A first law for entanglement rates

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arXiv:1612.07769
A. O’Bannon, J. Probst, RR, C. Uhlemann

Entanglement entropy

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Definition of entanglement entropy:

Split the Hilbert space of a quantum system into subspace $\mathcal{A}$ and its complement $\mathcal{B}$. Reduced density matrix:

$$\rho_{\mathcal{A}} = \sum_{|\psi\rangle \in \mathcal{B}} \langle\psi|\rho|\psi\rangle$$

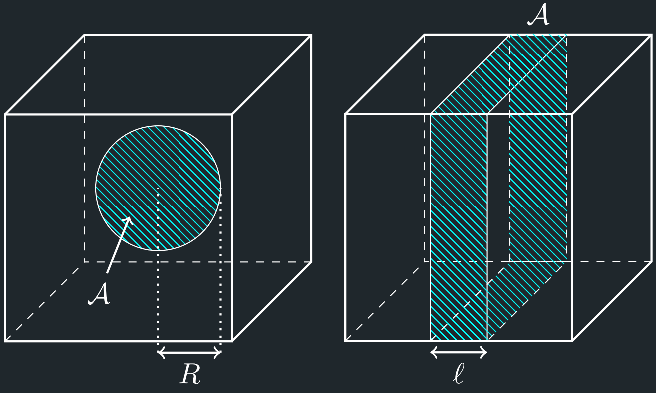
EE = von Neumann entropy of reduced density matrix

$$\boxed{\mathrm{S}_{\mathrm{EE}} = -\mathrm{Tr}\,\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}}$$

Entanglement entropy

For a QFT, it is natural to divide the Hilbert space by dividing a spatial slice of the manifold on which the QFT lives.

We make two choices of $\mathcal{A}$:



Sphere

Strip

The first law of entanglement

Modular Hamiltonian H: $\rho_{\mathcal{A}} \equiv e^{-H}$

For small perturbations of the state of a QFT:

$\delta S_{\mathrm{EE}} = \delta \langle H \rangle$ “First law of entanglement”

D. D. Blanco, H. Casini, L. Hung,
R. C. Myers, 1305.3182 [hep-th]

Sometimes the change in the modular Hamiltonian is proportional to the change in energy

$$\delta S_{\mathrm{EE}} = \frac{\delta E}{T_{\mathrm{ent}}}$$

J. Bhattacharya, M. Nozaki, T. Takayanagi,
T. Ugajin, 1212.1164 [hep-th]

Comparing different states in different theories, the first law may not hold. We will study certain time-dependent perturbations to Hamiltonians.

Entanglement entropy is difficult to calculate in QFT, so we use holography.

Entanglement entropy in holography

In AdS/CFT: a gravity theory on AdS $\Leftrightarrow$ CFT on its boundary

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Time-dependent holographic EE: V. E. Hubeny, M. Rangamani, T. Takayanagi, 0705.0016 [hep-th]

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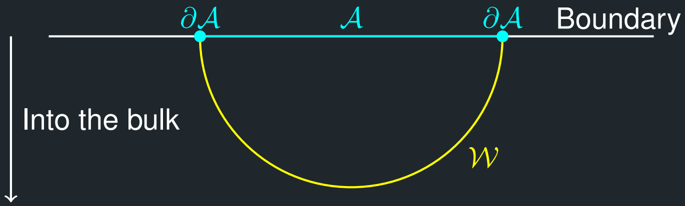

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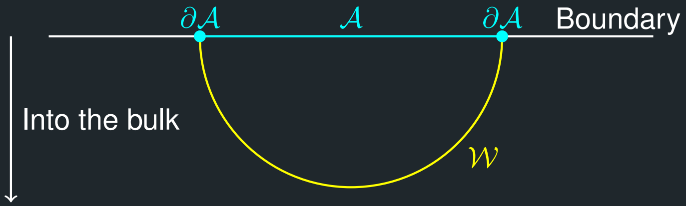

$$S_{\mathrm{EE}} = \frac{A[\mathcal{W}]}{4G_N}$$

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Perturbative holographic entanglement entropy

