Moonshine: Unexpected Symmetries in String Theory

Sam Fearn

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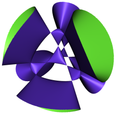

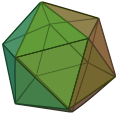
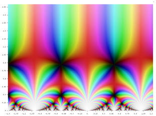

‘The energy produced by the breaking down of the atom is a very poor kind of thing. Anyone who expects a source of power from the transformation of these atoms is talking **moonshine**’

Ernest Rutherford (1937), The Wordsworth Book of Humorous Quotations

Outline

1. Introduction & Motivation
2. Mathieu Moonshine
3. Large $\mathcal{N} = 4$ Theories
4. Conclusions and Summary

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The numbers of the left hand sides are coefficients of the j-invariant, a modular function (form of weight 0, where we allow meromorphicity) for $SL(2, \mathbb{Z})$ with $q \equiv e^{2\pi i \tau}$ expansion

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots \tag{2}$$

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The numbers on the right are dimensions of irreducible representations of the Monster group $\mathbb{M}$, the largest finite sporadic group of order $\approx 8 \times 10^{53}$.

Monstrous Moonshine - A Monster Module

The previous equalities suggest the existence of a graded Monster module $V^\natural$

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Frenkel, Lepowsky and Meurman (FLM)[1] constructed the Moonshine module $V^\natural$.

[1] Igor B Frenkel, James Lepowsky, and Arne Meurman. "A natural representation of the Fischer-Griess Monster with the modular function J as character". In: *Proceedings of the National Academy of Sciences* 81.10 (1984), pp. 3256–3260.

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They showed that this module has automorphism group $\mathbb{M}$ and graded dimension $j(\tau) - 744$. That is, the CFT has $j(\tau) - 744$ as the partition function and has $\mathbb{M}$ symmetry.

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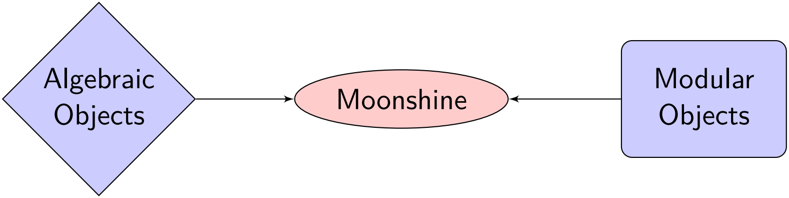

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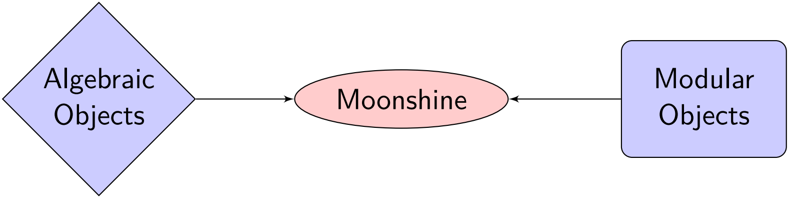

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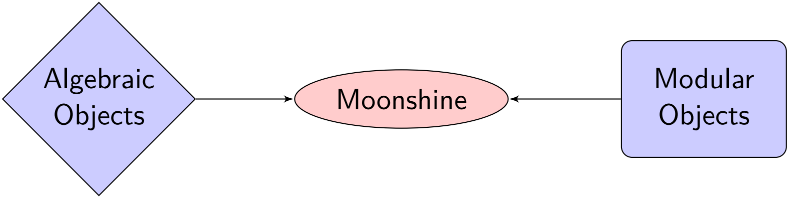

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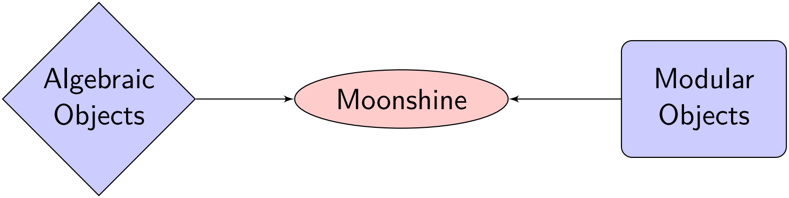

