A journey beyond multiple polylogarithms. What is there and how do we handle it?

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TTP - KIT, Karlsruhe

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Based on collaboration with *A. von Manteuffel*, *E. Remiddi*, *A. Primo*

[arXiv:1602.01481], [arXiv:1610.08397]

INTRODUCTION

1. Feynman integrals and scattering amplitudes are present fascinating mathematical structures

   $\Downarrow$

   Multiple polylogarithms, Hopf algebras, symbols and coproduct, elliptic polylogarithms ... (?)

2. Starting to understand these structures allowed us to complete precision calculations for the LHC that were unthinkable before !

3. Unique perspective on the mathematics of Feynman integrals provided by the **differential equations method**
   [A.V.Kotikov '90; E.Remiddi '97; T. Gehrmann, E.Remiddi '00; ....]

A revolution in multi-loop calculations has started when physicists have re-discovered the so-called **multiple polylogarithms**
[E.Remiddi, J.Vermaseren '99; T. Gehrmann, E.Remiddi '00; ....]

$$G(0;x) = \ln(x)\,, \qquad G(a;x) = \ln\left(1 - \frac{x}{a}\right) \quad \text{for} \quad a \neq 0$$

$$G(\underbrace{0,\ldots,0}_{n};x) = \frac{1}{n!}\ln^n(x)\,, \qquad G(a,\vec{w};x) = \int_0^x \frac{dy}{y-a}\,G(\vec{w};y)\,.$$

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Multiple polylogarithms are *special* because they satisfy
first order differential equations with rational coefficients

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Interestingly enough, many multiloop Feynman integrals (mostly with massless propagators) satisfy **first order differential equations** in the mandelstam invariants, at least in the limit $d \to 4$!

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As a result, the coefficients of their **Laurent series** for $d \approx 4$ can be written as linear combinations of rational functions and multiple polylogarithms!

a- First order differential equations are easy to solve order by order in $\epsilon = (4-d)/2$

b- Results are written in terms of **Chen iterated integrals** [K-T. Chen '77]

c- Everything changes if the differential equations are of **higher order**...

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Dimensionally regularised Feynman Integrals fulfil differential equations!
[Kotikov '90, Remiddi '97, Gehrmann-Remiddi '00,...]

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Direct consequence of **Integration-by-parts (IBPs)** identities in d-dimensions!

$$\int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d} \left(\frac{\partial}{\partial k_j^{\mu}}\, v_\mu\, \frac{S_1^{\sigma_1} \ldots S_s^{\sigma_s}}{D_1^{\alpha_1} \ldots D_n^{\alpha_n}} \right) = 0\,, \qquad v^\mu = k_j^\mu, p_k^\mu$$

Reduced to N master integrals, $I_i(d;x_k)$ with $i = 1, \ldots, N$.

$\Downarrow$

Differentiating the masters and using the **IBPs** we get a system of
N coupled differential equations

$$\frac{\partial}{\partial\, x_k} I_i(d;x_k) = \sum_{j=1}^{N} c_{ij}(d;x_k)\, I_j(d;x_k)\,.$$

Let's look more in detail - *in reality* equations are in block form

$$I_j(d;x_k) = (m_j(d;x_k)\,,\ \ sub_j(d;x_k))$$

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$$\frac{\partial}{\partial x_k} m_i(d;x_k) = \sum_{j=1}^{N} h_{ij}(d;x_k)\, m_j(d;x_k) + \sum_{j=1}^{M} nh_{ij}(d;x_k)\, sub_j(d;x_k)\,.$$

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homogeneous piece is MAIN source of complexity - whether differential equations are coupled

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No way to solve this in general...
We must use some other "physical" insight...

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non-homogeneous piece is the second source of complexity – we must integrate over it!

$$\Downarrow$$

Can be symplified using differential equations and dispersion relations [E.Remiddi, LT '16]

Let's focus on the most difficult part: **homogeneous equations**

1- In the many cases, it is possible to find a basis of master integrals for which the homogeneous system decouples (becomes triangular) for $d \to 4$.

