Thermal Conductivity on Curved Manifolds in the Hydrodynamic Limit

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YTF9

Based on "Thermal backflow in CFTs" [arXiv: 1610.00392] by E. Banks, A. Donos, J. Gauntlett, T. Griffin and L. Melgar

and

work in collaboration with A. Donos, J. Gauntlett

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- Calculation of transport coefficients for strongly coupled field theories [1].
- "Hydrodynamic expansion" for gravity: fluid/gravity correspondence [2].

[1] [Kovtun et al. '05, $\cdots$]
[2] [Bhattacharyya et al. '08 (1), Bhattacharyya et al. '08(2), $\cdots$]

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Motivation from experiment:

- Recent experiments with strained graphene [5].

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$$T_{\mu\nu} = \epsilon u_\mu u_\nu + \left[P(\epsilon) - \zeta_b(\epsilon) D_\lambda u^\lambda\right](g_{\mu\nu} + u_\mu u_\nu) - 2\eta(\epsilon)\sigma_{\mu\nu} \quad (1)$$

where the shear tensor is

$$\sigma_{\mu\nu} = D_{(\mu} u_{\nu)} + u_{(\mu} u^\lambda D_\lambda u_{\nu)} - (g_{\mu\nu} + u_\mu u_\nu)\frac{D_\lambda u^\lambda}{d-1} \quad (2)$$

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P is the pressure, ζ_b is the bulk viscosity and η is the shear viscosity.

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Imposing the tracelessness condition (4) on the stress tensor (1), we find $\zeta_b = 0$, $\epsilon = (d-1)P$, and so we get

$$T_{\mu\nu} = P\left(g_{\mu\nu} + d u_\mu u_\nu\right) - 2\eta\sigma_{\mu\nu} \tag{5}$$

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By dimensional analysis we have $P = c_0 T^d$ and $\eta = c_1 T^{d-1}$, where c_0 and c_1 depend on the microscopics of the CFT.

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For example, we can take g_{ij} to be periodic in the spatial directions (i.e. metric on torus).

We perturb the CFT by introducing a thermal gradient $\zeta \equiv -T^{-1}dT$. We take ζ to be a closed 1-form, so *locally* we can write $\zeta = d\phi$, where $\phi = -\ln T$.

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$$ds^2 = -(1-2\phi)dt^2 + g_{ij}(x)dx^i dx^j \tag{7}$$

and the perturbed fluid velocity becomes

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This gives rise to a temperature variation around the equilibrium value $T = T_0 + \delta T$.

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$$T_0\partial_t\delta u_i - 2\frac{c_1}{dc_0}\left(\nabla^j\nabla_{(j}\delta u_{i)} - \frac{1}{d-1}\nabla_i\nabla_j\delta u^j\right) + \nabla_i\delta T = T_0\zeta_i \tag{9a}$$

$$(d-1)\,T_0^{-1}\partial_t\delta T + \nabla_i\delta u^i = 0 \tag{9b}$$

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- ∇_i is the covariant derivative with respect to g_{ij}.
- Only the ratio $c_1/dc_0 = \eta_0/s_0$ depends on the microscopic CFT.

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In terms of our perturbations we have

$$c_0 d T_0^{d-1} \sqrt{g} \delta u^i = \kappa^{ij} \zeta_j \tag{12}$$

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$$-i\omega\beta_i - 2\nabla^j\nabla_{(j}\beta_{i)} + \partial_i\sigma = \xi_i \tag{13a}$$

$$\nabla_i\beta^i = 0 \tag{13b}$$

where we have traded the perturbations $(\delta u_i, \delta T, \zeta_i)$ for (β_i, σ, ξ_i) respectively.

AC Thermal Conductivity

We can now write (13a) in the following way

$$(\mathcal{D} - i\omega)\,\beta_i = \mathcal{G}\xi_i \tag{14}$$

where the linear operators

$$\begin{aligned} \mathcal{D}\beta_i &= -2\nabla^j\nabla_{(j}\beta_{i)} + 2\nabla_i\Box^{-1}\left(\nabla^{(k}\nabla^{l)}\nabla_{(k}\beta_{l)}\right) \\ \mathcal{G}\xi_i &= \xi_i - \nabla_i\Box^{-1}\left(\nabla_l\xi^l\right) \end{aligned} \tag{15}$$

act on closed 1-forms ξ_i and co-closed 1-forms β_i.

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- For one-dimensional lattices $ds^2 = \gamma(x)dx^2 + g_{ab}(x)dx^a dx^b$, i.e. when our system depends only on 1 coordinate x.

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- Thermoelectric conductivities?
- Holography?

Thank you for your attention

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