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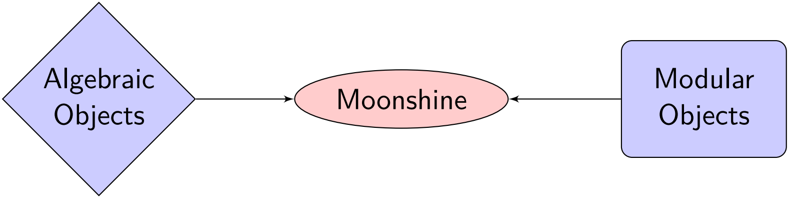

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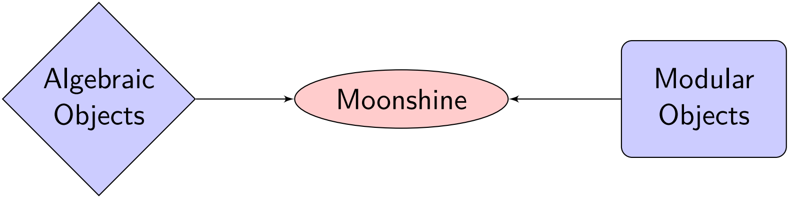

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Therefore,

$$Z(\gamma\tau) = Z(\tau), \qquad \gamma \in SL_2(\mathbb{Z})$$

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We begin by considering the *partition function*:

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The partition function for an $\mathcal{N} = (4, 4)$ theory is given by

$$Z(\tau, z; \bar{\tau}, \bar{z}) = \mathrm{Tr}_{\mathcal{H}}(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L_0} - \frac{c}{24}} y^{2J_0^3} \bar{y}^{2\bar{J}_0^3}) \qquad q = e^{2\pi i \tau}, y = e^{2\pi i z}. \quad (5)$$

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Although the partition function is an important quantity containing the information about all states, it depends on where we are in the (80-dimensional) moduli space of $K3$, $\mathcal{M}_{K3}$.

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For some purposes it is convenient to consider a related quantity known as the Elliptic Genus. This is moduli space independent.

$$\varepsilon_{\mathcal{M}}(\tau, z) := Z_{\tilde{R}}(\tau, z; \bar{\tau}, \bar{z} = 0) \quad (6)$$

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$$\varepsilon_{K3}(\tau, z) := Tr_{\mathcal{H}^R}\left((-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L_0} - \frac{\bar{c}}{24}} y^{2J_0^3}\right) \tag{7}$$

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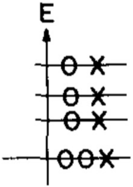

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The elliptic genus of $K3$ can be shown[3] to be a weak Jacobi form of weight 0 and index 1.

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$$\varepsilon_{K3} = 8\left[\frac{\theta_2(\tau, z)^2}{\theta_2(\tau)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau)^2}\right] \tag{8}$$

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ε is a topological invariant and can be related to other invariants,

$$\varepsilon_{K3}(\tau, z = 0) = \chi(K3) = 24 \tag{9}$$

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The Elliptic Genus in $\mathcal{N} = 4$ Characters

Alvarez-Gaumé and Freedman[4] showed that a sigma model on a hyperkähler manifold has $\mathcal{N} = 4$ symmetry.

[4] Luis Alvarez-Gaume and Daniel Z Freedman. "Geometrical structure and ultraviolet finiteness in the supersymmetric σ-model". In: *Communications in Mathematical Physics* 80.3 (1981), pp. 443–451.

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In terms of $\mathcal{N} = 4$ characters we can expand the elliptic genus as

$$\varepsilon_{K3}(\tau, z) = 24\mathrm{ch}^{\tilde{R}}_{l=0}(\tau, z) + \Sigma(\tau) q^{\frac{1}{8}} \hat{\mathrm{ch}}^{\tilde{R}}_{l=1/2}(\tau, z) \tag{10}$$

where

$$\Sigma(\tau) = q^{-\frac{1}{8}}(-2 + \mathbf{90}q + \mathbf{462}q^2 + \mathbf{1540}q^3 + \mathbf{4554}q^4 + \ldots). \tag{11}$$

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These coefficients are all sums of dimensions of irreducible representations of M_{24}.