- Decoupling due to degeneracy of IBPs in **even integer numbers of dimensions**, $d = 2\,n$ [E.Remiddi, L.T. '13; L.T. '15]

- In this case, it was conjectured that a **canonical basis** exists [Henn '13]

$$\frac{\partial}{\partial x_k} I_i(d; x_k) = (d-4) \sum_{j=1}^{N} c_{ij}(x_k)\, I_j(d; x_k)\,, \quad c_{ij}(x_k) \quad \text{in } d\text{-}log \text{ form}\,.$$

- MAIN PROPERTY: the homogeneous equation becomes trivial in $d = 4$

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2- Unfortunately, in many cases the equations remain coupled in $d = 4$.
[*More than one master integral remain linearly independent as* $d \to 4$]

It's enough to start putting some masses in the loops!

Interesting because:

1- LHC is pushing precision beyond 5%

2- High energies and High p_T $\to$ probe massive particles in the loops
   a- Top quark corrections to Hj, HH, $\gamma\gamma$, jj, ...
   b- New massive states?

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We know a couple of examples now

$S(d;p^2) =$

- $p^2 \neq 0$, three massive lines
- 2 master integrals
- Satisfy 2 coupled diff. eqs.
- Needed for NNLO $t\bar{t}$

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massive 3-particle cut

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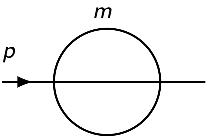
$S(d; p^2) =$ 

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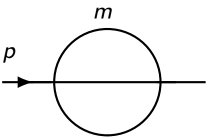
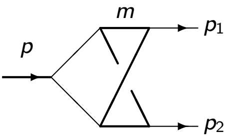
$S(d; p^2) =$ 

$T(d; p^2) =$ 

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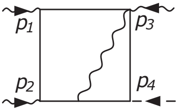
More recently, impressive calculation of two-loop box with five massive propagators [R.Bonciani et al '16]



with $p_1^2 = p_2^2 = p_3^2 = 0\,,\qquad p_4^2 = m_h^2\,,$ and propagator mass m^2

- There are 4 master integrals
- A basis can be found where 3 master integrals are coupled
- The 3×3 system can be reduced to a **second order** differential equation, with the third integral as inhomogeneous term (not obvious!)
- Needed for NLO corrections to Hj production through a top loop

$\Downarrow$

Again, **NO** 3-particle cut!

What do these examples have in common?

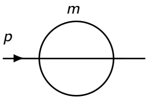
- In all three cases, bulk of complexity is a second-order homogeneous differential equation
- In all three cases, the result can be written as integrals over **complete elliptic integrals** and **multiple polylogarithms**

$$\Downarrow$$

Until now, second order differential equation solved with methods ad hoc, mostly trial-and-error, based on inspecting the differential equations and try to map it to the equation satisfied by the elliptic integrals!

- Interesting approach developed in [R.Bonciani et al '16], based on reparametrizing mandelstam invariants in order to make structure of second order differential equation manifest!

But I want to go back to an older idea, in the context of the Sunrise graph
[S.Laporta, E.Remiddi '04]

$$\left(\frac{d^2}{d\,s^2} + A(d;s)\,\frac{d}{d\,s} + B(d;s)\right)$$


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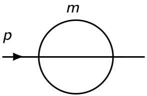
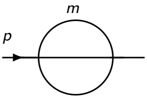


$$+\, G(d;s)\,\mathrm{Tad}(d;m^2) = 0\,.$$

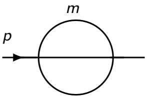
They realized that the imaginary part of the sunrise is precisely the solution homogeneous equation, since the Tadpole doesn't have a branch cut in s!

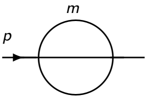
$$\left(\frac{d^2}{d\,s^2} + A(d;s)\,\frac{d}{d\,s} + B(d;s) \right) \mathrm{Im} \left(\right.$$


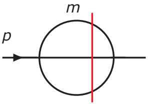
$$\left. \right) = 0\,.$$

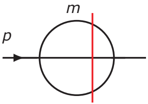
How do we generalize this?

Computing the imaginary part means cutting the graph!