Area is a functional of embedding: $A[X, G]$

Metric G = background + perturbation

Perturbing the background metric changes EE

$$\delta S_{\mathrm{EE}} = \frac{1}{4G_N} \int \mathrm{d}^{d-2}\xi \left(\frac{\delta A}{\delta X^m} \underset{\substack{\uparrow \\ \text{Change in embedding}}}{\delta X^m} + \frac{\delta A}{\delta G_{mn}} \underset{\substack{\uparrow \\ \text{Change in metric}}}{\delta G_{mn}} \right)$$

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$$\underset{\text{Change in embedding}}{\uparrow} \qquad\qquad \underset{\text{Change in metric}}{\uparrow}$$

$$\boxed{\delta S_{\mathrm{EE}} = \frac{1}{4G_N} \int \mathrm{d}^{d-2}\xi \frac{\delta A}{\delta G_{mn}} \delta G_{mn}}$$

$\dfrac{\delta A}{\delta G_{mn}}$ and the integral are evaluated on the extremal surface in the background metric.

A first law for rates

Asymptotically AdS metric

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\left(dz^2 + g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\right)$$

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(2)}_{\mu\nu} z^2 + \ldots + g^{(d)}_{\mu\nu} z^d + \ldots$$

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Our setup:

Background: $\mathrm{d}s^2 = \frac{L^2}{z^2}\left(\mathrm{d}z^2 + g_{tt}(z)\mathrm{d}t^2 + g_{xx}(z)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j\right)$

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Perturbation:

$\partial_x \delta g_{\mu\nu} = 0$ Translational invariance

$\delta g_{ij} = \delta g_{xx}\delta_{ij}$ Rotational invariance

$\partial_t \delta g^{(n<d)}_{\mu\nu} = 0$ Boundary conditions

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Subregion small compared to other scales $\Rightarrow z$ small

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Subregion small compared to other scales ⇒ z small

First law of entanglement rates: $\partial_t \delta S_{\mathrm{EE}} \propto \partial_t \delta E$

A first law for rates

Under our assumptions, rate of change of EE

$$\boxed{\partial_t \delta S_{\mathrm{EE}} = \partial_t \delta g_{xx}^{(d)} \frac{d}{4G_N} \int \mathrm{d}^{d-2}\xi \, z^d \frac{\delta A}{\delta g_{xx}}}$$

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Boundary stress tensor

$$\langle T_{\mu\nu} \rangle = \frac{dL^{d-1}}{16\pi G_N} g^{(d)}_{\mu\nu} + X_{\mu\nu}\left[g^{(n<d)}\right]$$

S. de Haro, K. Skenderis,
S. N. Solodukhin, hep-th/0002230

$\delta g^{(d)}$ traceless $\Rightarrow \delta\langle T_{tt}\rangle = d\delta\langle T_{xx}\rangle - \delta X^{\mu}{}_{\mu}$

With our assumptions, the Weyl anomaly is time-independent, $\partial_t \delta\langle T^{\mu}{}_{\mu}\rangle = \partial_t \delta X^{\mu}{}_{\mu} = 0$, so rate of change of the energy:

$$\boxed{\partial_t \delta E = \partial_t \delta g^{(d)}_{xx} \frac{d(d-1)L^{d-1}}{16\pi G_N} \mathrm{Vol}(\mathcal{A})}$$

A first law for rates

Rate of change of energy is proportional to rate of change of EE, “First law of entanglement rates”:

$$\partial_t \delta S_{\mathrm{EE}} = \partial_t \delta E \, \underbrace{\frac{1}{4\pi(d-1)L^{d-1}\mathrm{Vol}(\mathcal{A})} \int d^{d-2}\xi \, z^d \frac{\delta A}{\delta g_{xx}}}_{T_{\mathrm{ent}}^{-1}}$$

Massless scalar field

Einstein gravity coupled to massless scalar field

$$S = \frac{1}{16\pi G_N} \int \mathrm{d}^{d+1}x \, \sqrt{-G} \left(R - 2\Lambda - 8\pi G_N \left(\partial\phi\right)^2 \right)$$

Massless scalar $\phi \Leftrightarrow$ Marginal scalar operator $\mathcal{O}$

$$\phi = \underset{\substack{\uparrow \\ \text{Source}}}{\phi_0} + \phi_2 z^2 + \ldots + \underset{\substack{\uparrow \\ \text{Determines } \langle \mathcal{O} \rangle}}{\phi_d} z^d + \ldots$$

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Choose a source $\phi_0 = -ct$.