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Outline

A Larger Algebra

Spindel et al.[5] studied σ-models on group manifolds. They found that it was possible to construct a model with $\mathcal{N} = 4$ superconformal symmetry with an $SU(2) \oplus SU(2) \oplus U(1)$ Kac-Moody subalgebra and four free fermions. This algebra is known as A_γ, or the Large $\mathcal{N} = 4$ algebra.

[5] Philippe Spindel et al. "Complex structures on parallelised group manifolds and supersymmetric σ-models". In: *Physics Letters B* 206.1 (1988), pp. 71–74.

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The fermions and $U(1)$ algebra can be decoupled, such that A_γ factorises

$$\mathrm{Ch}^{A_\gamma} = \mathrm{Ch}^{A_{QU}} \times \mathrm{Ch}^{\tilde{A}_\gamma} \tag{12}$$

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Characters for the Large $\mathcal{N} = 4$ algebra are defined by

$$\mathrm{Ch}^{A_\gamma, R} = \mathrm{Tr}_{\mathcal{H}^R}(q^{L_0 - c/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}} \chi^{iU_0}). \tag{13}$$

In terms of the levels, $c = \frac{6k^+k^-}{k}$, where $k = k^+ + k^-$.

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This can be seen as a consequence of the 0-modes of the free fermions.

$$\mathrm{Ch}^{A_{QU}, \tilde{R}}\Big|_{z_+ = z_-} = 0 \tag{15}$$

Gukov's Index

Gukov et al.[6] used the fact that massive characters have a double zero at $z_+ = z_-$ while massless characters only have a single zero, to define a new index which I refer to as the Gukov Index.

[6] Sergei Gukov et al. "An index for 2D field theories with large $\mathcal{N} = 4$ superconformal symmetry". In: *arXiv preprint hep-th/0404023* (2004).

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$$I_L(\mathcal{C}) := -z_+ \frac{\partial}{\partial z_-} Z\Big|_{z_+=z_-=z} \tag{16}$$

$$= Tr\left(2T_0^{-3}(-1)^{2T_0^{-3}} q^{L_0-c/24} z^{2T_0^{+3}+2T_0^{-3}}\right) \tag{17}$$

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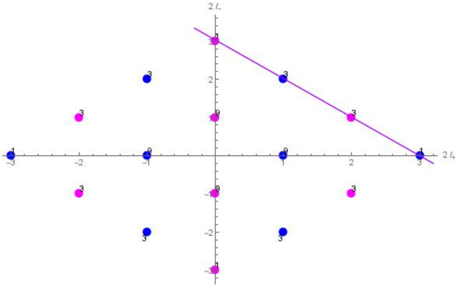

Gukov's Index

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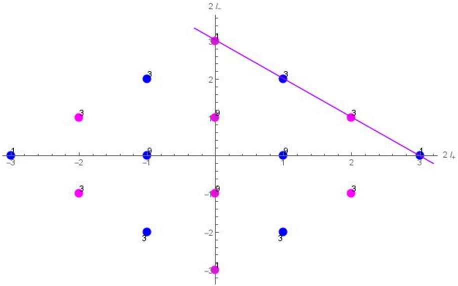

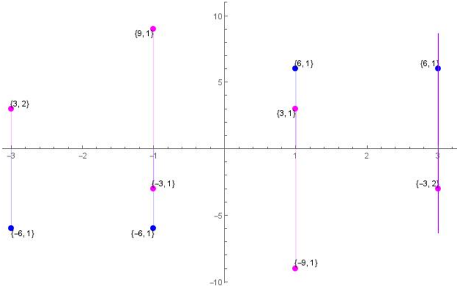

Charges of Gukov states

Using the characters calculated by Petersen and Taormina we can obtain[6]

$$-z_+ \frac{\partial}{\partial z_-}\bigg|_{z_+=z_-=z} \mathrm{Ch}_0^{A_\gamma,\tilde{R}}(k^+,k^-,l^+,l^-,u) = (-1)^{2l^-1} q^{u^2/k} \Theta^-_{\mu,k}, \quad (18)$$

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$$\Theta^-_{\mu,k} = q^{\mu^2/4k}\sum_{n\in\mathbb{Z}} q^{kn^2+n\mu}(z^{2kn+\mu} - z^{-2kn-\mu}) \quad (19)$$

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Outline

1. Introduction & Motivation
2. Mathieu Moonshine
3. Large $\mathcal{N} = 4$ Theories
4. Conclusions and Summary

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- There exists a larger $\mathcal{N} = 4$ algebra A_γ, which is found by considering WZW models on Wolf spaces (certain group cosets). The Gukov index is an invariant of such theories, generalising the elliptic genus.
- The Gukov index receives contributions only from massless states, but unlike the elliptic genus, states throughout the representation.
- We're currently studying the Gukov index of some particular models in more detail.

