored with a triangle diagram where the horizontal line '1' and the bottom diagonal line 'e₁' are solid, while the top diagonal line 'e₂' and the vertical line 'e₃' are dashed red.
- Divergent (Term 2):** A diagram with a light blue rounded rectangle containing a bubble diagram with two horizontal lines labeled '1' and two curved lines labeled 'e₁' (bottom) and 'e₂' (top). This is tensored with a triangle diagram where the horizontal line '1' and the vertical line 'e₃' are solid, while the diagonal lines 'e₁' and 'e₂' are dashed red.
- Divergent (Term 3):** A triangle diagram where the horizontal line '1' and the diagonal line 'e₁' are solid, while the diagonal line 'e₂' and the vertical line 'e₃' are dashed red. This is tensored with a triangle diagram where the horizontal line '1' and the diagonal line 'e₁' are solid, while the diagonal line 'e₂' and the vertical line 'e₃' are dashed red.

- Poles cancel due to 'homological identity': $\Gamma_{\infty} = \Gamma_3 + \Gamma_{12} + \dots$

$$-\epsilon \left[\text{Triangle Diagram} \right] = \left[\text{Triangle Diagram 1} \right] + \left[\text{Triangle Diagram 2} \right]$$

The diagrams are as follows:

- Left side:** A triangle diagram with a thick horizontal line labeled '1' on the left, a thick vertical line labeled 'e₃' on the right, and two diagonal lines labeled 'e₁' (bottom) and 'e₂' (top). The entire diagram is enclosed in a light blue rounded rectangle.
- Right side (Term 1):** A triangle diagram where the horizontal line '1' and the bottom diagonal line 'e₁' are solid, while the top diagonal line 'e₂' and the vertical line 'e₃' are dashed red.
- Right side (Term 2):** A triangle diagram where the horizontal line '1' and the diagonal line 'e₁' are solid, while the diagonal line 'e₂' and the vertical line 'e₃' are dashed red.



The Master formula



$$\Delta \left(\int_{\gamma} \omega \right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$$

- Magic formula:
 - ➔ Coproduct = ‘insert a complete set of states’ $1 = \sum_i |\omega_i\rangle \otimes \langle \gamma_i|$
 - ➔ Knows about analytic continuation: $\Delta \text{Disc} = (\text{Disc} \otimes \text{id}) \Delta$
 - ➔ Knows about differential equations: $\Delta \partial_z = (\text{id} \otimes \partial_z) \Delta$
 - ➔ Knows about master integrands and master contours.
 - ➔ Works in DimReg.
- Works in ‘all known cases’ (MPLs & one-loop integrals).
- Does it make (correct!) predictions?
 - ➔ Beyond one loop? Elliptic functions?