$\mathrm{Im}\Bigg($ 

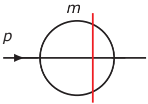
$\Bigg) \propto \mathrm{Cut}\Bigg($ 

$\Bigg) =$ 



$$= \int \mathfrak{D}^d k \, \mathfrak{D}^d l \, \delta(k^2 - m^2) \, \delta(l^2 - m^2) \, \delta((k - l - p)^2 - m^2)$$

For the **sunrise** the maximal cut (which is the discontinuity in s!) gives



$$= \frac{1}{\sqrt{(3m-\sqrt{s})(\sqrt{s}+m)^3}}\, \mathrm{K}\left(\frac{16m^3\sqrt{s}}{(3m-\sqrt{s})(\sqrt{s}+m)^3}\right)$$

where $\mathrm{K}(x)$ is the complete elliptic integral of the first kind.

$$\mathrm{K}(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-x\,t^2)}}$$

This is one solution of its second order differential equation.
The second one can be found algorithmically...

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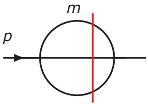

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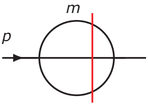

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Clearly, any cut of a graph will fulfil a similar *(simpler)* differential equation

But, since all subtopologies have fewer propagators, if we **cut all propagators** (maximal cut) we are necessarily left only with the homogeneous equation!

The Maximal Cut provides us with **ONE solution** of the homogeneous system!

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$$\frac{\partial}{\partial x_k} m_i(d;x_k) = \sum_{j=1}^{N} h_{ij}(d;x_k)\, m_j(d;x_k) + \sum_{j=1}^{M} nh_{ij}(d;x_k)\, sub_j(d;x_k)\,.$$

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Similar observation made for difference equations (DRA) method
[R. Lee, V. Smirnov '12]

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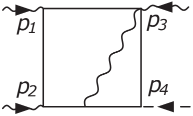

$$= I_{a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7\, a_8\, a_9}\Big|_{a_{7,8,9}<0}$$

$$= \int \mathfrak{D}^d k \mathfrak{D}^d l \frac{(k\cdot p_3)^{-a_7}(l\cdot p_1)^{-a_8}(l\cdot p_2)^{-a_9}}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5} D_6^{a_6}},$$

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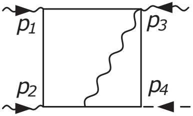

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Very non trivial example. There are 4 master integrals

$$\mathcal{I}_1 = I_{1\,1\,1\,1\,1\,1\,0\,0\,0}\,,\quad \mathcal{I}_2 = I_{1\,2\,1\,1\,1\,1\,0\,0\,0}\quad \mathcal{I}_3 = I_{1\,1\,1\,1\,2\,1\,0\,0\,0}\,,\quad \mathcal{I}_4 = I_{1\,1\,1\,1\,1\,1\,-1\,0\,0}.$$

$\mathcal{I}_4$ is completely decoupled in $d = 4$. The **differential equations** read

$$\begin{cases} \dfrac{\partial}{\partial x}\mathcal{I}_1(\mathbf{x}) = a_{11}(\mathbf{x})\,\mathcal{I}_1(\mathbf{x}) + a_{12}(\mathbf{x})\,\mathcal{I}_2(\mathbf{x}) + a_{13}(\mathbf{x})\,\mathcal{I}_3(\mathbf{x}) + \text{subtopos} \\[2ex] \dfrac{\partial}{\partial x}\mathcal{I}_2(\mathbf{x}) = a_{21}(\mathbf{x})\,\mathcal{I}_1(\mathbf{x}) + a_{22}(\mathbf{x})\,\mathcal{I}_2(\mathbf{x}) + a_{23}(\mathbf{x})\,\mathcal{I}_3(\mathbf{x}) + \text{subtopos} \\[2ex] \dfrac{\partial}{\partial x}\mathcal{I}_3(\mathbf{x}) = a_{33}(\mathbf{x})\,\mathcal{I}_3(\mathbf{x}) + \text{subtopos}\,, \end{cases}$$