Thanks for listening!

A sporadic group

Theorem

The Classification of Finite Simple Groups.

This theorem states that all finite simple groups fall into one of the following families:

1. Cyclic groups of order n for n prime.
2. Alternating groups of degree at least 5.
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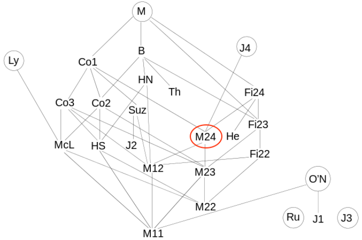
M_{24} is one of the sporadic finite simple groups. It is a subgroup of the Monster group M, as shown below.



The Golay Code

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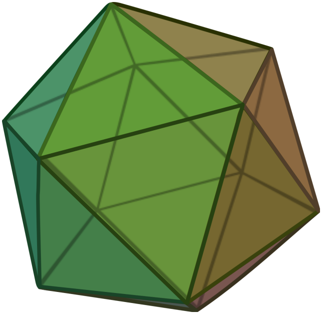

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Lexicographic Code

$$
\begin{aligned}
c_0 &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
c_1 &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1) \\
c_2 &= (0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,1,1,1,1) \\
c_3 &= (0,0,0,0,0,0,0,0,0,0,1,1,0,0,1,1,0,0,1,1,0,0,1,1) \\
&\vdots
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The Mathematical Game of Mogul

H H H H H H H H H H H H H H H H H H H H H H H H

$\downarrow$

H H T H H H T H H T H H H H H T H H H H H T H T

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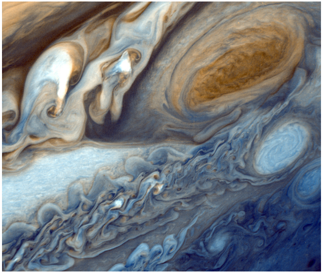
The Golay code was used to transmit photos back from the Voyager spacecraft.



M_{24}

We can define M_{24} in many different ways, however one that suits us is the following.

Definition

$$M_{24} := Aut(\mathcal{G}_{24}) \tag{21}$$

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M_{24} has order $2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 244823040$

Unitary HW Representations

In the Ramond sector, unitary highest weight representations are labelled by the quantum numbers of the SU(2)'s $I_R^{\pm}$, the U(1) charge u and the conformal dimension h.

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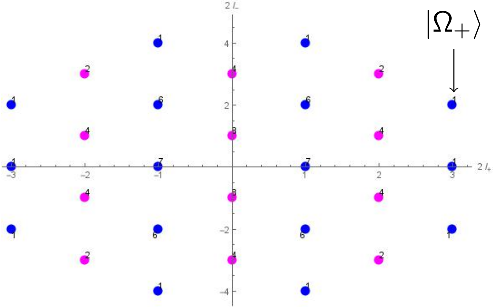

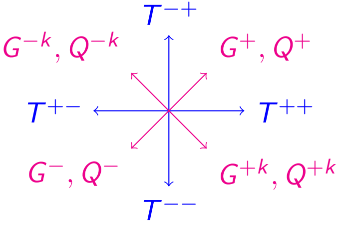

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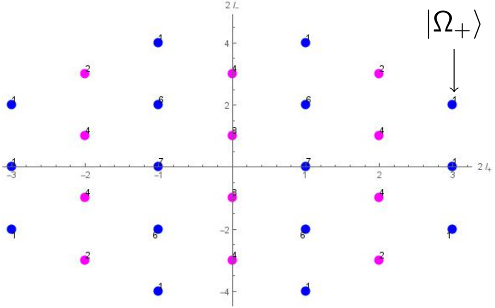