Shift symmetry of $\phi \Rightarrow g_{\mu\nu}^{(n<d)}$ depend only on derivatives of ϕ
$\Rightarrow$ are time independent.

Massless scalar field

To be concrete, choose $d = 3$.

Solving Einstein's equations near the boundary:

$$g_{\mu\nu} = \eta_{\mu\nu} + \left(8\pi G_N c^2 \delta^t_\mu \delta^t_\nu + 2\pi G_N c^2 \eta_{\mu\nu}\right) z^2 + \delta g^{(3)}_{\mu\nu} z^3 + \ldots$$
$$\phi = -ct + \phi_3 z^3 + \ldots$$

Coefficients of z^3 undetermined by near boundary analysis.

Time-independent contribution to EE from coefficient of z^2.
Time-dependent contribution from coefficient of z^3.

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\begin{aligned}
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\end{aligned}
$$

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Boundary energy momentum tensor

$$
\langle T_{\mu\nu} \rangle = -\frac{2}{\sqrt{-g^{(0)}}} \frac{\delta S}{\delta g^{(0)}_{\mu\nu}} = \frac{3L^2}{16\pi G_N} g^{(3)}_{\mu\nu}
$$

Contribution to energy comes from the coefficient of z^3.

Massless scalar field

$$\begin{aligned} \delta S_{\mathrm{EE}}^{\mathrm{sphere}} &= \frac{\delta E}{T_{\mathrm{ent}}^{\mathrm{sphere}}} + \frac{2}{3}\pi c^2 R^2 \\ \delta S_{\mathrm{EE}}^{\mathrm{strip}} &= \frac{\delta E}{T_{\mathrm{ent}}^{\mathrm{strip}}} + \frac{\pi^2 \Gamma\left(\frac{1}{4}\right)}{6\sqrt{2}\Gamma^3\left(\frac{3}{4}\right)} c^2 \ell \mathrm{Vol}(\mathbb{R}) \end{aligned}$$

First law of entanglement is violated, but taking time derivatives gives a first law for the rates.

Entanglement temperature depends on geometry:

$$\begin{aligned} T_{\mathrm{ent}}^{\mathrm{sphere}} &= \frac{2}{\pi R} \\ T_{\mathrm{ent}}^{\mathrm{strip}} &= \frac{16\Gamma\left(\frac{3}{4}\right)}{\pi \Gamma^2\left(\frac{1}{4}\right)} \frac{1}{\ell} \end{aligned}$$

The same as for a time-independent perturbation of the state of a 3d CFT.

Massless scalar field

Can carry out the same calculation in other dimensions.

In $d = 2$

$$\boxed{\delta S_{\mathrm{EE}} = \frac{\delta E}{T_{\mathrm{ent}}} + \frac{2\pi}{9} c^2 R^2 \left(4 - 3\log(2R/L) + 6\eta\right)}$$

η comes from scheme dependence in renormalization.

In $d = 4$, for a sphere

$$\boxed{\delta S_{\mathrm{EE}} = \frac{\delta E}{T_{\mathrm{ent}}} - \frac{\pi^2}{9} c^2 R^2 \left(5 - 6\log(2R/\epsilon)\right)}$$

ϵ is a short distance cutoff.

An example with an electric field

On $\mathrm{AdS}_5 \times \mathrm{S}_5$, insert a D7-brane filling the $\mathrm{AdS}_5 \times \mathrm{S}_3$ subspace.

$$S_{\mathrm{D7}} \propto \int \mathrm{d}^8\zeta \sqrt{-\det\left(\Gamma_{ab} + 2\pi\alpha' F_{ab}\right)}$$

Metric on D7-brane

U(1) gauge field strength

Make ansatz for x component of gauge field:

$$A_x = -\,\mathcal{E}\,t + a_x(u)$$

Electric field in CFT

Some function of radial coordinate

To compute entanglement entropy and energy in the CFT, we need the linearised backreaction of the brane.

We find the first law for rates is satisfied for sphere and strip.

Conclusions

For time dependent perturbations under certain assumptions a first law for rates holds, but the first law in the form $\delta S_{\mathrm{EE}} \propto \delta E$ can be violated.

Explicit examples of a first law for rates: marginal scalar operators with time-dependent sources, electric field.

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How general is the first law for rates? Answering this could help to understand non-equilibrium physics.