Very non trivial example. There are 4 master integrals

$$\mathcal{I}_1 = I_{111111000}\,, \quad \mathcal{I}_2 = I_{121111000} \quad \mathcal{I}_3 = I_{111121000}\,, \quad \mathcal{I}_4 = I_{111111-100}.$$

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$$\Downarrow$$

$$\frac{\partial^2}{\partial x^2}\mathcal{I}_1(\mathbf{x}) + A(\mathbf{x})\,\frac{\partial}{\partial x}\mathcal{I}_1(\mathbf{x}) + B(\mathbf{x})\,\mathcal{I}_1(\mathbf{x}) + C(\mathbf{x})\,\mathcal{I}_3(\mathbf{x}) = \text{subtopos} + \mathcal{O}(\epsilon)\,.$$

Very non trivial example. There are 4 master integrals

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Let's perform a maximal cut now on $\mathcal{I}_1$ and $\mathcal{I}_3$ in $d = 4$, $\epsilon = 0$

$\mathrm{Cut}\,(\mathcal{I}_3) = \mathrm{Cut}$

$=$

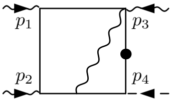
$$= \int \mathfrak{D}^{\,4}k\, \delta(k^2 - m^2)\delta((k-p_1)^2 - m^2)\delta((k-p_1-p_2)^2 - m^2)$$

$\times\ \mathrm{Cut}$

$= 0.$

Let's perform a maximal cut now on $\mathcal{I}_1$ and $\mathcal{I}_3$ in $d = 4$, $\epsilon = 0$

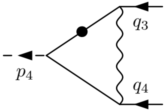
$$\mathrm{Cut}\left(\mathcal{I}_3\right) = \mathrm{Cut}\Bigg($$



$$\Bigg) =$$

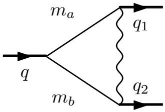
$$= \int \mathfrak{D}^{\,4} k \, \delta(k^2 - m^2)\delta((k - p_1)^2 - m^2)\delta((k - p_1 - p_2)^2 - m^2)$$

$$\times \ \mathrm{Cut}\Bigg($$



$$\Bigg) = 0.$$

The reason is very simple. We have

Cut


$$= \frac{1}{\sqrt{(q^2 + q_1^2 - q_2^2)^2 - 4\; q^2\, q_1^2}}\,.$$

and

Cut

$$= \frac{\partial}{\partial\, m_a^2}\ \mathrm{Cut}$$

$$= 0\ \ !$$

The maximal cut of this one loop triangle in $d = 4$ does not depend on the masses in the propagators!!!!

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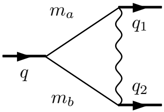
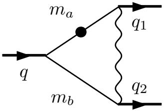


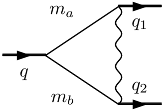
) $= \dfrac{1}{\sqrt{(q^2+q_1^2-q_2^2)^2-4\;q^2\,q_1^2}}$.

and

Cut (



) $= \dfrac{\partial}{\partial\, m_a^2}$ Cut (



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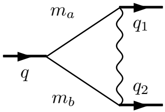

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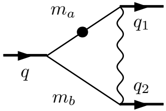

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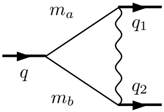


) $= 0$!

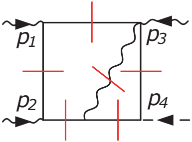
The maximal cut of this one loop triangle in $d = 4$ does not depend on the masses in the propagators!!!!

This finally means that, if we cut everything in $d = 4$ we are left with

$$\frac{\partial^2}{\partial x^2}\mathrm{Cut}(\mathcal{I}_1(\mathbf{x})) + A(\mathbf{x})\,\frac{\partial}{\partial x}\mathrm{Cut}(\mathcal{I}_1(\mathbf{x})) + B(\mathbf{x})\,\mathrm{Cut}(\mathcal{I}_1(\mathbf{x})) = 0\,.$$

I.e. the maximal cut of $\mathcal{I}_1(\mathbf{x})$ provides us with one of the homogeneous solution of the second order differential equation that we want to solve!