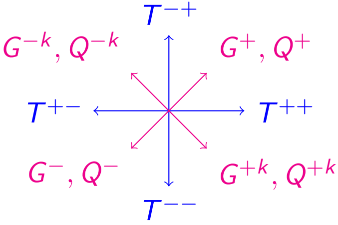

We can see that there is no unique highest weight state, instead we have a state $|\Omega_+\rangle$ which is a highest weight of $SU(2)^+$ and satisfies

$$T_0^{+3}|\Omega_+\rangle = l_+^+|\Omega_+\rangle = l_R^+|\Omega_+\rangle \tag{22}$$

$$T_0^{-3}|\Omega_+\rangle = l_+^-|\Omega_+\rangle = (l_R^- - 1)|\Omega_+\rangle \tag{23}$$

Massive and Massless Representations

Considering the norm $|Q_0^{-k} G_0^{-k} |\Omega_+\rangle\,|$ leads to a unitarity bound

$$hk \geq u^2 + (l_+^+ + l_+^-)^2 + \frac{k^+ k^-}{4} \tag{24}$$

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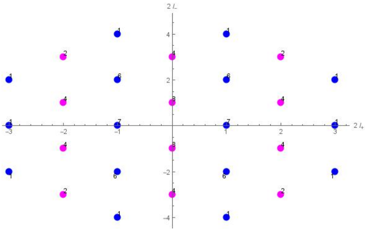
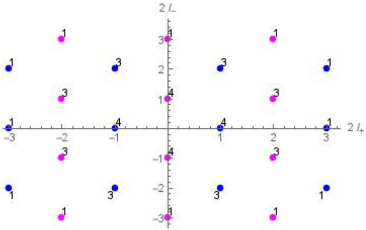
When this bound is saturated, we call the representation a massless representation. Representations which are not massless are massive.



Massive Representation



Massless Representation

$SU(2|2)$

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In the Ramond sector, the zero-mode subalgebra is given by the Lie superalgebra $SU(2|2)$. This can be described[7] by block matrices

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \tag{25}$$

with the elements of A,D (B,C) in the even (odd) part of a complex Grassman algebra such that

$$\mathrm{STr}(M) := \mathrm{Tr}(A) - \mathrm{Tr}(D) = 0 \tag{26}$$
$$M + M^{\ddagger} = 0. \tag{27}$$

[7] JF Cornwell. "Group theory in physics Vol III: supersymmetries and infinite-dimensional algebras". In: *Techniques of Physics* 10 (1989).

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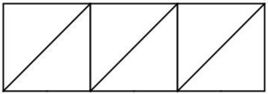
These conditions can be used to show that $SU(2|2)$ has 7 bosonic generators and 8 fermionic generators which satisfy the 0-mode algebra of A_γ.

[7] Cornwell, "Group theory in physics Vol III: supersymmetries and infinite-dimensional algebras".

Representations and Supertableaux

Representations of $SU(2|2)$ can be classified by supertableaux[8], in a similar manner to SU(N) representation; we consider symmetrised and anti-symmetrised tensor products of vectors. These vectors now live in a graded vector space.



[8] A Baha Balantekin and Itzhak Bars. "Dimension and character formulas for Lie supergroups". In: *Journal of Mathematical Physics* 22.6 (1981), pp. 1149–1162.

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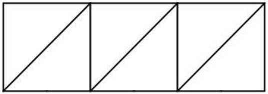
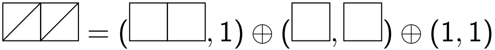


The bosonic subalgebra of $SU(2|2)$ is $SU(2) \times SU(2) \times U(1)$. Representations of $SU(2|2)$ can therefore be branched into irreducible representations of $SU(2) \times SU(2) \times U(1)$.



(28)

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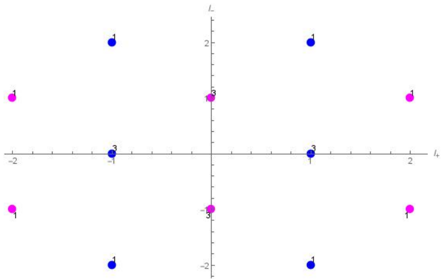
A_γ in supertableaux



Ground level for $k^+ = 3$, $k^- = 2$, $l^+ = l^- = 1$.

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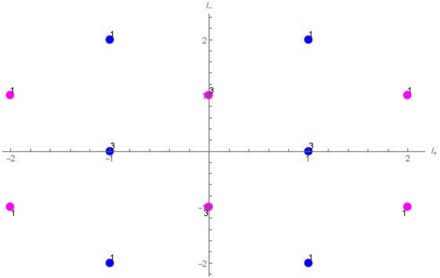

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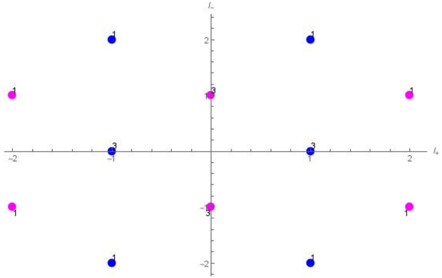

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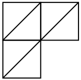
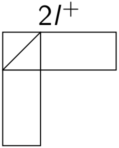


$$= 2(\square, 1) \oplus 2(1, \square) \oplus (\square\square, \square) \oplus (\square, \square\square) \qquad (29)$$

In general the ground level is always described by a single tableau

$$\mathrm{Ch}_0^{A_\gamma, R} = \left(2l^- \right.$$



$$\left. \right) q^{h-c/24} + \dots \qquad (30)$$

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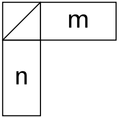
$I_L\Bigg($ 

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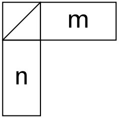
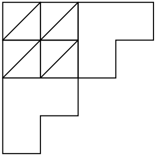
$I_L\Big($  $\Big) = (-1)^n(z^{-m-n-1} - z^{m+n+1})$ (31)

$I_L\Big($  $\Big) = 0$ (32)

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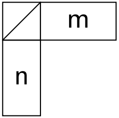

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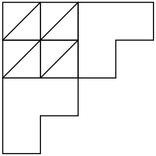

$$I_L\left(\quad \right) = 0 \tag{32}$$

These are the only possible types of supertableau for $SU(2|2)$, so all contributions to the Gukov index of an A_γ representation comes from tableaux of the first type. The first type of tableau is the massless $SU(2) \times SU(2)$ multiplet, the second type is the massive multiplet.

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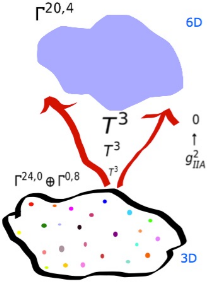
- Monstrous Moonshine was hidden in the partition function of a particular CFT
- The Elliptic Genus of $K3$, which revealed Mathieu Moonshine when written in terms of $\mathcal{N} = 4$ characters, described the right-moving ground states of the theory.

Umbral moonshine can also be seen in terms of the elliptic genus of K3: Recall that we split the elliptic genus into massless and massive characters of $\mathcal{N} = 4$. We can instead split the elliptic genus into a part corresponding to some surface singularities of the $K3$ and the remaining 'Moonshine' part which encodes the moonshine form[9].

[9] Miranda CN Cheng and Sarah Harrison. "Umbral Moonshine and K3 Surfaces". In: *arXiv preprint arXiv:1406.0619* (2014).

Hidden Physics

Kachru et al.[10] consider 3d gravity theories by for instance compactifying the Type II string on $K3xT^3$. The moduli space of such theories can be thought of as the space of 32-dimensional even unimodular lattices of signature (8,24). In a neighbourhood of some particular points in this moduli space the theory has Umbral symmetry.



[10] Shamit Kachru, Natalie M Paquette, and Roberto Volpato. "3D String Theory and Umbral Moonshine". In: *arXiv preprint arXiv:1603.07330* (2016).

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- Monstrous moonshine can be explained in terms of a string propagating on an orbifold of the 'Leech Torus' $\mathbb{R}^{24}/\Lambda$ where the j-invariant describes the partition functions for the theory. In Mathieu Moonshine we don't consider the full partition function but the elliptic genus which only counts half BPS states (right moving ground states).