Let's compute it then! The simplest way is cutting first the triangle



$$= \int \mathfrak{D}^4 k\, \delta(k^2 - m^2)\delta((k-p_1)^2 - m^2)\delta((k-p_1-p_2)^2 - m^2)$$

$\times$ Cut (

)

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$$\times \frac{1}{\sqrt{m^4 + ((k+p_3)^2 - m_h^2)^2 - 2m^2((k+p_3)^2 + m_h^2)}}$$

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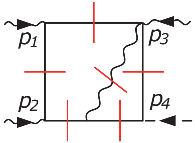

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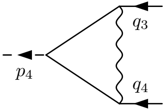
$\times$ Cut $\Bigg($  $\Bigg)$

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In this way we are left with three more delta functions, which can be used to localized 3 out of 4 components of the momentum k. We find

$$F_1 = \mathrm{Cut}(\mathcal{I}_1) = \frac{1}{\sqrt{s\, R(s,t,m_h^2,m^2)}}\, \mathsf{K}\left(\frac{16m^2\sqrt{-s\,t\,u\,m_h^2}}{R(s,t,m_h^2,m^2)}\right)$$

with

$$R(s,t,m_h^2,m^2) = s\left(m_h^2 - t\right)^2 - 4m^2\left(m_h^2(s-t) + t(s+t) - 2\sqrt{-s\,t\,u\,m_h^2}\right)$$

$$\Downarrow$$

The similarity of the structure of the solution to that of the sunrise is striking!

We don't need to do anything to compute the second solution.
If $K(x)$ is a solution, then also $K(1-x)$ must be a solution [known stuff]!

The **second independent solution** is therefore

$$F_2 = \frac{1}{\sqrt{s\, R(s,t,m_h^2,m^2)}}\, \mathrm{K}\left(1 - \frac{16 m^2 \sqrt{-s\,t\,u\,m_h^2}}{R(s,t,m_h^2,m^2)}\right)$$

More, equivalent, solutions can be found using
the invariance of the differential equations under $x \to 1/x$

$$\widetilde{F}_1 = \frac{1}{\sqrt{s\,m^2}(-s\,t\,u\,m_h^2)^{1/4}}\, \mathrm{K}\left(\frac{1}{\omega}\right), \qquad \widetilde{F}_2 = \frac{1}{\sqrt{s\,m^2}(-s\,t\,u\,m_h^2)^{1/4}}\, \mathrm{K}\left(1 - \frac{1}{\omega}\right).$$

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Once the homogeneous solution is known, we can write integral representation for inhomogeneous one. This will contain integrals over elliptic integrals and simpler functions in general.

Understanding the properties of these integrals, which class of functions they form, do they fulfill an algebra and so on, is still matter of study. Many developments happened and are happening in this very moment:

- Elliptic polylogarithms [Bloch, Vanhove '13]
- Alternative (equivalent??) definition [Weinzierl et al '14,'15,'16]
- Relation with hypergeometric functions [Passarino '16]

For now applied only to very simple $1 \to 1$ kinematics of the sunrise graph and the Kite integral. New developments to come hopefully soon!

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CONCLUSIONS

1- We understand much more than just a few years ago on the functions required to compute multiloop feynman integrals

2- A lot is due to differential equations method, and the possibility of decouple them for $d \to 4$

3- Until recently, if decoupling not possible, we were "hopeless"
Elliptic functions $\to$ *hic sunt leones*

4- I think we can say, today we understand something more:
maximal cut can be practically used to solve the homogeneous solution: it is a fundamental step towards a complete solution!

5- What is missing?

a) Understanding of these new functions –
– some promising steps have been taken [S. Weinzierl et al. '14,'15,'16]

b) Handling the algebraic complexity and swell of expressions at intermediate stages –
– brute force (more RAM, faster CPUs,... ?)
– more clever reduction codes?
– finite fields? [A.Manteuffel, R.Schabinger '15; T.Peraro '16]
– Two- (multi-)loop unitarity? [S.Badger et al '13,'14,'15; H. Ita '16; P.Mastrolia et al '11,'13,'16; ....]

THANKS!